AN APPROACH TO NONLOCAL, NONLINEAR ADVECTION

QIANG DU*, JAMES R. KAMM†, R.B. LEHOUCQ†, AND MICHAEL L. PARKS†

Abstract. We describe an approach to nonlocal, nonlinear advection in one dimension that extends the usual pointwise concepts to account for nonlocal contributions to the flux. The spatially nonlocal operators we consider do not involve derivatives. Instead, the spatial operator involves an integral that, in a distributional sense, reduces to a conventional nonlinear advective operator. In particular, we examine a nonlocal inviscid Burgers equation, which gives a basic form with which to characterize properties associated with well-posedness, and to examine numerical results for specific cases. We describe the connection to a nonlocal viscous regularization, which mimics the viscous Burgers equation in an appropriate limit. We present numerical results that compare the behavior of the nonlocal Burgers formulation to the standard local case. The developments presented in this paper form the preliminary building blocks upon which to build a theory of nonlocal advection phenomena consistent within the peridynamic theory of continuum mechanics.

Key words. conservation laws, advection, nonlocal operator, integral operator, Burgers equation, peridynamics

AMS subject classifications. 35L65,45A05,45G05,45J05,65M06,65M08,65M12

1. Introduction.

This paper describes an approach to nonlocal, nonlinear advection in one dimension. This development extends the usual pointwise concepts to account for nonlocal contributions to the flux. The spatially nonlocal operator we consider does not involve a spatial derivative (weak or otherwise). Instead, the nonlocal operator is an integral operator that, in a distributional sense, reduces to the familiar nonlinear advection operator.

Our nonlocal approach to advection is motivated by the nonlocality of force interaction postulated in peridynamics [34]. Peridynamics is a nonlocal continuum theory that has been developed and successfully applied to elastic material response, including critical phenomena such as material failure; see [36] for a recent review. As an illustration of the relevancy of our nonlocal approach, a linear peridynamic model for an infinite one-dimensional bar [37] can be written as

$$u_{tt}(x,t) = \int_{\mathbb{R}} \big(u(y,t) - u(x,t)\big)\phi_\varepsilon(y-x)\,dy + b(x,t)\,, \tag{1.1}$$

where $u = u(x,t)$ denotes the displacement and $\phi_\varepsilon = \phi_\varepsilon(z)$ is an even kernel function reflecting the material properties. In peridynamics, ϕ_ε is taken to be a function supported in a spherical neighborhood with a radius ε (the peridynamic horizon), to account for the nonlocal interactions. We will demonstrate in §4.5 that (1.1) can be re-written as a system of two first-order in time nonlocal equations—in direct analogy to the system of first order advection equations associated with the second order linear wave equation.

*Department of Mathematics, Pennsylvania State University,University Park, PA 16802, qdu@math.psu.edu. This author is supported in part by the U.S. Department of Energy grant DE-SC0005346 and the U.S. NSF grant DMS-1016073

†Sandia National Laboratories, P.O. Box 5800, MS 1320, Albuquerque NM 87185-1320, {jrkamm,rblehou,mlparks}@sandia.gov. This work was supported by the Laboratory Directed Research and Development program at Sandia National Laboratories. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94-AL85000.

A similar transformation for the nonlinear version of (1.1) is significantly more involved. We postulate a scalar nonlocal nonlinear advection equation with conventional nonlinear advection replaced by

$$\int_{\mathbb{R}} \psi\left(\frac{u(y,t)+u(x,t)}{2}\right)\phi_a(y,x)\,dy\,, \tag{1.2}$$

where the kernel[1] ϕ_a is antisymmetric in its arguments: $\phi_a(y,x) = -\phi_a(x,y)$. A lemma is presented in §4 explains that the above nonlocal nonlinear advection is equivalent to conventional nonlinear advection in a distributional sense. Our model serves as a fundamental building block for the study of more general second-order nonlocal nonlinear systems. Thus, instead of the usual second-order in time peridynamic equation corresponding to, say, elastic waves, we consider a first order in time, nonlocal, nonlinear advection equation in this paper. Our ultimate goal is to develop a consistent approach to nonlocal shock physics that is congruent with the existing peridynamics framework. As a first step, $\psi(u) \equiv u^2/2$ leads to a nonlocal, inviscid Burgers equation as a nonlinear example of our ideas. However, we show in §6 that the choice of kernel function ϕ_a can regularize the solution of the nonlocal inviscid Burgers' equation such that it will not develop a shock.

This paper is structured as follows. We review the local 1D advection equation in §2. Section §3 contains a brief review of previous approaches to nonlocal advection, which are to be contrasted to the new approach presented in §4. This section also discusses the notion of a nonlocal flux, its relation with the conventional flux, regularization of the nonlocal, nonlinear conservation law (4.1), the special case of linear advection and a derivation that the linear one-dimensional peridynamic equation can be written as two linear advection equations, respectively, in §§4.1–4.5. A basic, conservative, numerical method (and corresponding linear stability analysis) for this equation is developed in §5. A discussion of a new nonlocal Burgers equation in given in §6, including results explaining when the nonlocal, nonlinear advection equation is well-posed, and numerical results for specific cases. We summarize our contribution in §7.

2. The local advection equation. The concept of advection and use of advection equations is pervasive in applied mathematics and computational physics. The fundamental representation of this concept is contained in a balance equation between the time rate-of-change of some quantity u and the corresponding spatial divergence of some function f—the flux function—of that same quantity:

$$u_t + \operatorname{div} f(u) = 0\,. \tag{2.1}$$

This equation follows from standard balance arguments over a fixed domain for differentiable fields. We assume that there is no external source or sink, so that the right-hand side of (2.1) is identically zero. While this concept extends to higher dimensions, we consider exclusively the one-dimensional case, for which this equation reduces to

$$u_t + f_x(u) = 0 \quad \text{or} \quad u_t + f'(u)\,u_x = 0\,, \tag{2.2}$$

where the second equality holds for f differentiable in u and u differentiable in x. We restrict our attention to convex flux functions, so that $f'' \geq 0$.

[1]Referred to as the *micromodulus* in the peridynamics literature.

In the simplest case, the flux function is proportional to the field itself, $f(u) = c\,u$, which leads to the linear one-way wave equation,

$$u_t + c\,u_x = 0\,, \quad c \in \mathbb{R}\,. \tag{2.3}$$

This equation possesses the traveling wave solution, $u(x,t) = g(x-ct)$, where $g(x) = u(x,t=0)$ is the initial condition; for $c > 0$, this represents uniform translation of the initial wave profile to the right at constant speed c. The simplest nonlinear case is given by a flux function quadratic in u, $f(u) = u^2/2$, yielding the well-known inviscid Burgers equation,

$$u_t + \left(u^2/2\right)_x = 0 \quad \text{or} \quad u_t + u\,u_x = 0\,, \tag{2.4}$$

where, again, the second form holds for u differentiable in x. This equation is an elementary yet powerful model for many shock phenomena, as it has a convex flux function, leads to the development of shocks in finite time for smooth-but-nontrivial initial conditions, and forms a basis for exploring fundamental shock wave concepts, such as the entropy. Moreover, both the linear advection equation and Burgers equation are valuable test cases with which to evaluate numerical schemes for hyperbolic conservation laws.

3. Previous approaches to nonlocal advection. Implicit in the discussion so far is the notion of locality. That is, all expressions depend exclusively on the point under evaluation. Specifically, these equations describe behavior that is governed by point-values of the state u, its derivatives u_t and u_x, and the flux function f.

In contrast, there exist physical theories in which values of some quantity at a point are strongly influenced by values of the field in a neighborhood of that point. Such theories are generically referred to as "nonlocal." A noteworthy example is the field of nonlocal continuum mechanics, which has a significant history; see, e.g., the work of Eringen [15] and references therein. Intrinsic to these nonlocal theories is the concept that the state at a particular point is related to values in some set of points whose extent is bounded away from zero. This influence can be represented as an integral over the appropriate domain of some function of state values, modulated by a kernel function. One instance of such a nonlocal continuum theory is the peridynamic theory introduced by Silling [34]. Peridynamic theory is based on a uniform description encompassing both smooth and discontinuous (e.g., fracture) behavior.

Examination of nonlocal continuum theories leads one to consider the implications of such nonlocality for advection. The literature contains several instances of what can be broadly termed "nonlocal advection" (also referred to as "nonlocal convection"). Below is a brief discussion of some of the published models related to nonlocal advection, many of which examine nonlocal forms of Burgers equation. For consistency, we denote the independent variable as $u = u(x,t)$, with time t and lone spatial variable x.

Nonlocal advection through nonlocal wave speed. Logan [28] considers an straightforward extension of (2.3) on a finite domain in which the local wavespeed c is generalized to a nonlocal wavespeed, related to the value of u over a fixed domain Ω, so that the governing equation becomes $u_t + \left(\int_\Omega G(u)\,dy\right)u_x = 0$ where G is a known (specified) function of u. Logan analyzes this initial-boundary-value problem by the method of characteristics for various functions G.

Nonlocalization through integral operators. This corresponds to a nonlocality introduced for the flux function in the conventional advection equation (2.1). For instance, in a recent paper, Bentancourt et al. [7] postulate a nonlinear nonlocal flux function for the purpose of modeling sedimentation. The nonlocality in the flux function is given by a integral operator with a radial, nonnegative kernel of compact support.

Several papers consider the Hilbert transform for the flux function, defined as $\mathbb{H}[u] := (1/\pi)\,⨍_{-\infty}^{\infty} dy\, u(y)/(x-y)$, where the integral is interpreted in the principal value sense. Baker et al. [5] consider the following locally regularized forms for nonlocal advection, motivated as models for vortex sheet dynamics:

$$u_t + \big(\,\mathbb{H}\,(u)\big)_x = \epsilon\, u_{xx}\,, \tag{3.1a}$$

$$u_t - \mathbb{H}(u)\, u_x = \epsilon\, u_{xx}\,. \tag{3.1b}$$

The expression (3.1a) uses the (local) divergence operator on the (nonlocal) flux function $\mathbb{H}[u]$; alternatively, (3.1b) is closer to Logan's approach, with a nonlocal effective wavespeed given by $\mathbb{H}[u]$, but with nonzero RHS. An equation similar to the inviscid form of these equations was considered by Castro and Córdoba [10], who examine existence, finite time blow-up, and ill-posedness issues. Parker [32] considers similar equations and provides a derivation that builds upon Hunters's equation $u_X + Q_\theta = 0$, where Q is a nonlocal, quadratically nonlinear functional of u, to obtain a specific evolution equation of the form $iJ\,C_X + \int_{\mathbb{R}} \Gamma(k-\ell,\ell)(k-\ell)\ell C(k-\ell,X)C(\ell,X)\,d\ell$, leading to a discussion of the local vs. a nonlocal form of Burgers equation. Alternatively, Biello and Hunter [8] focus on an inviscid Burgers-Hilbert equation, $u_t + (u^2/2)_x = \mathbb{H}[u]$.

Nonlocalization through discrete operators. Veksler and Zarmi [39, 40] consider a nonlocal form of the Burgers equation, i.e., related to (2.4), together with asymptotics of its solutions. They also discuss the Burgers hierarchy, together with its relation to the generalized Lax pair approach (see, e.g., the notes of Miller [29]), extended to a form related to Burgers equation that is "discretely nonlocal" in that it involves function values at discrete points.

Nonlocalization through fractional differential operators. In some studies, the local advection operator is maintained, and the fractional operator modifies the diffusive term in the viscous Burgers equation. For example, Droniou [12] examines an equation with local advection and fractional derivative regularization and discusses the notion of an entropy solution for convex flux functions. Alibaud and coworkers [2, 3] (and references therein) also consider a similar 1D form with the usual 1D Burgers flux and a fractional derivative regularization. Ervin, Heuer and Roop [16] replace the regularization in (3.1a) with a weighted, directional fractional diffusion operator and consider a flux function given by an integral operator. This work is based, in part, upon earlier work by Biler and Woyczyński [9] where the weighted, directional fractional diffusion operator is given by the fractional Laplacian. Woyczyński [41] provides detailed investigations into a generalized Burgers equation of the form $u_t + \nu\,(-\Delta)^{\alpha/2}\,u + a\cdot\nabla\,(u^r) = 0$, for $0 < \alpha \ll 2$, $r \geq 1$, and constant a; this monograph also contains an extensive discussion of the local Burgers equation. Alternatively, Miškinis [38] considers a fractional derivative (nonlocal) generalization of Burgers equation given by $u_t + (1/2)D_x^p(D_x^{1-p}u)^2 = \alpha\, u_{xx}$, where D_x^p is the fractional derivative of order p, so that here the advective term is nonlocal and the diffusive term is local.

Nonlocalization through generalized flux, no regularization. Benzoni-Gavage [6] considers questions of existence and stability for the generalized Burgers equation, $u_t + \mathcal{F}_x[u] = 0$, where the Fourier transform of the operator $\mathcal{F}$ is given by $\hat{\mathcal{F}}[u](k) = \int_{-\infty}^{\infty} \Lambda(k-l)\hat{u}(k-l)\hat{u}(l)\,dl$. Alì et al. [1] provide the motivation for and describes properties of the kernel $\Lambda(k,l)$ for a generalized Burgers equation of this form.

Standard, local flux, nonlocalized regularization. The system of equations $u_t + u\,u_x = \phi_x$, $\phi_{xx} - \phi = u$, are recast by Fellner and Schmeiser [17] as the single equation $u_t + u\,u_x = \phi_x[u]$, where $\phi[u]$ is expressed as the nonlocal operator $\phi[u] = \int_{\mathbb{R}} G(x-y)\,u(y)\,dy$. Liu [26] considers nonlocal Burgers equations of the form $u_t + u\,u_x + (G * B[u,u_x])_x = 0$, where G is the same kernel as in [17] (which is related to the Ostrovsky-Hunter equation). In [27], Liu et al. consider a version of the Ostrovsky-Hunter equation in the form $(u_t + u\,u_x)_x = \gamma\,u$, $\gamma > 0$, noting that this equation is integrable with the inverse scattering transform. Chmaj [11] considers traveling wave solutions to a generalized nonlocal Burgers equation of the form $u_t + (u^2/2)_x + u - K * u = 0$, where the convolution kernel K is symmetric in its argument. Duan et al. [14] develop energy methods to examine the existence and stability of solutions to the set of equations that are multi-dimensional generalizations of the equation studied by Chmaj [11]. Rohde [33] considers questions of existence and uniqueness of nonlocally regularized conservation laws of the form $u_t + \mathrm{div} f(u) = \mathcal{R}[\epsilon, u]$, where $\mathcal{R}$ is a nonlocal regularization operator. Kissling and Rohde [21] consider a generalization of this approach of the form $u_t^{\epsilon,\lambda} + f_x(u^{\epsilon,\lambda}) = \mathcal{R}^{\epsilon}[\lambda; u^{\epsilon,\lambda}]$, where ϵ is a scale parameter and λ is an auxiliary parameter; they consider the macroscale (shock) problem as the limit of microscale, regularized problems. Kissling et al. [20] provide a study that complements the work in [21], focusing on the multidimensional case for a particular form of nonlocal regularization.

Nonlocal convection-diffusion equation. Perhaps the approach most nearly aligned with the nonlocal advective approach introduced in the next section is that considered by Ignat and Rossi [19]. They analyze the nonlocal evolution equation

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}} \big(u(y,t) - u(x,t)\big) J(y-x)\,dy \\ &\quad + \int_{\mathbb{R}} \big(h\big(u\big)(y,t) - h\big(u\big)(x,t)\big) K(y-x)\,dy \quad (x,t) \in \mathbb{R}^d \times (0,\infty)\,, \\ u(x,0) &= g(x)\,, \qquad\qquad x \in \mathbb{R}^d\,, \end{aligned} \tag{3.2}$$

where $J(y-x) = J(x-y)$ and K are non-negative, normalized radial functions satisfying $\int_{\mathbb{R}^d} J = \int_{\mathbb{R}^d} K = 1$, and h is a non-decreasing, locally Lipschitz, continuous function such that $h(0) = 0$. In addition to proving existence, uniqueness, and continuous dependence with respect to the initial condition upon the solution, Ignat and Rossi demonstrate that, under an appropriate scaling, the solution of the re-scaled (3.2) converges to the solution of $v_t = \Delta v + b \cdot \nabla h\big(v\big)$, $v(x,0) = g(x)$.

4. Nonlocal advection: A new approach. The approaches of the previous section impose specific assumptions on the nonlocality, either in the advective operator or in the regularization term. Motivated by peridynamics, we consider a generalization of nonlinear advection given by (1.2) and posit the following integro-differential equation:

$$
\begin{aligned}
u_t(x,t) + \int_{\mathbb{R}} \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\phi_a(y,x)\,dy = 0\,, \qquad & (x,t)\in\mathbb{R}\times(0,\infty)\,,\\
u(x,0) = g(x)\,, \qquad & x\in\mathbb{R}\,,
\end{aligned}
\tag{4.1}
$$

where, again, the kernel ϕ_a is antisymmetric in its arguments: $\phi_a(y,x) = -\phi_a(x,y)$. We claim that (4.1) represents a nonlocal, nonlinear conservation law for advection. The following result demonstrates that for a particular kernel the nonlocal conservation law (4.1) reduces to its conventional counterpart (2.2). In particular, this lemma shows that the nonlocal and conventional conservation laws are equivalent in the sense of distributions.

LEMMA 4.1. *Assume that ψ, u, and u_x decay sufficiently fast to zero as their arguments approach infinity. If the kernel in* (4.1) *is the negative derivative of the Dirac delta distribution, i.e., $-\partial\delta(y-x)/\partial y$, then*

$$
\int_{\mathbb{R}} \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\big(-\frac{\partial}{\partial y}\delta(y-x)\big)\,dy = \psi_x\big(u(x,t)\big)\,, \quad (x,t)\in\mathbb{R}\times(0,\infty)\,.
$$

Proof. The conclusion follows by an integration by parts of (4.1), viz.,

$$
\begin{aligned}
\int_{\mathbb{R}} \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\big(-\frac{\partial}{\partial y}\delta(y-x)\big)\,dy &= -\Big[\psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\,\delta(y-x)\Big]\Big|_{y=-\infty}^{y=\infty}\\
&+ \int_{\mathbb{R}} \psi_y\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\,\delta(y-x)\,dy = \psi_x\big(u(x,t)\big)\,.
\end{aligned}
$$

□

The subsequent subsections discuss the notion of a nonlocal flux, its relation with the conventional flux, regularization of the nonlocal, nonlinear conservation law (4.1), the special case of linear advection and a derivation that the linear one-dimensional peridynamic equation can be written as two linear advection equations, respectively.

4.1. Nonlocal flux. We now show how (4.1) corresponds to a conservation law and identify the flux. The following result:

$$
\int_a^b\int_a^b \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\phi_a(y,x)\,dy\,dx = 0\,, \tag{4.2}
$$

is a consequence of the antisymmetry of the integrand. Using this result and integrating the first equation of (4.1) over the interval (a,b) yields

$$
\frac{d}{dt}\int_a^b u(x,t)\,dx + \int_a^b\int_{\mathbb{R}\setminus(a,b)} \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\phi_a(y,x)\,dy\,dx = 0\,. \tag{4.3}
$$

Extending the interval (a,b) to the entire line and using the asymmetry of this integrand with respect to x and y gives the result that

$$
\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = 0\,, \tag{4.4}
$$

demonstrating that $\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} g(x)\,dx$ is a conserved quantity for the nonlocal conservation law (4.1).

More generally, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be open intervals such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$. Define

$$\Psi\big(\mathcal{I}_1, \mathcal{I}_2, t\big) := \int_{\mathcal{I}_1} \int_{\mathcal{I}_2} \psi\bigg(\frac{u(y,t)+u(x,t)}{2}\bigg)\phi_a(y,x)\, dy\, dx \,. \tag{4.5}$$

With this notation, we rewrite (4.3) as

$$\frac{d}{dt}\int_a^b u(x,t)\, dx + \Psi\big((a,b), \mathbb{R}\setminus(a,b), t\big) = 0 \,. \tag{4.6}$$

The antisymmetry of the integrand leads to the following result.

Lemma 4.2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be open intervals such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$. Then*

$$\Psi\big(\mathcal{I}_1, \mathcal{I}_2, t\big) + \Psi\big(\mathcal{I}_2, \mathcal{I}_1, t\big) = 0 \,, \tag{4.7a}$$

$$\Psi\big(\mathcal{I}_1, \mathcal{I}_1, t\big) = 0 \,. \tag{4.7b}$$

Proof. The conclusion follows from the antisymmetry of the integrand of Ψ given in (4.5). See [13, Section 6] for the proof in a more general setting. □

We therefore identify $\Psi\big(\mathcal{I}_1, \mathcal{I}_2, t\big)$ with the flux of u from $\mathcal{I}_1$ into $\mathcal{I}_2$. Evidently (4.7) explains that the flux is equal and opposite among disjoint intervals and there is no flux from an interval into itself. This is in contrast to the conventional notion of the flux where a unit normal on an orientable surface separating the volumes $\mathcal{I}_1$ and $\mathcal{I}_2$ carries the direction for the flux. We conclude that (4.6) is an instance of an abstract balance law—the production of an extensive quantity inside of an interval is balanced by the flux of the same quantity out of the same interval. Both the production and flux are additive and biadditive, respectively, over disjoint intervals; e.g.,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{I}_1} u(x,t)\, dx + \frac{d}{dt}\int_{\mathcal{I}_2} u(x,t)\, dx &= \frac{d}{dt}\int_{\mathcal{I}_1\cup\mathcal{I}_2} u(x,t)\, dx \\
&= -\Psi\big(\mathcal{I}_1\cup\mathcal{I}_2, \mathbb{R}\setminus\big(\mathcal{I}_1\cup\mathcal{I}_2\big), t\big) \\
&= \Psi\big(\mathbb{R}\setminus\big(\mathcal{I}_1\cup\mathcal{I}_2\big), \mathcal{I}_1, t\big) + \Psi\big(\mathbb{R}\setminus\big(\mathcal{I}_1\cup\mathcal{I}_2\big), \mathcal{I}_2, t\big) \\
&= \Psi\big(\mathbb{R}\setminus\mathcal{I}_1, \mathcal{I}_1, t\big) + \Psi\big(\mathbb{R}\setminus\mathcal{I}_2, \mathcal{I}_2, t\big) \,,
\end{aligned}$$

where the last equality follows by adding 0 as given by (4.7a). These additive and biadditive properties for the production and flux of a quantity can be shown to be a necessary and sufficient condition for the antisymmetry of the integrand of Ψ given in (4.5); see [13, Section 6] for details.

4.2. Conventional flux. The following result demonstrates that the flux given by (4.5) may be expressed in a more conventional form as the flux out of an interval through its endpoints.

Lemma 4.3. *If the map*

$$(x,y) \to \psi\bigg(\frac{u(y,t)+u(x,t)}{2}\bigg)\phi_a(y,x)$$

is integrable, then

$$\Psi\big((a,b), \mathbb{R}\setminus(a,b), t\big) = \mathcal{F}_u^+\big(a,b,t\big) - \mathcal{F}_u^-\big(a,b,t\big) \,,$$

where

$$\mathcal{F}_u^-(a,b,t) = \int_0^{b-a}\int_0^{\infty} \psi\left(\frac{u(a+y,t)+u(a-z,t)}{2}\right)\phi_a(a+y,a-z)\,dz\,dy\,, \tag{4.8a}$$

$$\mathcal{F}_u^+(a,b,t) = \int_0^{b-a}\int_0^{\infty} \psi\left(\frac{u(b+y,t)+u(b-z,t)}{2}\right)\phi_a(b+y,b-z)\,dy\,dz\,, \tag{4.8b}$$

each denote the flux emanating from the interval (a,b) in the negative and positive directions, respectively.

Proof. Biadditivity of the flux Ψ implies that

$$\Psi\big((a,b),\mathbb{R}\setminus(a,b),t\big) = \Psi\big((a,b),(-\infty,a),t\big) + \Psi\big((a,b),(b,\infty),t\big).$$

The change of variables $s := a+y, r := s-z$ and $r := b+y, s := b-z$ on the fluxes on the righthand side of the equality, respectively, yields

$$\Psi\big((a,b),(-\infty,a),t\big) = \int_0^{b-a}\int_0^{\infty} \psi\left(\frac{u(a+y,t)+u(a-z,t)}{2}\right)\phi_a(a-z,a+y)\,dz\,dy,$$

$$\Psi\big((a,b),(b,\infty),t\big) = \int_0^{b-a}\int_0^{\infty} \psi\left(\frac{u(b+y,t)+u(b-z,t)}{2}\right)\phi_a(b+y,b-z)\,dy\,dz,$$

after changing back to the original integration variables x and y. The conclusion now follows by invoking the antisymmetry of ϕ_a in $\Psi\big((a,b),(-\infty,a),t\big)$ and the integrability of the map $(x,y) \to \psi(1/2(u(y,t)+u(x,t)))\phi_a(y,x)$; e.g., see [4, p.421]. ∎

More generally, the equation for the flux from an interval $(x-x_1,x)$ to an interval $(x,x+x_2)$ through the point x takes the form

$$\int_0^{x_1}\int_0^{x_2} \psi\left(\frac{u(x+y,t)+u(x-z,t)}{2}\right)\phi_a(x+y,x-z)\,dy\,dz\,. \tag{4.9}$$

This expression can be construed as a sum of all interactions carried by ψ between two points such that their interaction passes through x, and is determined collectively by the values of u at points to the left of x (identified by z) *and* the values of u at points to the right of x (identified by y).[2] This stands in contrast to the conventional setting, where the flux through x depends only upon the value of u and its derivative at x, as in (2.2).

Further insight into the relationship between (4.1) and (4.9) can be gained from an application of Noll's Lemma I [31, 24] to scalar functions in one dimension. Namely, given continuously differentiable u there exists a scalar function q

$$q(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(\frac{u(x+y,t)+u(x-z,t)}{2}\right)\phi_a(x+y,x-z)\,dy\,dz$$

such that

$$\frac{d}{dx}q(x) = \int_{\mathbb{R}} \psi\left(\frac{u(y,t)+u(x,t)}{2}\right)\phi_a(y,x)\,dy\,.$$

This demonstrates that (4.1) can be written in the form (2.2) using the flux q for a continuously differentiable u.

[2]A similar interpretation within the context of the balance of linear momentum is given in [23, Sec. 6].

4.3. Regularization of nonlocal advection. The regularization of inviscid advection equations (such as (2.2)) plays an important role in the associated physics and mathematics. By regularization, we refer to the modification of the otherwise-zero righthand side of the conservation law with terms (often of a distinctly dissipative or dispersive nature) that are small relative to the scale of the solution, except in regions of steep gradients. We propose a regularization of the (inviscid) nonlocal advection equation (4.1) that is a variation on the convection-diffusion equation introduced by Ignat and Rossi (see §3) of the form:

$$\begin{aligned} u_t(x,t) + \int_{\mathbb{R}} \psi\Big(\frac{u(y,t)+u(x,t)}{2}\Big)\phi_a(y,x)\,dy &= \epsilon\,\mathcal{L}u(x,t), \quad (x,t)\in\mathbb{R}\times(0,\infty)\,, \\ u(x,0) &= g(x)\,, \qquad\qquad\qquad x\in\mathbb{R}\,, \end{aligned} \tag{4.10}$$

where

$$\mathcal{L}u(x,t) := \int_{\mathbb{R}} \big(u(y,t)-u(x,t)\big)\,\phi_s(y,x)\,dy\,, \qquad (x,t)\in\mathbb{R}\times(0,\infty)\,, \tag{4.11}$$

with the kernel ϕ_s symmetric in its arguments: $\phi_s(y,x)=\phi_s(x,y)$. Using an argument similar to that used to establish Lemma 4.1, we have

LEMMA 4.4. *Assume that u_{xx} is continuous and that u and u_x decay sufficiently fast to zero as their arguments approach infinity. If the kernel in* (4.11) *is the second derivative of the Dirac delta distribution, i.e., $\partial^2\delta(y-x)/\partial y^2$, then*

$$\mathcal{L}u = u_{xx}.$$

In this case, the regularized nonlocal equation (4.10), for ϕ_a as in Lemma 4.1, reduces to the viscous conservation law

$$u_t + \psi_x(u) = \epsilon\, u_{xx}\,.$$

In the local case, shock waves are typically idealized as inviscid structures, but it is only through an appropriate regularization of the governing conservation equations, associated with vanishing viscosity, that shock formation and propagation can be properly understood. The form of the equation given in (4.10) comprises a nonlocal approach to a regularized (viscous) nonlocal conservation law consistent with the (inviscid) nonlocal advection equation given in (4.1). See [18] for a more involved discussion on the relationship between $\mathcal{L}$ and the Laplacian Δ.

Integrating the first equation of (4.10) over the interval $\mathcal{I}$ implies the balance law

$$\frac{d}{dt}\int_{\mathcal{I}} u(x,t)\,dx + \Phi\big(\mathcal{I},\mathbb{R}\setminus\mathcal{I},t\big) = 0\,, \tag{4.12}$$

where

$$\Phi\big(\mathcal{I},\mathbb{R}\setminus\mathcal{I},t\big) := \Psi\big(\mathcal{I},\mathbb{R}\setminus\mathcal{I},t\big) - \epsilon\int_{\mathcal{I}}\int_{\mathbb{R}\setminus\mathcal{I}} \big(u(y,t)-u(x,t)\big)\,\phi_s(y,x)\,dy\,dx\,,$$

and Ψ is the flux given by (4.5). The balance law (4.12) states the production of an extensive quantity inside the interval $\mathcal{I}$ is balanced by the flux Φ out of the same interval. Evidently the balance law (4.12) augments the advective law (4.6) with diffusive flux. It is in this sense that the viscous conservation law (4.10) is a variation on the convection-diffusion equation introduced by Ignat and Rossi (see §3) because their equation is in nonconservative form. Lemma 4.2 and the ensuing discussion explain that the antisymmetry of the integrand corresponding to the flux is crucial for a balance law.

4.4. Linear advection. Consider the important special case when $\psi(v) = v$. If we further assume that ϕ_a is translation invariant, i.e., depends only on the difference of its arguments, then ϕ_a is antisymmetric if and only if it is an odd function, which we denote $\varrho_o(y-x)$. The conservation law (4.1) then becomes

$$\begin{aligned} u_t(x,t) + \int_{\mathbb{R}} u(y,t)\varrho_o(y-x)\,dy = 0\,, \qquad & (x,t) \in \mathbb{R}\times(0,\infty)\,, \\ u(x,0) = g(x)\,, \qquad & x \in \mathbb{R}\,, \end{aligned} \tag{4.13}$$

where the integral involving $u(x)$ vanishes identically because ϱ_o is an odd function. Let

$$\widehat{u}(\xi,t) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x,t)\,e^{-i\xi x}\,dx, \qquad u(x,t) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u}(\xi,t)\,e^{i\xi x}\,d\xi \tag{4.14}$$

denote the Fourier and inverse Fourier transforms, respectively, of the function u. Then the Fourier transform of the nonlocal evolution equation (4.13) is

$$\begin{aligned} \widehat{u}_t(\xi,t) &= \sqrt{2\pi}\,\widehat{\varrho}_o(\xi)\,\widehat{u}(\xi,t)\,, \\ \widehat{u}(\xi,0) &= \widehat{g}(\xi)\,. \end{aligned} \tag{4.15}$$

Because ϱ_o is an odd function,

$$\widehat{\varrho}_o(\xi) = i\,\mathrm{Im}\big(\widehat{\varrho}(\xi)\big) = \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} \varrho_o(x)\,\sin(\xi x)\,dx := i\frac{1}{\sqrt{2\pi}}\beta(\xi)\,.$$

Therefore, the solution to (4.15) is given by

$$\widehat{u}(\xi,t) = e^{i\beta(\xi)t}\,\widehat{g}(\xi)\,. \tag{4.16}$$

The inverse Fourier transform of (4.16) implies that

$$u(x,t) = \int_{\mathbb{R}} G(x,y,t)\,g(y)\,dy\,, \tag{4.17a}$$

$$\text{where} \quad G(x,y,t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\beta(\xi)t}\,e^{i\xi(x-y)}\,d\xi\,. \tag{4.17b}$$

If we consider the choice $\varrho_o(y-x) = -\partial\delta(y-x)/\partial y$, then $\beta(\xi) = -\xi$, so that

$$G(x,y,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-y-t)}\,d\xi = \delta(x-y-t)\,, \tag{4.18}$$

and, therefore, $u(x,t) = g(x-t)$. Thus, the solution to the nonlocal linear conservation law (4.13) is the same as that for the conventional linear conservation law in a distributional sense.

For the nonlocal linear conservation law (4.13), there are traveling wave solutions of the form[3] $u(x,t) = e^{ik(x-c(k)t)}$ where the wave speed $c = c(k)$ is determined from (4.15) by

$$c(k)k = \beta(k) = \int_{\mathbb{R}} \varrho_o(x)\,\sin(kx)\,dx\,. \tag{4.19}$$

[3]This is equivalent to the usual plane wave form $u(x,t) = \exp(ikx - \omega(k)t)$, where the angular frequency is given by the wavespeed times the wave number, i.e., $\omega(k) = c(k)\,k$.

For the specific choice of $\varrho_o = \phi_a^P \chi_{(-\varepsilon,\varepsilon)}$ given by[4],

$$\phi_a^P(x,y) = \frac{1}{\varepsilon^2}\begin{cases} 1\,, & y > x\,,\\ 0\,, & y = x\,,\\ -1\,, & y < x\,,\end{cases} \tag{4.20}$$

where $\chi_{(-\varepsilon,\varepsilon)}$ is the characteristic function on $(-\varepsilon,\varepsilon)$, we obtain

$$c(k) = \frac{\sin^2(k\varepsilon/2)}{(k\varepsilon/2)^2},$$

a particular case of the dispersion relation. Clearly, we have that $c(k) \le 1$, which implies that the wave speed in this nonlocal case is always less or equal to that of the linear local advection equation. This is reflected in the numerical simulations, see Figures 6.2 and 6.5.

4.5. Linear peridynamic equation. We now demonstrate that the peridynamic model for an infinite one-dimensional bar [37] given by (1.1) can be re-written as a system of two first-order in time nonlocal equations—in direct analogy to the system of first order advection equations associated with the second order linear wave equation.

Let ϕ_ε denote an even function with zero mean such that ϕ_ε approaches the second derivative of the Dirac delta distribution as $\varepsilon \to 0$. Lemma 4.4 demonstrates that such a ϕ_0 implies that the peridynamic equation (1.1) is equivalent to the classical second order wave equation in a distributional sense. For simplicity, we set $b(x,t) = 0$. Now let

$$\phi_\varepsilon(x) := \int_{\mathbb{R}} \rho_\varepsilon(z)\rho_\varepsilon(x-z)\,dz$$

for an odd function ρ_ε, that is, ϕ_ε is the self convolution of ρ_ε. Note that $\int_{\mathbb{R}} \phi_\varepsilon\,dx = 0$ so that ϕ_ε denotes an even function with zero mean. Define the function

$$v_t(x,t) := \int_{\mathbb{R}} u(z,t)\rho_\varepsilon(x-z)\,dz = \int_{\mathbb{R}} \big(u(z,t) + u(x,t)\big)\rho_\varepsilon(x-z)\,dz\,, \tag{4.21a}$$

the nonlocal analogue of the conventional splitting $v_t = u_x$. The linear peridynamic equation (1.1) and the relation (4.21a) imply that

$$\begin{aligned}
\frac{\partial}{\partial t}u_t(x,t) = u_{tt}(x,t) &= \int_{\mathbb{R}} \big(u(y,t) - u(x,t)\big)\phi_\varepsilon(x-y)\,dy\\
&= \int_{\mathbb{R}} u(y,t)\bigg(\int_{\mathbb{R}} \rho_\varepsilon(z)\rho_\varepsilon(x-y-z)\,dz\bigg)\,dy\\
&= \int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} u(y,t)\rho_\varepsilon(x-z-y)\,dy\bigg)\rho_\varepsilon(z)\,dz\\
&= \int_{\mathbb{R}} v_t(x-z,t)\rho_\varepsilon(z)\,dz = \frac{\partial}{\partial t}\int_{\mathbb{R}} v(x-z,t)\rho_\varepsilon(z)\,dz\,,
\end{aligned}$$

[4]The prefactor $1/\varepsilon^2$ has been chosen such that $\phi_a^P(x,y)$ converges in distribution to the negative derivative of the Dirac delta measure as $\varepsilon \to 0$.

where the second equality follows from ϕ_ε having zero mean. A compatible set of initial conditions implies

$$u_t(x,t) = \int_{\mathbb{R}} v(z,t)\rho_\varepsilon(x-z)\,dz = \int_{\mathbb{R}} \big(v(z,t)+v(x,t)\big)\rho_\varepsilon(x-z)\,dz\;, \tag{4.21b}$$

which is the nonlocal analogue of the conventional splitting $u_t = v_x$.

A change of variable with $w^\pm = u \pm v$ leads to a nonlocal linear advection equation for w of the form

$$w_t^\pm(x,t) = \pm\int_{\mathbb{R}} w^\pm(y,t)\rho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}} \big(w^\pm(x,t)+w^\pm(y,t)\big)\rho_\varepsilon(x-y)\,dy\;. \tag{4.22}$$

The results of §4.5 explain that

$$w^\pm(x,t) = e^{i(x \pm c(\xi)t)\xi}$$

are waves traveling to the right and left, respectively, and are the nonlocal analogue of waves $e^{i\xi(x\pm t)}$ for the conventional wave equation $w_{tt} = w_{xx}$.

5. A numerical scheme for the nonlocal advection equation. In this section we discuss a conservative numerical scheme and the associated numerical fluxes for the nonlocal equations we consider as analogues of classical conservative numerical schemes [22, 25].

The preponderance of peridynamic constitutive models utilize a ϕ_a that is compactly supported on intervals of length 2ε, so that the flux Ψ takes the form

$$\Psi\big((a,b), \mathbb{R}\setminus(a,b), t\big) = \int_a^b \int_{x-\varepsilon}^{x+\varepsilon} \psi\left(\frac{u(y,t)+u(x,t)}{2}\right)\phi_a(y,x)\,dy\,dx\,. \tag{5.1}$$

In the following we overload our notation and write

$$\Psi\big(a,b,t\big) = \Psi\big((a,b), \mathbb{R}\setminus(a,b), t\big)\,.$$

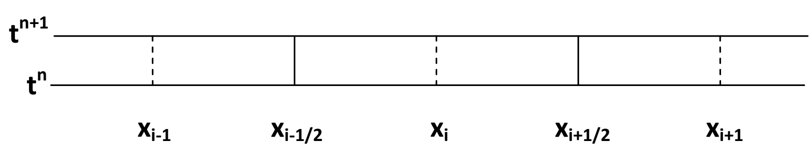


FIG. 5.1. *Finite volume discretization of a one-dimensional domain, showing cell* $[x_{i-1/2}, x_{i+1/2}]$.

5.1. A nonlocal Lax-Friedrichs method. As shown in Figure 5.1, we divide a one-dimensional domain into cells $(x_{i-1/2}, x_{i+1/2})$, each of width Δx. Suppose also that time is divided into discrete intervals (t^n, t^{n+1}). A numerical scheme is conservative if the change in the total conserved quantity in cell $(x_{i-1/2}, x_{i+1/2})$ in time interval (t^n, t^{n+1}) is equal to the net flux through the boundaries of $(x_{i-1/2}, x_{i+1/2})$ in the time interval (t^n, t^{n+1}). Utilizing (5.1), we express this relation mathematically as

$$\int_{x-1/2}^{x+1/2} \big(u(x,t^{n+1}) - u(x,t^n)\big)\,dx = -\int_{t^n}^{t^{n+1}} \Psi(x_{i-1/2}, x_{i+1/2}, t)\,dt\,,$$

which we rewrite as

$$\overline{u}_i^{n+1} = \overline{u}_i^n - \frac{\Delta t}{\Delta x}\overline{\Psi}^n(x_{i-1/2}, x_{i+1/2}, t)\,, \tag{5.2}$$

where

$$\overline{u}_i^n \approx \frac{1}{\Delta x}\int_{x-1/2}^{x+1/2} u(x,t^n)\,dx\,, \quad \text{and}$$
$$\overline{\Psi}^n(x_{i-1/2}, x_{i+1/2}, t) \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \Psi(x_{i-1/2}, x_{i+1/2}, t)\,dt\,.$$

In analogy with the classical Lax-Friedrichs method [22], we propose a nonlocal Lax-Friedrichs method:

$$\overline{u}_i^{n+1} = \frac{\overline{u}_{i-1}^n + \overline{u}_{i+1}^n}{2} - \frac{\Delta t}{\Delta x}\Psi(x_{i-1/2}, x_{i+1/2}, t^n). \tag{5.3}$$

This method is conservative, as it can be cast in the form of (5.2) by identifying

$$\overline{\Psi}(x_{i-1/2}, x_{i+1/2}, t^n) := \Psi(x_{i-1/2}, x_{i+1/2}, t^n) - \frac{1}{2}\frac{\Delta x}{\Delta t}\left(\overline{u}_{i+1}^n - 2\overline{u}_i^n + \overline{u}_{i-1}^n\right).$$

The rightmost term is immediately identified as introducing an an artificial viscosity. See [22, §14.2] for more details. For the remainder of the paper, we follow standard practice and drop the bar notation.

5.2. A discrete flux. We develop a quadrature for the nonlocal flux under the assumption that the numerical solution u_h approximating u is a piecewise constant function within each cell and there is no continuity of u_h across adjoining cells. For ease of exposition, we assume that $\Delta x < \varepsilon$. Of course, higher-order approximations to the function within each cell could be used, but we consider the simplest case in the illustrative examples that follow. Distinct from the classical case, all the fluxes we will consider are nonlocal.

To describe a quadrature for (5.1) we make the simplifying assumption that $\varepsilon/\Delta x \equiv r$ is a positive integer. That is, we assume that there are $r \in \mathbb{Z}^+$ computational cells in the interaction region of extent ε. This is not a limiting constraint, but simplifies the presentation. Since we assume the discretized solution to be piecewise constant over the cells, we can write an exact quadrature for (5.1) as

$$\Psi\left(x_{i-1/2}, x_{i+1/2}, t\right) = \sum_{j=-r}^{r} \omega_j \psi\left(\frac{u_h(x_{i+j},t) + u_h(x_i,t)}{2}\right)\phi_a(x_{i+j}, x_i)(\Delta x)^2\,, \tag{5.4}$$

where

$$\omega_j = \begin{cases} 0\,, & j = 0\,, \\ 1\,, & j = \pm 1, \ldots, \pm(r-1)\,, \\ 1/2\,, & j = -r, r\,. \end{cases} \tag{5.5}$$

Equation (4.2) explains why $\omega_0 = 0$. As for $\omega_{\pm j} = 1/2$, consider the limits of integration of the flux (5.1) through the points $x_{i+1/2}$ and $x_{i-1/2}$. The domain of integration shown in Figure 5.2 illustrates why $\omega_{\pm j} = 1/2$.

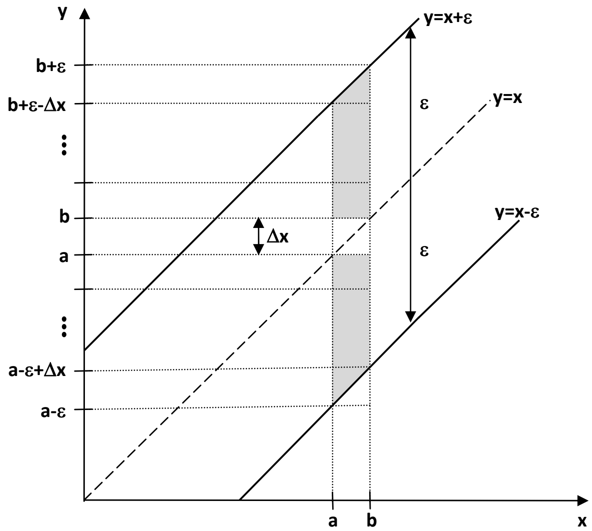


FIG. 5.2. *A graphical depiction of* (5.4)*, a discretization of* (5.1) *using a piecewise constant solution over each cell. The domain of integration is shown in grey. Cell* $(a, b) \times (a, b)$ *is zero by* (4.2)*. Only half of the cells farthest from* (a, b) *in* y *are integrated, as expressed in* (5.5)*.*

5.3. Linear stability analysis. We consider classic von Neumann stability analysis for the nonlocal linear advection equation, where $\psi(u) = u$ in (5.2). For simplicity, we assume the antisymmetric function ϕ_a^P given in (4.20). The numerical method of (5.3) may then written as

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{1}{\varepsilon^2}\frac{\Delta t}{\Delta x}\left(\sum_{j=1}^{r}\left(\frac{u_{i+j}^n + u_i^n}{2}\right)(\Delta x)^2 - \sum_{j=1}^{r}\left(\frac{u_{i-j}^n + u_i^n}{2}\right)(\Delta x)^2\right) .$$

We substitute the ansatz $u_i^n = \lambda^n e^{ikj\Delta x}$, where $x_j = j\Delta x$, into this expression. This gives, after considerable simplification, that

$$|\lambda|^2 = \cos^2(k\Delta x) + \left(\frac{\Delta t \Delta x}{\varepsilon^2}\right)^2 \left(\sum_{j=1}^{r} \sin(jk\Delta x)\right)^2 . \tag{5.6}$$

We now derive sufficiency relations for a CFL condition, that is conditions assuring $|\lambda| \leq 1$. First, we notice that for any $j \geq 1$, we have the following simple elementary inequality $|\sin(jk\Delta x)| \leq j|\sin(k\Delta x)|$ and so

$$\big(\sum_{j=1}^{r} \sin(jk\Delta x)\big)^2 \leq |\sin(k\Delta x)|^2 \big(\sum_{j=1}^{r} j\big)^2 = \big(\frac{r(r+1)}{2}\big)^2 |\sin(k\Delta x)|^2 .$$

With our assumption that $\varepsilon = r\,\Delta x$ for a positive integer r, we obtain

$$\Delta t \leq \frac{r}{(r+1)} 2\Delta x, \tag{5.7}$$

which, substituted into (5.6), gives $|\lambda| \leq 1$. The choice of $r = 1$ corresponds to the Lax-Friedrich scheme for conventional linear advection (with unit wave speed) and then the above restriction on the time-step and the conventional restriction are the same.

For the nonlinear nonlocal problem, by applying a linear stability analysis based on the frozen coefficient technique, the time-step restriction (5.7) is modified as

$$\Delta t \leq \frac{r}{c(r+1)} 2\Delta x, \tag{5.8}$$

where c is a suitable upper bound of the local wave speed.

Consequently, for given values of ε and Δx, one can always choose a timestep so that the scheme in (5.3) is linearly stable. Note that a direct implementation of (5.2) with $\overline{\Psi}$ replaced by Ψ is just a nonlocal analogue of the classical forward-time center-space (FTCS) discretization [22], which is well known to be unconditionally unstable.

6. Nonlocal Burgers Equation. This section presents analytical and numerical results for a special case of the nonlocal conservation law (4.1) where $\psi(u) = u^2/2$. This leads to the nonlocal Burgers equation:

$$\begin{aligned} u_t(x,t) + \mathcal{B}u(x,t) &= 0\,, \qquad (x,t) \in \mathbb{R} \times (0,\infty)\,,\\ u(x,0) &= g(x)\,, \qquad\qquad\qquad x \in \mathbb{R}\,, \end{aligned} \tag{6.1}$$

where the kernel is compactly supported and

$$\mathcal{B}u(x,t) := \frac{1}{2} \int_{\mathbb{R}} \left(\frac{u(y,t) + u(x,t)}{2} \right)^2 \phi(x-y)\, dy. \tag{6.2}$$

Lemma 4.1 demonstrates that for the special choice of the negative derivative of the Dirac delta distribution for the kernel, we have

$$\frac{1}{2} \int_{x-\varepsilon}^{x+\varepsilon} \left(\frac{u(y,t) + u(x,t)}{2} \right)^2 \big(-\frac{\partial}{\partial y} \delta(y-x) \big)\, dy = \frac{1}{2} \frac{\partial}{\partial x} \big(u(x,t) \big)^2.$$

That is, the nonlocal Burgers equation (6.1) and the conventional inviscid Burgers equations, respectively, are equivalent in the sense of distributions for such a singular kernel. We therefore anticipate shock formation when the kernel is sufficiently singular.

We will now show, however, that for integrable kernels, i.e., kernels of the form $\phi_a(y,x) = \phi(y-x)$ with ϕ an odd function in L^1, the solutions of the nonlocal Burgers equation do not develop a shock, i.e., they do not develop a discontinuity in finite time for a finite valued u. Before presenting the main result, Theorem 6.3, we present two needed technical results. The first result characterizes the regularity of the nonlocal quadratic spatial operator while the second result provides an energy estimate.

LEMMA 6.1. *For $\phi \in L^1(\mathbb{R})$ an odd function, $\mathcal{B}$ is a bounded operator from $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$ and satisfies for any $u \in H^1(\mathbb{R})$,*

$$\|\mathcal{B}u\|_{L^2} \leq \frac{1}{2} \|\phi\|_{L^1} \|u\|_{L^4}^2 \leq \frac{1}{2} \|\phi\|_{L^1} \|u\|_{L^\infty} \|u\|_{L^2}\ , \tag{6.3a}$$

$$\|\frac{d}{dx} \mathcal{B}u\|_{L^2} \leq \|\phi\|_{L^1} \|u\|_{L^\infty} \|\frac{d}{dx} u\|_{L^2}\ . \tag{6.3b}$$

$\mathcal{B}$ is also locally Lipshitz in $H^1(\mathbb{R})$, or more precisely, there exists a constant $C > 0$ such that, for any $u, v \in H^1(\mathbb{R})$,

$$\|\mathcal{B}u - \mathcal{B}v\|_{H_1} \leq C \big(\|u\|_{H^1} + \|v\|_{H^1} \big) \|\phi\|_{L^1} \|u - v\|_{H_1}\ . \tag{6.3c}$$

Proof. Given u and v in $H^1(\mathbb{R})$, by the Sobolev embedding theorem, we have $\|u\|_{L^\infty} \leq C\|u\|_{H^1}$ and $\|v\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{H^1(\mathbb{R})}$ for some constant $C > 0$. Thus, by Young's and the Cauchy-Schwartz inequalities, we arrive at (6.3a). Similarly, it is straightforward to see that

$$\frac{d}{dx}\mathcal{B}u(x) = \frac{1}{4}\int_{\mathbb{R}} \big(u(y)+u(x)\big)\big(u'(y)+u'(x)\big)\phi(x-y)\,dy\,. \tag{6.4}$$

We may again use Young's and the Cauchy-Schwartz inequalities to obtain (6.3b).

Next, let $M = \max\big(\|u\|_{L^\infty}, \|v\|_{L^\infty}\big)$, which gives

$$|\mathcal{B}u(x) - \mathcal{B}v(x)| \leq \frac{M}{2}\int_{\mathbb{R}} \big(|u(y)-v(y,t)| + |u(x)-v(x)|\big)|\phi(x-y)|\,dy\,.$$

By Young's and Hölder's inequalities, we get

$$|\mathcal{B}u(x) - \mathcal{B}v(x)|_{L^2} \leq M\|u-v\|_{L^2}\|\phi\|_{L^1}\,,$$

and moreover,

$$\begin{aligned}\frac{d}{dx}\mathcal{B}u(x) - \frac{d}{dx}\mathcal{B}v(x) =& \frac{1}{4}\int_{\mathbb{R}} \big[\big(u(y)+u(x)\big)\big(u'(x)+u'(y)\big)\\ &\qquad -\big(v(y)+v(x)\big)\big(v'(x)+v'(y)\big)\big]\,\phi(x-y)\,dy\,.\end{aligned}$$

After comparing, term by term, the difference in the integrand on the right hand side and using the Sobolev and Young's inequalities, we obtain

$$\begin{aligned}|\frac{d}{dx}\mathcal{B}u(x) - \frac{d}{dx}\mathcal{B}v(x)| \leq& \frac{1}{2}\int_{\mathbb{R}} [C\,M|u'(y)-v'(y)| + \|u-v\|_{L^\infty}|v'(y)|]\,|\phi(x-y)|dy\\ &+\frac{1}{4}\,[C\,M|u'(x)-v'(x)| + |u'(x)|\|u-v\|_{L^\infty}]\,\|\phi\|_{L^1}\,,\end{aligned}$$

which implies

$$\|\frac{d}{dx}\mathcal{B}u(x) - \frac{d}{dx}\mathcal{B}v(x)\|_{L^2} \leq (3C\,M + C\,\|u\|_{H^1} + C\,\|v\|_{H^1})\|\phi\|_{L^1}\|u-v\|_{H_1}\,.$$

Since $M \leq C\max(\|u\|_{H^1}, \|v\|_{H^1})$, the inequality (6.3c) is established, so leading to the local Lipshitz property of $\mathcal{B}$. ◻

The next results explains that while $\int_R u(x,t)dx$ is a conserved quantity, the spatial L^2 norm of the solution in time for the nonlocal Burgers equation (e.g., the energy of the system) is not conserved, in contrast to the energy associated with the conventional Burgers equation.

Lemma 6.2. *Given a time interval $(0,T)$ and $\phi \in L^1(\mathbb{R})$ an odd function, let $u = u(x,t)$ be a solution of the nonlocal Burgers equation (6.1). Then*

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t)dx = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} u^2(y,t)u(x,t)\phi(x-y)\,dy\,dx\,. \tag{6.5a}$$

Thus, if $K = \|u\|_{L^\infty(\mathbb{R}\times(0,T))}$ is finite, then for any $t \in (0,T)$,

$$\|u(\cdot,t)\|_{L^2} \leq \|g\|_{L^2}e^{\|\phi\|_{L^1}Kt/4}\,, \tag{6.5b}$$

$$\|\frac{\partial}{\partial x}u(\cdot,t)\|_{L^2} \leq \|\frac{dg}{dx}\|_{L^2}\; e^{\|\phi\|_{L^1}Kt}. \tag{6.5c}$$

Proof. By multiplying (6.1) by u, the equation (6.5a) follows from a change of variable in the integration (or rather, from the nonlocal Green's identity [13]).

Young's inequality implies

$$\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} u^2(y,t)u(x,t)\phi(x-y)\,dy \le \frac{1}{4}\|u(\cdot,t)\|_{L^2}^2\|u(\cdot,t)\|_{L^\infty}\|\phi\|_{L^1}\ ,$$

establishing (6.5b) by integrating in time.

Similarly, by differentiating (6.1) in space and multiplying with $\frac{\partial}{\partial x}u(x,t)$, we get

$$\frac{d}{dt}\int_{\mathbb{R}}\big(\frac{\partial u}{\partial x}(x,t)\big)^2 dx = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} u_x(x,t)(u(y,t)+u(x,t))(u_x(y,t)+u_x(x,t))\phi(x-y)\,dy\ .$$

We may thus combine Young's inequality with Cauchy-Schwartz and Hölder's inequalities to get (6.5c). ∎

Lemmas 6.1–6.2 lead to our main result, the well-posedness the nonlocal Burgers equation.

THEOREM 6.3. *Assume $\phi \in L^1(-\varepsilon,\varepsilon)$, $g \in H^1(\mathbb{R})$, then there exists a time interval $(0,T)$ such that the nonlocal Burgers equation (6.1) has a unique solution in $C(0,T;H^1(\mathbb{R}))\cap H^1(\mathbb{R}\times(0,T))$. Moreover, let $(0,T)$ be the maximum time interval on which such a solution exists, then $\limsup_{t\to T}\|u(\cdot,t)\|_{L^\infty}=\infty$.*

Proof. The proof follows standard steps as for the case of existence of solutions for abstract ODEs in Banach space and we only outline the key ingredients. First, by integrating (6.1) in time, we may define a Picard iteration that, for a sufficiently small time interval, (depending on the spatial H^1 norm of the initial data), gives a contraction in $L^2(0,T;H^1(\mathbb{R}))$ due to the local boundedness and the local Lipshitz continuity of the operator $\mathcal{B}$. This leads to the local existence and uniqueness of solution; one may obtain further regularity from the nonlocal equation In the second step, based on the a priori estimates given in Lemma 6.2, we see that for a time interval $(0,T)$, as long as $\limsup_{t\to T}\|u(\cdot,t)\|_{L^\infty}$ remains finite, we have the uniform boundedness of the solution in $L^\infty(0,T;H^1(\mathbb{R}))$, thus the solution can be further extended to a larger interval. Combining these steps together, we get the global well-posedness result and the possible blow-up criterion as stated in the theorem. ∎

The above theorem implies in particular that if we start with smooth data, while the solutions remains pointwise bounded in space and time, the solution maintains H^1 regularity in space (further regularity may also be derived) so that there is no shock formation for the nonlocal Burgers equation (6.1) with a L^1 kernel ϕ. Moreover, if H^1 regularity of the solution is lost, then this can only occur because of finite-time blowup.

Theorem 6.3 leads to a striking conclusion that for an L^1 kernel ϕ_a, the solution of the nonlocal Burgers equation (6.1) maintains H^1 regularity without the addition of any viscous regularization. The regularization introduced in §4.3 is unnecessary for L^1 kernel ϕ_a. For instance, selecting $\phi_a \equiv \phi_a^P\chi_{(-\varepsilon,\varepsilon)}$ where ϕ_a^P is given by (4.20) implies that the solution of the nonlocal Burgers equation possesses spatial H^1 regularity for all positive ε. But since the limit of $\phi_a^P\chi_{(-\varepsilon,\varepsilon)}$ as $\varepsilon\to 0$ is only defined in the sense of distributions, the conclusion of Theorem 6.3 is invalid for the limit. This behavior is consistent with local Burgers equation where the solution is known not to posses spatial H^1 regularity.

Table 6.1

The set of numerical experiments run for the Δx-refinement study and the ε-refinement study. The domain has a characteristic length L with $N-1$ cells each of width Δx. The smallest and largest values of ε constitute 0.4% of L and 10% of L, respectively.

	Δx-refinement study					ε-refinement study			
N	2000	4000	8000	16000	32000	10000	10000	10000	10000
Δx	3.14e-3	1.57e-3	7.86e-4	3.93e-4	1.97e-4	6.28e-4	6.28e-4	6.28e-4	6.28e-4
ε	5.02e-2	5.02e-2	5.02e-2	5.02e-2	5.02e-2	1.26e-2	6.28e-2	1.57e-1	3.14e-1
ε/L	1.60e-2	1.60e-2	1.60e-2	1.60e-2	1.59e-2	4.00e-3	2.00e-2	5.00e-2	1.00e-1
$\varepsilon/\Delta x$	16	32	64	128	256	20	100	250	500

6.1. Numerical experiments. Motivated by basic results for the local inviscid Burgers equation, we consider numerical simulations for two simple initial conditions. The first of these is used to illustrate shock formation from continuous initial conditions in the local inviscid Burgers equation, and the second shows how an initially discontinuous profile evolves.

A nondimensional measure of the nonlocality is given by the ratio of the nonlocal radius ε to a characteristic length L of the problem under consideration. For the problems considered, the latter value is assigned to the half-length of the interval. Another nondimensional measure of the nonlocality of the discrete problem is given by $\varepsilon/\Delta x$, the number of mesh cells within the interaction radius. We thus conduct both a Δx refinement study, holding ε fixed, and also a ε refinement study, holding Δx fixed. In the Δx-refinement study, the solution is computed with a fixed nonlocal interaction distance $\varepsilon \approx 0.05$ on five meshes, with $N = 2000, 4000, 8000, 16000$, and 32000 nodes, respectively. In the ε refinement study, the solution is computed with $N = 10000$ nodes for nonlocal interaction distances defined by $\varepsilon/\Delta x = 20, 100, 250$, and 500, respectively. The nondimensionalized measures of nonlocality for these problem parameters are shown in Table 6.1. The smallest and largest values of ε constitute 0.4% and 10% of L, respectively.

All of the following numerical experiments are performed using the nonlocal Lax-Friedrichs scheme (5.3) on the interval $[-\pi, \pi)$ with periodic boundary conditions. As we mainly consider cases where $r \gg 1$, the right-hand side of (5.8) is well-approximated by $2\Delta x/c$. Consequently, we utilize for all simulations a mesh refinement path where $\Delta t/\Delta x = 2/c$ is fixed, with $c = 80$.

To explore the effect of the kernel function $\phi_a(y,x)$, three different kernel functions are chosen, and all studies described in Table 6.1 are run for each kernel. The first kernel we consider is (4.20); perhaps the simplest antisymmetric kernel function possible, all points interact equally. The second kernel function we consider is the C^∞ derivative of a Gaussian:

$$\phi_a^C(y,x) = \frac{1}{\alpha}\frac{y-x}{\sqrt{2\pi}\sigma^3}\, e^{(y-x)^2/2\sigma^2}\,, \qquad \alpha = \operatorname{erf}(2\sqrt{2}) - 4\, e^{-8}\sqrt{\frac{2}{\pi}}\,, \tag{6.6}$$

where we choose $\sigma = \varepsilon/4$. The final kernel function we consider is the singular function

$$\phi_a^S(y,x) = \frac{1}{2\varepsilon}\frac{1}{y-x}. \tag{6.7}$$

This kernel function is the antisymmetric analogue to the symmetric micromodulus function described in [35], the most commonly used micromodulus function in peridynamics. As with the kernel (4.20), the functions (6.6) and (6.7) contain prefactors so that these kernels converge to the negative of the first derivative of the Dirac delta distribution as $\sigma \to 0$ and $\varepsilon \to 0$, respectively, in the sense of distributions. Unlike the other kernel functions considered, (6.7) is not in L^1. Further, the numerical scheme used does not integrate the singularity (c.f. Figure 5.2).

6.2. Nonlocal Burgers shock formation. The following problem leads to shock formation from a smooth initial state for the local inviscid Burgers equation, as discussed by Muraki [30]. Consider the following initial conditions:

$$u(x,0) = -\sin x\,,\ -\pi \le x \le \pi, \qquad u(x \pm 2k\pi, t) = u(x,t)\,,\ \forall k \in \mathbb{Z}^+\,. \tag{6.8}$$

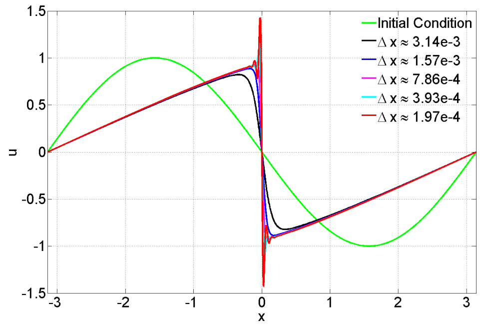
The nonlocal Lax-Friedrichs scheme is run for these initial conditions to a final simulation time $t = 2.0$. Computed results are presented in Figs. 6.1, 6.2, and 6.3.

Figures 6.1(a), 6.1(b) show the outcome of the mesh refinement study, holding the horizon ε constant ($\varepsilon \approx 0.05$, $\varepsilon/L \approx 1.59\times10^{-2}$). The classical Lax-Friedrichs method, whose solution for this initial condition can be seen in Figure 6.2(a), has a well-established shock front at $x = 0$ at this time (and for which shock formation begins at $t = 1$). The nonlocal solutions are qualitatively similar to an N-wave, with additional oscillations appearing around $x = 0$. The horizon refinement findings are shown in Figures 6.1(c), 6.1(d), where more oscillations are observed for larger values of ε. In this case the the smallest ("least nonlocal") horizon ($\varepsilon \approx 1.26\times10^{-2}$, $\varepsilon/L \approx 4\times10^{-3}$) exhibits no oscillations, while the largest ("most nonlocal") horizon ($\varepsilon \approx 3.14\times10^{-1}$, $\varepsilon/L \approx 0.1$) exhibits pronounced oscillations. A study of the effect of different kernel functions is given in Figure 6.2, which shows solutions generated by the three different kernels on a relatively fine mesh ($\Delta x \approx 6.28\times10^{-4}$, $\Delta x/L \approx 2\times10^{-4}$), the top row having the smallest horizon ($\varepsilon \approx 1.26\times10^{-2}$, $\varepsilon/L \approx 4.00\times10^{-3}$), and the bottom row having the largest horizon ($\varepsilon \approx 3.14\times10^{-1}$, $\varepsilon/L \approx 0.1$) used in the horizon refinement study. For comparison, the solution generated by the classical Lax-Friedrichs method for the local inviscid Burgers equation (2.4) is also plotted. The wave speed of the classical model exceeds that of the nonlocal model, as discussed in §4.4. Figures 6.2(a), 6.2(b) show that for the smallest horizon, results very close to the local Lax-Friedrichs result are produced, while Figures 6.2(c), 6.2(d) show solutions for the largest horizon are distinctly different from the local solution. Additionally, differences in the three kernel functions are revealed only for larger horizons. We see in Figure 6.2(d) that the solution computed using kernel (4.20), which is in a sense the "most nonlocal" kernel (weighting all nonlocal interactions equally), to be visually distinct from solutions computed using the other two kernels, as it has a larger peak value and more oscillations near $x = 0$. Figures 6.3(a) and 6.3(b) show that the energy integral $\int_{-\pi}^{\pi} u^2(x,t)\,dx$ is not conserved. These plots indicate that energy is strictly nonincreasing in time, as we expect from a stable discretization, and also consistent with the dissipation associated with the artificial viscosity introduced by the nonlocal Lax-Friedrichs scheme.

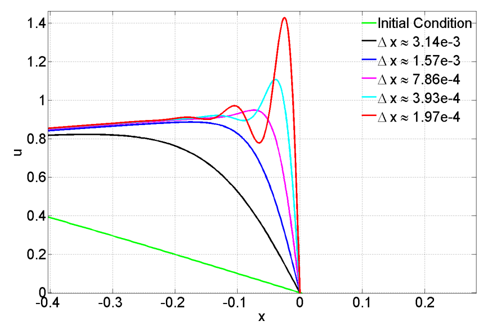
6.3. Nonlocal Burgers shock/rarefaction propagation. We also consider a different initial condition given by the periodic "tophat" function:

$$u(x,0) = H(x+\pi/2) - H(x-\pi/2)\,, -\pi < x < \pi\,,\ u(x \pm 2k\pi, t) = u(x,t)\,, \forall k \in \mathbb{Z}^+. \tag{6.9}$$

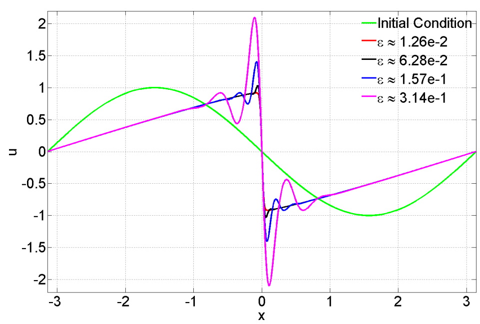
For these initial conditions, the local Burgers equation possesses an exact solution consisting of a growing, linear rarefaction wave (on the left) together with a uniformly propagating shock wave (on the right); these independent structures persist up to $t = 2\pi$, when the leading edge of the rarefaction overtakes the shock.



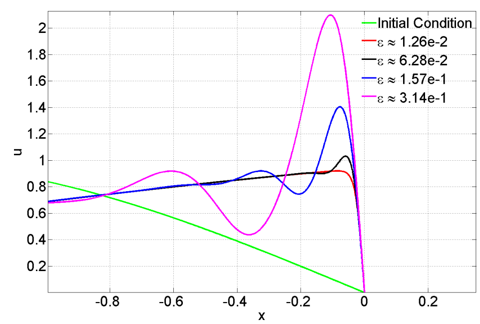
(a) Δx-refinement study, $\varepsilon \approx$ 5.02e-2



(b) Δx-refinement study, $\varepsilon \approx$ 5.02e-2



(c) ε-refinement study, $\Delta x \approx$ 6.28e-4



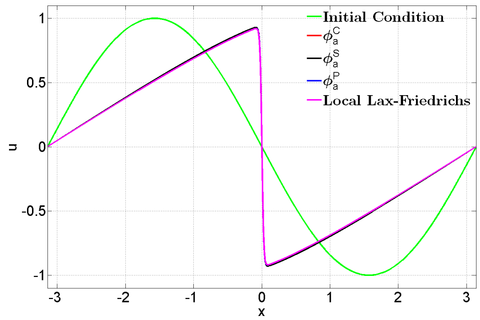
(d) ε-refinement study, $\Delta x \approx$ 6.28e-4

FIG. 6.1. *Numerical results for initial condition* (6.8)*, as described in §6.2. The mesh refinement study of Figs. (a,b) and nonlocal horizon study of Figs. (c,d) are based on the kernel* (4.20).

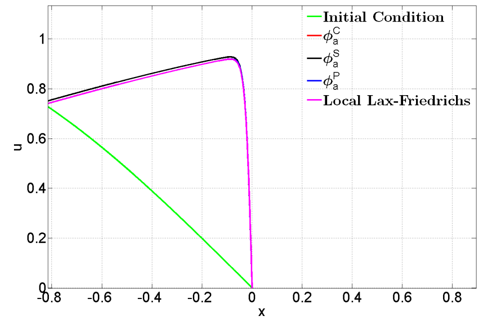
We repeat the mesh and horizon refinement studies of the previous section. For this initial condition, the nonlocal Lax-Friedrichs scheme is run to a final simulation time $t = 1.0$. The mesh refinement and horizon refinement studies for the kernel (4.20) and this initial condition are shown in Figure 6.4, and are qualitatively similar to the previous test case, with increased oscillations seen for larger values of ε. Plots of the energy are not reported for these initial conditions, as they are similar to those in Figure 6.3.

A study of the effect of different kernel functions is given in Figure 6.5, which again shows the solutions generated by the three different kernel on a relatively fine mesh ($\Delta x \approx 6.28\times10^{-4}$, $\Delta x/L \approx 2\times10^{-4}$) for both the smallest and largest values of ε used in the horizon refinement study. For comparison, the solution generated by the classical Lax-Friedrichs method for the local inviscid Burgers equation (2.4) is again plotted. Figures 6.5(a), 6.5(b) show that for the smallest horizon, results very close to the local Lax-Friedrichs result are produced, while Figures 6.5(c), 6.5(d) again show nonlocal solutions distinctly different from the local solution. Again, differences in the three kernel functions are revealed only for larger horizons. The nonlocal solution component corresponding to the rarefaction wave (on the left) is qualitatively similar to the local solution even for a large horizon, unlike the shock (on the right), where the local and nonlocal solutions are manifestly different.

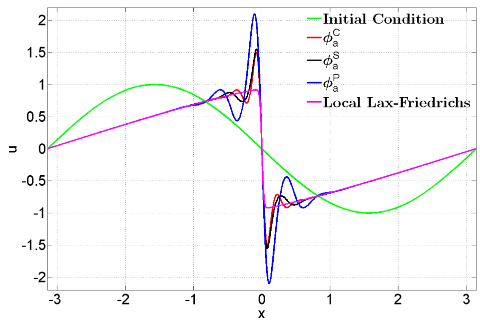
7. Summary. We have presented a new approach to nonlocal, nonlinear advection in one dimension. The motivation for this research is provided by the peridynamic theory of continuum mechanics [36]. The development contained in this paper com-



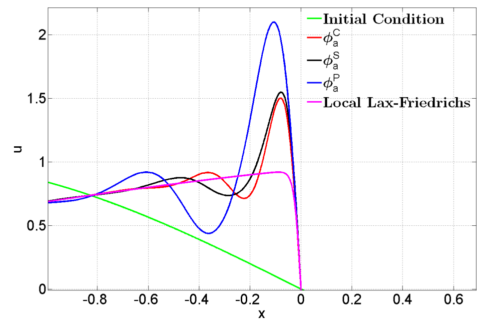
(a) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2



(b) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2

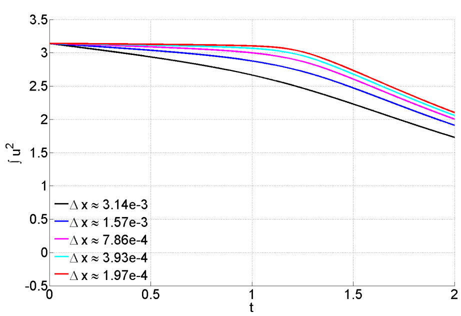


(c) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1

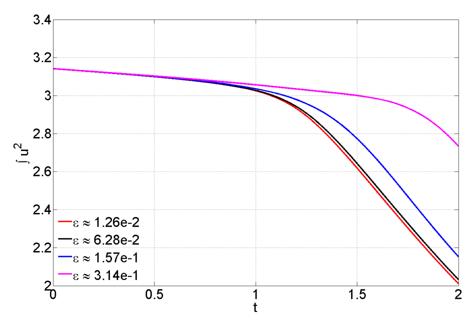


(d) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1

FIG. 6.2. *Numerical results for initial condition* (6.8)*, as described in* §*6.2. Results for all three kernel functions are shown. Figs. (a,b) show the smallest value of ε used, and Figs.(c,d) show the largest value of ε used. The solution produced by the classical Lax-Friedrichs method applied to the local inviscid Burgers equation is included for comparison. For small ε, all kernels display results close to the classical result, but this is not the case for larger values of ε.*



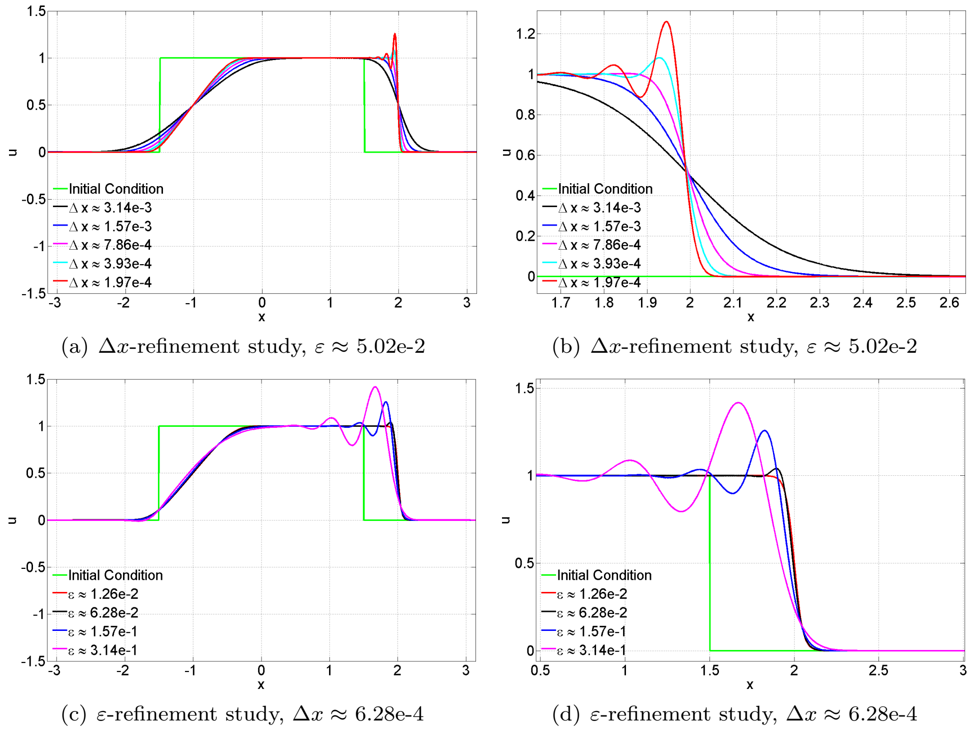
(a) Δx-refinement study, $\varepsilon \approx$ 5.02e-2



(b) ε-refinement study, $\Delta x \approx$ 6.28e-4

FIG. 6.3. *Plots of $\int_{-\pi}^{\pi} u^2(x,t)\,dx$ as a function of time, for the studies shown in Figure 6.1. Both plots confirm that the energy is strictly nonincreasing.*

prises the first steps toward formulation of a coherent mathematical approach for nonlocal advective phenomena consistent with existing peridynamics theory. Using integral operators, we proposed a nonlocal advection equation that we showed to be equivalent to the corresponding local advection equation in the sense of distributions, analyzed the specific case of nonlocal linear advection and demonstrated that the linear peridynamic equation can be written in terms of two nonlocal linear advective



(a) Δx-refinement study, $\varepsilon \approx$ 5.02e-2 (b) Δx-refinement study, $\varepsilon \approx$ 5.02e-2

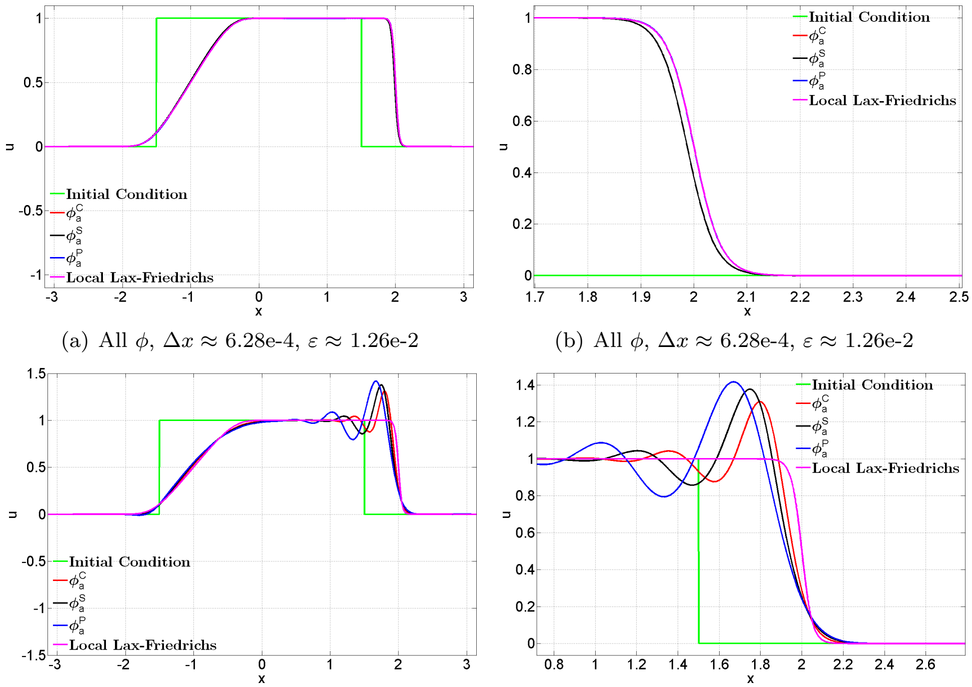
(c) ε-refinement study, $\Delta x \approx$ 6.28e-4 (d) ε-refinement study, $\Delta x \approx$ 6.28e-4

FIG. 6.4. *Numerical results for initial condition* (6.9)*, as described in* §*6.3. The mesh refinement study of Figs. (a,b) and nonlocal horizon study of Figs. (c,d) are based on the kernel* (4.20).

equations. Moreover, we posited a nonlocal regularization that, likewise, reduces to the usual local case in the sense of distributions. Our analysis suggested a generalized concept of a flux that applies, in 1D, to disjoint open intervals on the line. We developed a simple conservative numerical method that was shown to be linearly stable under the usual von Neumann analysis. We performed basic computational experiments with this method on a nonlocal Burgers equation for initial conditions that correspond, in the local case, to shock formation and shock/rarefaction propagation. We also established the well-posedness of the nonlocal Burgers equation over finite time intervals when the antisymmetric kernel associated with the nonlocal advective operator is an element of L^1. Two important conclusions are that a shock cannot develop in finite time and the resulting nonlocal Burgers equation naturally incorporates regularization. The results of these calculations showed that dissipation on the coarser meshes considered significantly damped the solution structure. For the parameters considered, the effect of different nonlocal horizons, however, was slight, and, likewise, the influence of different kernel functions was minor.

Our numerical results are dependent not only on the parameters and kernel functions considered, but on the numerical scheme, as well. In future work, we intend to develop more sophisticated, less dissipative numerical methods with which to investigate solution behavior and go beyond L^1 kernels and seek exact solutions so that we may verify our numerical results.

Acknowledgments. This work was supported by the Laboratory Directed Research and Development Program of Sandia National Laboratories. The authors ac-



(a) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2 (b) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2

(c) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1 (d) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1

FIG. 6.5. *Numerical results for initial condition* (6.9)*, as described in* §*6.3. Results for all three kernel functions are shown. Figs. (a,b) show the smallest value of* ε *used, and Figs.(c,d) show the largest value of* ε *used. The solution produced by the classical Lax-Friedrichs method applied to the local inviscid Burgers equation is included for comparison. For small* ε*, all kernels display results close to the classical result, but this is not the case for larger values of* ε*.*

knowledge helpful discussions with P. Bochev, M. Gunzburger, L. Romero, M. Sears, S. Silling, and T. Trucano.

REFERENCES

[1] G. Alì, J. K. Hunter, and D. F. Parker. Hamiltonian equations for scale-invariant waves. *Stud. Appl. Math.*, 108:305–321, 2002.
[2] N. Alibaud and B. Andreianov. Non-uniqueness of weak solutions for the fractal Burgers equation. *Annal. Inst. H. Poincare C: Non-Linear Analysis*, 27:997–1016, 2010.
[3] N. Alibaud, C. Imbert, and G. Karch. Asymptotic properties of entropy solutions to fractal Burgers equation. *SIAM J. Math. Anal.*, 42:354–376, 2010.
[4] T. M. Apostol. *Mathematical Analysis.* Addison-Wesley series in mathematics. Addison-Wesley, second edition, 1974.
[5] G. R. Baker, X. Li, and A. C. Morlet. Analytic structure of two 1D-transport equations with nonlocal fluxes. *Physica D*, 91:349–375, 1996.
[6] S. Benzoni-Gavage. Local well-posedness of nonlocal Burgers equations. *Diff. Int. Eq.*, 22:303–320, 2009.
[7] F. Betancourt, R. Bärger, K. H. Karlsen, and E. M. Tory. On nonlocal conservation laws modelling sedimentation. *Nonlinearity*, 24(3):855, 2011.
[8] J. Biello and J. K. Hunter. Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities. *Comm. Pure Appl. Math.*, LXIII:303–336, 2010.
[9] P. Biler and W. A. Woyczyński. Global and exploding solutions for nonlocal quadratic evolution problems. *SIAM J. Appl. Math.*, 59(3):845–869, 1998.
[10] A. Castro and D. Córdoba. Global existence, singularities and ill-posedness for a nonlocal flux.

Adv. Math., 219:1916–1936, 2008.

[11] A. J. J. Chmaj. Existence of traveling waves for the nonlocal Burgers equation. *Appl. Math. Lett.*, 20:439–444, 2007.

[12] J. Droniou. Fractal conservation laws: Global smooth solutions and vanishing regularization. *Prog. Nonlin. Diff. Eq. Appl.*, 63:235–242, 2005.

[13] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. Technical report 2010-8353J, Sandia National Laboratories, 2010.

[14] R. Duan, K. Fellner, and C. J. Zhu. Energy method for multi-dimensional balance laws with non-local dissipation. *J. Math. Pures Appl.*, 93:572–598, 2010.

[15] A. C. Eringen. *Nonlocal Continuum Field Theories.* Springer-Verlag, New York, NY, 2002.

[16] V. J. Ervin, N. Heuer, and J. P. Roop. Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation. *SIAM Journal on Numerical Analysis*, 45(2):572–591, 2007.

[17] K. Fellner and C. Schmeiser. Burgers-Poisson: A nonlinear dispersive model equation. *SIAM J. Appl. Math.*, 64:1509–1525, 2004.

[18] M. Gunzburger and R. B. Lehoucq. A nonlocal vector calculus with application to nonlocal boundary value problems. *Multscale Model. Simul.*, 8:1581–1620, 2010.

[19] L. I. Ignat and J. D. Rossi. A nonlocal convection-diffusion equation. *J. Funct. Anal.*, 251:399–437, 2007.

[20] F. Kissling, P. LeFloch, and C. Rohde. A kinetic decomposition for singular limits of non-local conservation laws. *J. Diff. Eq.*, 247:3338–3356, 2009.

[21] F. Kissling and C. Rohde. The computation of nonclassical shock waves with a heterogeneous multiscale method. *Networks Hetero. Media*, 5:661–674, 2010.

[22] C. B. Laney. *Computational Gasdynamics.* Cambridge University Press, New York, NY, 1998.

[23] R. B. Lehoucq and S. A. Silling. Force flux and the peridynamic stress tensor. *J. Mech. Phys. Solids*, 56(4):1566–1577, 2008.

[24] R. B. Lehoucq and O. A. von Lilienfeld. Translation of Walter Noll's derivation of the fundamental equations of continuum thermodynamics from statistical mechanics. *J. Elast.*, 100:5–24, 2010.

[25] R. J. LeVeque. *Numerical Methods for Conservation Laws.* Birkhäuser, Basel, 1992.

[26] H. L. Liu. Wave breaking in a class of nonlocal dispersive wave equations. *J. Nonlin. Math. Phys.*, 13:441–466, 2006.

[27] Y. Liu, D. Pelinovsky, and A. Sakovich. Wave breaking in the Ostrovsky-Hunter equation. *SIAM J. Math. Anal.*, 42:1967–1985, 2010.

[28] J. D. Logan. Nonlocal advection equations. *Int. J. Math. Edu. Sci. Tech.*, 34:271–277, 2002.

[29] P. Miller. Integrable systems, nonlinear waves, and soliton theory, 2006. http://math.arizona.edu/~mcl/Miller/.

[30] D. J. Muraki. A simple illustration of a weak spectral cascade. *SIAM J. Appl. Math.*, 67:1504–1521, 2007.

[31] W. Noll. Die Herleitung der Grundgleichungen der Thermomechanik der Kontinua aus der statistischen Mechanik. *J. Rat. Mech. Anal.*, 4:627–646, 1955.

[32] D. F. Parker. Nonlinearity in elastic surface waves acts nonlocally. In R. V. Goldstein and G. A. Maugin, editors, *Surface Waves in Anisotropic and Laminated Bodies and Defects Detection*, pages 79–94. Kluwer Academic Publishers, Dordrecht, the Netherlands, 2004.

[33] C. Rohde. Scalar conservation laws with mixed local and nonlocal diffusion-dispersion terms. *SIAM J. Math. Anal.*, 37:103–129, 2005.

[34] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48:175–209, 2000. doi:10.1016/S0022-5096(99)00029-0.

[35] S. A. Silling and E. Askari. A meshfree method based on the peridynamic model of solid mechanics. *Comp. Struct.*, 83:1526–1535, 2005.

[36] S. A. Silling and R. B. Lehoucq. Peridynamic Theory of Solid Mechanics. *Advances in Applied Mechanics*, 44:73–69, 2010. doi:10.1016/S0065-2156(10)44002-8.

[37] S. A. Silling, M. Zimmermann, and R. Abeyaratne. Deformation of a peridynamic bar. *Journal of Elasticity*, 73:173–190, 2003.

[38] P. M. Skinis. Some properties of fractional Burgers equation. *Math. Model. Anal.*, 7:151–158, 2002.

[39] A. Veksler and Y. Zarmi. Traveling wave solutions of a simple non-local Burgers-like equation, 2002. arXiv:nlin/0207014v1.

[40] A. Veksler and Y. Zarmi. On spatially non-local Burgers-like dynamical systems. *Nonlinearity*, 16:1367–1380, 2003.

[41] W. A. Woyczyński. *Burgers-KPZ Turbulence*, volume 1700 of *Lecture Notes in Mathematics.*

Springer-Verlag, Berlin, 1998.