What I had planned to say on February 24th

I wanted to begin with problem 4 from homework 3: solve the system of differential equations

$$\dot{x} = y - x, \quad \dot{y} = 3y - 4x.$$

The corresponding matrix is $\left(\begin{smallmatrix} -1 & -4 \\ 1 & 3 \end{smallmatrix}\right)$, which has a repeated eigenvalue of 1. Only one independent eigenvector can be found for this eigenvalue, namely the vector $\left(1//2 \right)$ or multiples thereof.

The technique which we discussed last week, which is based on there being a full set of independent eigenvectors, therefore breaks down and *cannot be applied* without modification. One object of this class is to understand what linear algebra does tell us in the case of repeated eigenvalues. But before getting to the theory, let's solve problem 4 by hand.

Taking the hint $z = y - 2x$, we find

$$\dot{z} = \dot{y} - 2\dot{x} = (3y - 4x) - 2(y - x) = y - 2x = z.$$

We can solve this differential equation for z to get

$$z = ae^t$$

where a is a constant. Now write

$$\dot{x} = y - x = (y - 2x) + x = z + x = ae^t + x.$$

This is a differential equation for x which can be solved by the method of integrating factors:

$$a = e^{-t}(\dot{x} - x) = d/dt(e^{-t}x)$$

and so $e^{-t}x = at + b$, where b is another constant. Thus we get the general solution

$$x = (at + b)e^t, \quad y = z + 2x = (2at + (a + 2b))e^t.$$

Notice the appearance of the term te^t; if you think back to Math 250 you may remember that similar terms can show up in constant coefficient second order linear equations when the characteristic equation has repeated roots.

Now we will do some more linear algebra theory. Suppose that M is a square matrix. We know how to multiply matrices so we can define $M^2 = M \cdot M$, $M^3 = M \cdot M \cdot M$, and so on. We can then make the following definition:

DEFINITION: The *exponential* of the square matrix M, written $\exp(M)$ or e^M, is the sum of the infinite series (of matrices)

$$\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots .$$

Exercise for the virtuous: Figure out what it means for a series of matrices to converge, and prove that this one does.

Notice that if M is a *diagonal* matrix, say with $\lambda_1, \ldots, \lambda_n$ down the diagonal, then its powers are all diagonal too and thus in fact e^M is also a diagonal matrix, with diagonal entries $e^{\lambda_1}, \ldots, e^{\lambda_n}$.

PROPOSITION: Let t be real. Then

$$\frac{d}{dt}\left(e^{tM}\right) = Me^{tM}.$$

To prove this, differentiate the series term-by-term. What it means is that $\mathbf{x} = e^{tM}\mathbf{x}_0$ is the unique solution to the linear differential equation $\dot{\mathbf{x}} = M\mathbf{x}$ with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. The problem of solving such a linear system is therefore reduced to: how do we actually *calculate* these matrix exponentials?

PROPOSITION: If $N = R^{-1}MR$, then $e^{tN} = R^{-1}e^{tM}R$.

To prove this, the key calculation is

$$N^k = (R^{-1}MR)^k = R^{-1}MR \cdot R^{-1}MR \cdots R^{-1}MR = R^{-1}M^kR$$

because the adjacent R and R^{-1} pairs cancel. (Beware that matrix multiplication is not commutative in general!).

Matrices related like M and N above are called *similar*. The proposition tells us that if M is similar to N, and we can calculate the exponential of N, then we can also calculate the exponential on M. On the other hand, we saw above that it is very easy to calculate the exponential of a diagonal matrix, and therefore we can calculate the exponential of any matrix that is similar to a diagonal matrix. Linear algebra tells us that an $n \times n$ matrix with n distinct eigenvalues is similar to a diagonal matrix, so we can calculate its expoential. This is nothing but a fancy algebraic presentation of the method we covered last week.

The matrix $\left(\begin{smallmatrix} -1 & -4 \\ 1 & 3 \end{smallmatrix}\right)$, however, is *not* similar to *any* diagonal matrix (exercise: why not?). The best that we can do is to make just one of the two off-diagonal terms equal to zero: you can check that if $R = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ then

$$N = R^{-1}MR = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is then easy to prove by induction that

$$N^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$e^{tN} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} k \cdot \frac{t^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$

Notice how the term te^t falls out of the power series manipulations. You can now calculate $e^{tM} = Re^{tN}R^{-1}$ and check that it gives the same general solution that we obtained by hand above.

In general the *Jordan Normal Form Theorem* tells us that any square matrix is similar to one which is 'almost' diagonal; the eigenvalues (including possible repetitions) appear down the diagonal, and the only possible nonzero off-diagonal entries may be some 1s, just above the diagonal, between two occurrences of the same eigenvalue. The consequence for dynamics is that the solutions of $\dot{\mathbf{x}} = M\mathbf{x}$ for an n-dimensional vector $\mathbf{x}$ are just superpositions of the kinds of behaviors we have already described: exponential growth or decay in lines corresponding to real eigenvectors, oscillatory or spiraling motion in planes corresponding to complex eigenvectors, and possible polynomial-times-exponential terms arising from the off-diagonal 1s in a Jordan block. Basically, we have already seen in two dimensions all the possible kinds of phenomena that can arise.