Math 411 Homework 4 Due Thursday, March 17th

1. Find the solution of the linear system

$$\dot{x} = 4x - y, \quad \dot{y} = 2x + y$$

with initial condition $x(0) = 2$, $y(0) = 1$.

2. A square matrix M is called *skew-symmetric* if its transpose is equal to its negative (the transpose of a matrix is obtained by reflecting the matrix in its leading diagonal). For example, a 2×2 skew-symmetric matrix has the form $\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right)$.

Show that if M is a skew-symmetric square matrix, of any size, then the solutions to the differential equation

$$\dot{\mathbf{x}} = M\mathbf{x}$$

have the property that $|\mathbf{x}|^2$ is independent of time. (This is a version of the principle of conservation of energy.) What does this tell you about the nature of the fixed point at the origin?

3. A resistor (R ohms), a capacitor (C farads) and an inductor (L henrys) are connected in parallel. The voltage V across the system and the current I flowing in the inductor are related by

$$L\frac{dI}{dt} = V, \quad C\frac{dV}{dt} = -(I + V/R).$$

(Extra work for EE majors: Derive these equations.) Write down the matrix for this linear system. Show that (damped) oscillations can occur if and only if R is greater than some critical value (depending on L and C).

4. Consider the linear system

$$\dot{x} = -3x + 4y, \quad \dot{y} = 6x - 8y.$$

Is the fixed point at the origin attracting? Is it Liapunov stable? Give reasons.

5. Let $\mathbf{v}$ be a constant unit vector in 3-dimensional space. Show that the linear differential equation

$$\dot{\mathbf{x}} = \mathbf{v} \times \mathbf{x}$$

has solutions

$$\mathbf{x} = a\mathbf{v} + \mathbf{w}\cos(t) + \mathbf{v} \times \mathbf{w}\sin(t),$$

where a is a constant scalar and $\mathbf{w}$ is a constant vector that is perpendicular to $\mathbf{v}$. Describe these solutions geometrically.

(Here $\times$ denotes the vector ('cross') product of two given vectors.)