MATH 230H HOMEWORK 9 DUE NOVEMBER 29th

Please make sure to <u>staple all pages together</u> and to write your name on your homework.

(1) Find the moment of inertia about the z-axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.

(2) A planet P takes the form of a thin circular disc of uniform density. A spacecraft S moves on the axis of P (the straight line through the center of P and perpendicular to it). Show that the gravitational attraction exerted by P on S is proportional to $1 - \cos\alpha$, where 2α is the angle subtended by P at S.

(3) Use Stokes' theorem to evaluate

$$\int_C (y^2 + z^2)\,dx + (z^2 + x^2)\,dy + (x^2 + y^2)\,dz,$$

where C is the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, traversed counterclockwise when viewed from above.

(4) A function $f(x, y, z)$ is *homogeneous of degree* n, which means that for all $t > 0$, $f(tx, ty, tz) = t^n f(x, y, z)$. Show that $\mathbf{r} \cdot \nabla f(x, y, z) = n\, f(x, y, z)$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Deduce that

$$\iiint_B \nabla^2 f \, dx\,dy\,dz = n \iint_S f \, d\sigma$$

where B denotes the unit ball $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ and S denotes its boundary.

(5) Continuing with the set-up of question 4, show that $\nabla \cdot (f\mathbf{r}) = (n + 3)f$. Deduce that

$$\iiint_B f \, dx\,dy\,dz = \frac{1}{n+3} \iint_S f \, d\sigma = \frac{1}{n(n+3)} \iiint_B \nabla^2 f \, dx\,dy\,dz$$

and hence show that

$$\iiint_B (x - y + z)^4 \, dx\,dy\,dz = 36\pi/35.$$