

Math 411 Homework 6 Due Thursday, April 14th

1. A system of differential equations $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ in the plane is said to be a *gradient system* if $\mathbf{F} = \nabla V$ for some nontrivial smooth function V . Show that a gradient system cannot have any closed orbits. Use this principle to show that the system

$$\dot{x} = (1 + x - y^2)e^x, \quad \dot{y} = -2ye^x$$

has no closed orbits.

2. Use Dulac's criterion to show that the system

$$\dot{x} = x^3(1 + y^2) - e^y, \quad \dot{y} = -3e^{-x} + 2y^5$$

has no closed orbits.

3. Show that the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has a periodic solution. (PPLANE output by itself is not sufficient; you should give an argument using the Poincaré-Bendixson theorem.)

4. Consider the nonlinear oscillator equation

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0.$$

Suppose that the function F has the property that there are two positive constants, a and b , such that $F(x, \dot{x}) < 0$ if $x^2 + \dot{x}^2 < a$ and $F(x, \dot{x}) > 0$ if $x^2 + \dot{x}^2 > b$. Show that the system has at least one periodic solution for which $x^2 + \dot{x}^2$ remains between a and b . Can you give a physical interpretation of the assumptions on the function F ?

5. Consider the nonlinear oscillator

$$\ddot{x} + \mu(x^4 - 1)\dot{x} + x = 0.$$

Show that the equation has a unique stable limit cycle for $\mu > 0$. What happens as μ approaches 0? If $\mu < 0$, does the system still have a limit cycle? If so, is it stable or unstable?

You may illustrate your solution with PPLANE plots.