

## Math 411 Homework 2 Due Thursday, February 3rd

1. Consider the system of differential equations

$$\dot{x} = -x(x^2 + y^2)^{-1/2}, \quad \dot{y} = -y(x^2 + y^2)^{-1/2}.$$

Let  $r = (x^2 + y^2)^{1/2}$ . Calculate  $dr/dt$ , and hence show that every solution curve eventually passes through the point  $(0, 0)$ . Explain how this behavior is compatible with the Existence and Uniqueness Theorem for ordinary differential equations.

2. Show that the differential equation

$$\dot{x} = x(r - e^x)$$

undergoes a transcritical bifurcation at a certain value of the parameter  $r$ . Find this value, and sketch the bifurcation diagram.

3. The differential equation

$$\dot{N} = aN(1 - N/k) - h$$

provides a very simple model of the behavior of a natural resource  $N$  (like the number of fish in the sea) which naturally grows according to a logistic law (growth rate  $a$ , carrying capacity  $k$ ) but which is also ‘harvested’ at a fixed rate  $h$ . Imagine that  $a$  and  $k$  are fixed but that  $h$  is controllable. Investigate the dynamics of this system and show that a bifurcation occurs for a certain critical value of  $h$  (which you should find). What kind of bifurcation is it? Comment on the long-term behavior of the population when  $h$  is below, and when  $h$  is above, the threshold value.

In what ways is this model unrealistic?

4. Show that the differential equation

$$\dot{x} = x + \frac{rx}{1 + x^2}$$

undergoes a bifurcation at a certain value of the parameter  $r$ . Find this value, and sketch the bifurcation diagram. What kind of bifurcation takes place here?

5. Consider the standard form for a one-dimensional saddle-node bifurcation

$$\dot{x} = r + x^2.$$

Show that for  $r > 0$  there exist solution curves running from  $x = -1$  to  $x = +1$ , and that when  $r$  is small the length of time that the solution takes to get from  $x = -1$  to  $x = +1$  is proportional to  $r^{-1/2}$ . Sketch the graph of  $x$  against  $t$  for such a solution curve.