Math 497C Homework 4 — Due October 3rd

(1) A *lune* is the region of a sphere enclosed between two great circular arcs. Use Archimedes' tombstone theorem to find the area of a lune in terms of the angle at which the two arcs meet and the radius of the sphere.

(2) Let ABC be a triangle on a sphere of radius r, whose sides are great circular arcs and whose angles are α, β and γ. Let A', B', and C' be the points antipodal to A, B and C respectively. Show using the previous question that the areas of ABC and $A'BC$ add up to 2α, and find two similar identities involving the points B' and C'.

Explain why the areas of ABC, $A'BC$, $AB'C$, and ABC' add up to $2\pi r^2$. Deduce a formula for the area of ABC in terms of α, β and γ.

A sailor circumnavigates Australia by a route consisting of a triangle of great circular arcs. Show that one angle of the triangle is at least 63 degrees. (Assume radius of earth $= 4000$ miles, area of Australia $= 3$ million square miles.)

(3) A tensor in three-dimensional Euclidean space is called *isotropic* if its components remain unchanged by any rotation of the axes (i.e. any transformation in $SO(3)$). Verify that

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if two or three of } i, j, k \text{ are equal} \\ +1 & \text{if } i, j, k \text{ are all different and appear in cyclic order } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ are all different and appear in cyclic order } 1, 3, 2 \end{cases}$$

are homogeneous tensors. Show that the vector product of vectors A_j and B_k is the vector $\epsilon_{ijk} A_j B_k$.

Verify the identity

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp};$$

what does this say in terms of vector and scalar products?

Note: As we are working in Euclidean space the difference between 'up' and 'down' indices disappears; we have written them all 'down'.