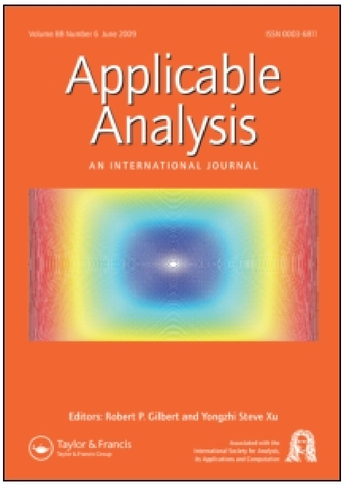

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Global existence and uniqueness of solutions of the time-dependent ginzburg-landau model for superconductivity

Qiang Du[a]
[a] Department of Mathematics,, Michigan State University,, Lansing, USA

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Global Existence and Uniqueness of Solutions of The Time-Dependent Ginzburg-Landau Model for Superconductivity

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QIANG DU
Department of Mathematics, Michigan State University, East Lansing, MI 48824, U. S. A.

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Abstract. We consider the initial-boundary value problems of the time-dependent nonlinear Ginzburg-Landau equations in superconductivity. It is assumed that the material sample occupies a bounded domain in two and three dimensional spaces. We illustrate that the original equations are not well-posed. In order to fix the lack of uniqueness of the solutions, possible choices of the gauge are identified. Global existence and uniqueness of solutions are proved in a proper gauge. A by-product is the convergence of finite-dimensional Galerkin approximations which may be used in the numerical study of superconductivity phenomena.

KEY WORDS: Superconductivity, time-dependent Ginzburg-Landau equations, choice of gauge, initial-boundary value problems, global existence and uniqueness, finite dimensional approximations

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1. INTRODUCTION.

The superconductivity of certain metals at very low temperatures was discovered in 1911 by H. Kamerlingh-Onnes. In the last few years, there has been a rebirth of interest in superconductivity among the physics, engineering, and mathematics communities, following the discovery of materials that retain superconductive properties at "high" temperatures. Mathematical and numerical studies of models of superconductivity have also received more and more attention.

One of the very popular macroscopic model for superconductivity used by researchers in their studies is the Ginzburg-Landau model proposed in 1950. In 1959, Gor'kov[1] showed that, in the appropriate limit, the steady state macroscopic Ginzburg-Landau model may be derived from the BCS theory. Well-known results based on the Ginzburg-Landau model include the prediction made by Abrikosov[2] on the existence of *type-II* superconductors ten years before it was observed experimentally. Earlier mathematical analyses of the steady state Ginzburg-Landau equations can be found, for example, in Carroll and Glick[3] and Odeh[4]. Recent studies can be found in[5-15] and the references sited therein.

Based on an averaging of the BCS theory, a time-dependent G-L model was derived by Gor'kov and Eliashberg[16] in 1968 (see also [17,18] for further discussions). The equations are nonlinear differential equations for the complex order parameter, the real vector magnetic potential and the real scalar electric potential.

Studies of the time-dependent Ginzburg-Landau model for superconductivity may give a better understanding of the dynamics of the superconducting transition, especially for the *type-II* superconductors. The generation and the interaction of "flux vortices" are of great interest. The fully nonlinear time-dependent model has been used in the recent numerical simulations[19-20] to study the growth of superconducting phase and various time-scales related to the transition in two-dimensional thin films. Other studies may be found, for example, in[21,22]. Extensive numerical simulations of the time-dependent G-L models are presently under way. However, when one attempts to set up numerical algorithms and simulation procedures, there are several questions remain to be answered for the time-dependent model. In particular, the question of global existence and uniqueness of the solutions has not been discussed in the literature except for some much simplified models in one space dimension. In fact, the original Ginzburg-Landau time-dependent equations are *not* mathematically well-posed unless *gauge fixing* has been done.

In this paper, we present results concerning the global existence and uniqueness of solutions of the time-dependent nonlinear Ginzburg-Landau equations for superconductivity in a two or three dimensional bounded domain. These results rely on the gauge fixing. The approach we adopt is a combination of estimates for a set of modified problems and their finite dimensional approximations and compactness arguments. The use of finite dimensional approximations is standard and is largely influenced by our interests in the numerical solution of the Ginzburg-Landau equations. A by-product of the approach taken here is a convergence result concerning the finite dimensional approximation which is helpful in further numerical studies of the time-dependent model. We are presently performing several numerical simulations that are based on the finite element approximations of the time-dependent G-L models and their variants that can take into account the inhomogeneity, anisotropy, thickness variation, normal inclusion, layered structures and other features that may be applicable to high-T_c superconductors. More results on numerical simulations will be reported elsewhere.

The paper is organized as follows. In section 2, we introduce the time-dependent equation and its nondimensionalized form. Then, we go through the steps of *gauge fixing* which are essential for the numerical simulation as well as the discussion on the well-posedness. In section 3, we study a set of modified equations and their finite dimensional Galerkin approximations. Energy type estimates are derived and compactness arguments are used to show the existence and the uniqueness of the modified equations. Then, we show the global existence and uniqueness of the solution for the original equations based on additional estimates. Finally, we mention in section 4 a convergence result for the finite dimensional approximations.

2. THE TIME-DEPENDENT GINZBURG-LANDAU MODEL.

There are many excellent references that may be consulted for detailed descriptions of both the microscopic and macroscopic theories of superconductivity, for example, Tinkham[18] and Abrikosov[23]. For a recent survey on the steady state model, we refer to Du, Gunzburger and Peterson[11].

2.1. The Ginzburg-Landau free energy.

In the presence of an applied magnetic field $\mathbf{H}$, Ginzburg and Landau postulated that the Gibbs free energy per unit volume of a superconducting material is given by

$$g = f - \frac{\mathbf{h}\cdot\mathbf{H}}{4\pi} = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_s}\left|\left(i\hbar\nabla + \frac{e_s\mathbf{A}}{c}\right)\psi\right|^2 + \frac{|\mathbf{h}|^2}{8\pi} - \frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}\,. \tag{2.1}$$

Here, the constant f_n is the free energy of the normal (non-superconducting) state in the absence of magnetic fields, ψ is the (complex-valued) order parameter, $\mathbf{A}$ is the magnetic potential, $\mathbf{h} = \operatorname{curl}\mathbf{A}$ is the magnetic field, α and β are constants (with respect to the space variable $\mathbf{x}$) whose values depend on the temperature, c is the speed of light, e_s and m_s are the charge and mass, respectively, of the superconducting charge-carriers, and $2\pi\hbar$ is Planck's constant. The temperature is taken to be constant in our discussion.

If Ω denotes the region occupied by the superconducting sample with boundarty Γ, the Gibbs free energy $\mathcal{G}$ of the sample is then given by

$$\begin{aligned}\mathcal{G}(\psi,\mathbf{A}) = \int_\Omega g\,d\Omega &= \int_\Omega \left(f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4\right)d\Omega \\ &+ \int_\Omega \left[\frac{1}{2m_s}\left|\left(i\hbar\nabla + \frac{e_s\mathbf{A}}{c}\right)\psi\right|^2 + \frac{|\mathbf{h}|^2}{8\pi} - \frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}\right]d\Omega\,.\end{aligned} \tag{2.2}$$

A nondimensionalized form of the free energy functional is given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left[|\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\operatorname{curl}\mathbf{A} - \mathbf{H}|^2 \right] d\Omega . \tag{2.3}$$

Here, κ, known as the Ginzburg-Landau parameter, is a very important material constant representing the ratio of penetration length to the coherence length. A critical value of κ is $1/\sqrt{2}$. For the *type-I* superconductors, we have $\kappa < 1/\sqrt{2}$, while for the *type-II* superconductors, we have $\kappa > 1/\sqrt{2}$.

The square of the magnitude of the complex order parameter, $|\psi|^2$, represents the density of the superconducting carriers. $\psi = 0$ corresponds to normal state and in a perfect superconducting state $|\psi| = 1$.

2.2. The evolution model.
According to Gorkov and Eliashberg[16], see also [18,23], the evolution Ginzburg-Landau model is given by:

$$\hbar\frac{\partial\psi}{\partial t} + ie\Phi\psi - D(\hbar\nabla - ie\mathbf{A})^2\psi + (\beta|\psi|^2 + \alpha)\psi = 0 \quad \text{in } \Omega ,$$

$$-\nu \operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{E} - 2\tau\left[|\psi|^2\mathbf{A} - \frac{i\hbar}{2e}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \operatorname{curl}\mathbf{H}\right] \quad \text{in } \Omega ,$$

where Φ is the scalar electric potential, τ and D are microscopic parameters and ν^{-1} measures the conductivity of normal electrons and $\mathbf{E}$ is the electric field given by

$$\mathbf{E} = \frac{\partial\mathbf{A}}{\partial t} + \nabla\Phi .$$

The boundary conditions are

$$(\hbar\nabla - ie\mathbf{A})\psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma , \tag{2.4}$$

and

$$\mathbf{E} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma .$$

A boundary condition that is more general than (2.4) is given by

$$(\hbar\nabla - ie\mathbf{A})\psi \cdot \mathbf{n} = -\gamma\psi \quad \text{on } \Gamma , \tag{2.4'}$$

where γ is small for insulators and large for magnetic materials, with normal metals lying between them[18]. Our discussion essentially extends to this case with the modification that an extra term

$$\int_\Gamma \hbar\gamma|\psi|^2 d\Gamma$$

should be added to the free energy functional in the discussion. Such a term is known to describe the proximity effect. For convenience, we only state results associated with the condition (2.4). Numerical studies of proximity effect will be given in a future report.

2.3. The nondimensionalized equations.
After proper non-dimensionalization, the time dependent equations may be given as

$$\eta\frac{\partial\psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega , \tag{2.5}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = -\mathbf{E} - \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl}\mathbf{H} \quad \text{in } \Omega , \tag{2.6}$$

where $\operatorname{curl} \mathbf{H} = 0$ if we consider just constant external field and

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \ . \tag{2.7}$$

The boundary conditions are

$$(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.8}$$

$$\operatorname{curl} \mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \Gamma \tag{2.9}$$

and

$$\mathbf{E}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,. \tag{2.10}$$

The initial conditions are

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}) \quad \text{in } \Omega.$$

and

$$\mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}) \quad \text{in } \Omega.$$

We assume that $|\psi_0(x)| \leq 1, a.e.$, which means that the magnitude of the initial order parameter does not exceed the value at superconducting state.

Throughout, for any non-negative integer s, $H^s(\mathcal{D})$ will denote the Sobolev space of real-valued functions having square integrable spatial derivatives of order up to s in the domain Ω. The corresponding spaces of complex-valued functions will be denoted by $\mathcal{H}^s(\mathcal{D})$. Corresponding spaces of vector-valued functions, each of whose d components belong to $H^s(\mathcal{D})$, will be denoted by $\mathbf{H}^s(\mathcal{D})$, *i.e.*, $\mathbf{H}^s(\mathcal{D}) = [H^s(\mathcal{D})]^d$. Norms of functions belonging to $H^s(\mathcal{D})$, $\mathbf{H}^s(\mathcal{D})$, and $\mathcal{H}^s(\mathcal{D})$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_s$. For details concerning these spaces, one may consult Adams[24]. A similar notational convention will hold for the Lebesgue spaces $L^p(\Omega)$ and their complex and vector-valued counterparts $\mathcal{L}^p(\Omega)$ and $\mathbf{L}^p(\Omega)$, respectively. We sometimes use $\|\cdot\|_B$ to denote the norm defined on the space B.

We make a convention that $(\cdot,\cdot)$ denotes the standard L^2 inner-product in the real function spaces while for complex valued functions

$$(\psi, \tilde{\psi}) = \int_\Omega \psi\tilde{\psi}^*\, d\Omega\,, \quad \forall\, \psi, \tilde{\psi} \in \mathcal{L}^2(\Omega)\,.$$

We will also make use of the following subspaces of $\mathbf{H}^1(\Omega)$:

$$\mathbf{H}^1_n(\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}$$

and

$$\mathbf{H}^1_n(\operatorname{div};\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \ : \ \operatorname{div}\mathbf{Q} = 0 \text{ in } \Omega \text{ and } \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}\,.$$

We note that $(\|\operatorname{div}\mathbf{Q}\|_0^2 + \|\operatorname{curl}\mathbf{Q}\|_0^2)^{1/2}$ and $\|\operatorname{curl}\mathbf{Q}\|_0$ define norms on $\mathbf{H}^1_n(\Omega)$ and $\mathbf{H}^1_n(\operatorname{div};\Omega)$, respectively, that are equivalent to the standard $\mathbf{H}^1(\Omega)$-norm $\|\mathbf{Q}\|_1$; see, *e.g.*, Adams[24], Girault and Raviart[25]. To take into account the time-dependence, we define the following spaces: for any given $T > 0$ and given Hilbert space B,

$$L^p(0,T;B) = \{f : f(\cdot,t) \in B\,, \forall\, t \in (0,T)\ a.e.\,,\ \int_0^T \|f(\cdot,t)\|_B^p\, dt < \infty\}\,.$$

Spaces like $L^\infty(0,T;B)$ and $H^m(0,T;B)$ are defined in similar ways. In particular, we let $\mathbf{S} = \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$ and

$$\mathbf{V} = \mathbf{L}^\infty(0,T;\mathbf{H}^1_n(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))\,.$$

Also, we let $\mathcal{S} = \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ and

$$\mathcal{V} = \mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega)) .$$

We assume that Ω is a bounded subdomain of $\mathbf{R}^d$ with smooth boundary, where $d = 2$ or 3. Results remain valid if Ω is a convex polygon or convex polyhedron. Basically, we require these conditions on the domain and its boundary in order to get the standard H^2 regularity estimates for the solutions of the equation

$$\Delta\psi = f \in \mathcal{L}^2(\Omega)$$

with homogeneous Neumann boundary condition and the related H^1 regularity results for the solutions of the system

$$\operatorname{div}\mathbf{A} = f \in L^2(\Omega) ,$$

$$\operatorname{curl}\mathbf{A} = \mathbf{g} \in \mathbf{L}^2(\Omega)$$

with condition $\mathbf{A}\cdot\mathbf{n} = 0$ on the boundary. For the solution $\mathbf{A}$ of

$$\lambda\nabla\operatorname{div}\mathbf{A} - \sigma\operatorname{curl}(\operatorname{curl}\mathbf{A} - \mathbf{H}) = \mathbf{g} \in \mathbf{L}^2(\Omega) \;, \quad (\lambda, \sigma > 0)$$

with boundary conditions (2.9) and $\mathbf{A}\cdot\mathbf{n} = 0$, H^1 regularity may be obtained for $\operatorname{div}\mathbf{A}$ and $\operatorname{curl}\mathbf{A}$ using the results of Girault and Raviart[25] on the decomposition of vector field in $\mathbf{L}^2(\Omega)$. For domains with smooth boundary, related H^2 regularity results are discussed in Georgescu[26].

2.4. Gauge invariance.

The Ginzburg-Landau equations have an important property, namely, that of gauge invariance. In order to prove the existence and uniqueness of the solutions, it is clear that a suitable gauge choice must be made first.

Formally, given a function χ, the linear transformation G_χ is defined by

$$G_\chi(\psi, \mathbf{A}, \Phi) = (\zeta, \mathbf{Q}, \Theta)$$

where

$$\zeta = \psi e^{i\kappa\chi} \;, \quad \mathbf{Q} = \mathbf{A} + \nabla\chi \quad \text{and} \quad \Theta = \Phi - \frac{\partial\chi}{\partial t} .$$

Note that if $(\zeta, \mathbf{Q}, \Theta) = G_\chi(\psi, \mathbf{A}, \Phi)$, then $(\psi, \mathbf{A}, \Phi) = G_{-\chi}(\zeta, \mathbf{Q}, \Theta)$, in this case, $(\zeta, \mathbf{Q}, \Theta)$ and $(\psi, \mathbf{A}, \Phi)$ are called *gauge equivalent*.

The time-dependent Ginzburg-Landau equations with the prescribed boundary conditions are gauge invariant in the sense that if $(\psi, \mathbf{A}, \Phi)$ is a solution to the equations, so is $(\zeta, \mathbf{Q}, \Theta) = G_\chi(\psi, \mathbf{A}, \Phi)$. Of course, the initial conditions may need to be modified. Notice that if at $t = 0$, $\chi \equiv 0$, then even the initial conditions remain unchanged. This indicates that the time-dependent Ginzburg-Landau equations given earlier lack uniqueness and thus are not well-posed.

To obtain mathematically well-posed equations, we go through a procedure commonly known as *fixing the gauge*, that is, enforcing extra constraints on the solutions. There are many well-known gauge choices, for example, the Coulomb gauge. Here, we briefly describe a few popular choices.

2.4.1. The Coulomb gauge.

The Coulomb gauge is a gauge in which $\mathbf{A}$ is taken to be divergence free. Given a solution $(\psi, \mathbf{A}, \Phi)$, it can be obtained by a gauge transformation G_χ where χ satisfies

$$\Delta\chi = -\operatorname{div}\mathbf{A} \quad \text{in } \Omega ,$$

and

$$\nabla\chi\cdot\mathbf{n} = -\mathbf{A}\cdot\mathbf{n} \quad \text{on } \Gamma .$$

More precisely, one gets the following equations in the current gauge:

$$\eta\frac{\partial\psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{2.11}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \operatorname{curl}\operatorname{curl}\mathbf{A} + \nabla\Phi = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl}\mathbf{H} \quad \text{in } \Omega\,, \tag{2.12}$$

$$-\Delta\Phi = \operatorname{div}\left[\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A}\right] \quad \text{in } \Omega\,. \tag{2.13}$$

and $\operatorname{div}\mathbf{A} = 0$ in Ω. The boundary conditions are

$$\nabla\psi\cdot\mathbf{n} = 0\,,\quad \nabla\Phi\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.14}$$

and

$$\operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n}\,,\quad \mathbf{A}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.15}$$

We may also fix the average of Φ in Ω to be zero, i.e.,

$$\int_\Omega \phi\, d\Omega = 0.$$

Though the Coulomb gauge is very convenient in the studies of steady state solutions[11], it may be more difficult to deal with the divergence free condition in numerical simulations in the unsteady case.

2.4.2. The $\Phi = \pm\operatorname{div}\mathbf{A}$ gauge.
In most cases, one would like to eliminate the electric potential Φ. Given a solution $(\psi, \mathbf{A}, \Phi)$, one choice to to use gauge transformation G_χ where χ satisfies

$$\frac{\partial\chi}{\partial t} - \Delta\chi = \Phi \mp \operatorname{div}\mathbf{A} \quad \text{in } \Omega\,,$$

with boundary condition:

$$\nabla\chi\cdot\mathbf{n} = \pm\mathbf{A}\cdot\mathbf{n} \quad \text{on } \Gamma\,,$$

and at $t = 0$, χ is determined, up to a constant, by $\Delta\chi = \pm\operatorname{div}\mathbf{A}$ in Ω, $\nabla\chi\cdot\mathbf{n} = \pm\mathbf{A}\cdot\mathbf{n}$ on Γ.

We have the following equations in the current gauge :

$$\eta\frac{\partial\psi}{\partial t} \pm i\eta\kappa\operatorname{div}\mathbf{A}\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{2.16}$$

$$\frac{\partial \mathbf{A}}{\partial t} - \Delta\mathbf{A} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl}\mathbf{H} \quad \text{in } \Omega\,, \tag{2.17}$$

with the boundary conditions

$$\operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n}\,,\quad \mathbf{A}\cdot\mathbf{n} = 0 \text{ and } \nabla\psi\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,. \tag{2.18}$$

At $t = 0$, we have $\operatorname{div}\mathbf{A} = 0$ in Ω.

2.4.3. The zero electric potential gauge.

Here, we give another gauge corresponding to $\Phi = 0$. This is one of most frequently used gauge choice in numerical simulations, see, for example, Frahm et. al. [19,20] Given a solution $(\psi, \mathbf{A}, \Phi)$, we use the gauge transformation G_χ where χ satisfies

$$\frac{\partial \chi}{\partial t} = \Phi$$

and at $t = 0$, $\Delta\chi = -\operatorname{div} \mathbf{A}$ in Ω with $\nabla\chi \cdot \mathbf{n} = -\mathbf{A} \cdot \mathbf{n}$ on Γ.

In this gauge, the equations becomes:

$$\eta\frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{2.19}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \operatorname{curl}\operatorname{curl} \mathbf{A} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl} \mathbf{H} \quad \text{in } \Omega\,, \tag{2.20}$$

The boundary conditions are

$$\nabla\psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.21}$$

$$\operatorname{curl} \mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \Gamma\,, \tag{2.22}$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.23}$$

and at $t = 0$, $\operatorname{div} \mathbf{A} = 0$ in Ω.

Notice that in the current gauge, the vector potential $\mathbf{A}$ need not be divergence free. In fact, taking the divergence of equation (2.20) and using equation (2.19), one may get

$$\frac{\partial}{\partial t}\operatorname{div} \mathbf{A} = -\frac{i}{2}\eta\kappa\left[\psi\frac{\partial \psi^*}{\partial t} - \psi^*\frac{\partial \psi}{\partial t}\right]\,. \tag{2.24}$$

Therefore, having the vector potential divergence free implies that either the order parameter is zero or its phase is independent of time. Neither is true in general. On the other hand, when we have an unsteady divergence of the vector potential $\mathbf{A}$, the charge density, given in the current gauge by

$$\operatorname{div} \mathbf{E} = \operatorname{div}\frac{\partial \mathbf{A}}{\partial t}\,,$$

need not vanish. This is only connected to the superconducting current in the unsteady case[22,23].

In this paper, we concentrate on this particular gauge choice. One of the reason is that the equations (2.19)-(2.20) may be viewed as a gradient flow with energy functional $\mathcal{G}(\psi, \mathbf{A})$. We like to point out that in the current gauge, the functional $\mathcal{G}$ is not coercive. This difficulty is resolved by introducing a set of modified problems and then deriving uniform estimates based on (2.24) and passing to the limit.

2.5. Steady state equations.

To make the discussion complete, we briefly mention the steady state equations. Here, discussions of the gauge choices may be done in a similar way. In fact, simple calculations show that[11] one can choose a gauge to have both $\Phi = 0$ and $\operatorname{div} \mathbf{A} = 0$, which cannot be achieved in the unsteady case. Thus, we have the following standard nondimensionalized steady state Ginzburg-Landau equations.

$$\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\,, \tag{2.25}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl}\mathbf{H} \quad \text{in } \Omega\,, \tag{2.26}$$

and $\operatorname{div}\mathbf{A} = 0$ in Ω. The boundary conditions are

$$\nabla\psi\cdot\mathbf{n} = 0\,,\ \operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n} \text{ and } \mathbf{A}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.27}$$

Solutions to the above steady state equations, commonly referred as the G-L equations in the Coulomb gauge, exist and are not unique in general. The convergence of the solutions of the time-dependent equations to the steady state equation as $t\to\infty$ remains to be studied.

3. GLOBAL EXISTENCE AND UNIQUENESS OF THE SOLUTION.

In this section, we study the strong solution of the weak forms corresponding to the time-dependent Ginzburg-Landau equations. We may choose to work in any one of the gauges mentioned earlier. However, for numerical simulation purposes, we choose the zero electric potential gauge. The global existence and uniqueness of strong solution in this gauge are given later in Theorem 3.13 and Theorem 3.15. The rationale of such choice in actual implementation will be given in Du[27]. We merely point out that there may be some difficulties associated with the choice of Coulomb gauge in unsteady computations (there are related discussions in the context of Navier-Stokes equations[25]), though the difficulty can be overcome in the steady state case[3]. Results related to the existence and uniqueness of solutions in other gauges may be obtained from the results presented here through gauge transformations.

3.1. Weak forms in the zero electric potential gauge.

We now concentrate on the zero electric potential gauge, there, the strong solution $(\psi, \mathbf{A})$ satisfies the following weak formulation of equations (2.19)-(2.23):

Find $(\psi, \mathbf{A}) \in \mathcal{V}\times\mathbf{V}$ such that

$$\begin{aligned}&\eta\frac{d}{dt}(\psi,\tilde{\psi}) + \left(\left[-\frac{i}{\kappa}\nabla\psi - \mathbf{A}\psi\right], \left[-\frac{i}{\kappa}\nabla\tilde{\psi} - \mathbf{A}\tilde{\psi}\right]\right)\\ &\quad + ([|\psi|^2 - 1]\psi, \tilde{\psi}) = 0 \quad \forall\, \tilde{\psi}\in\mathcal{H}^1(\Omega)\,;\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\frac{d}{dt}(\mathbf{A},\tilde{\mathbf{A}}) + (\operatorname{curl}\mathbf{A}, \operatorname{curl}\tilde{\mathbf{A}}) + (|\psi|^2\mathbf{A}, \tilde{\mathbf{A}})\\ &\quad + \left(\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*), \tilde{\mathbf{A}}\right) = (\mathbf{H}, \operatorname{curl}\tilde{\mathbf{A}}) \quad \forall\, \tilde{\mathbf{A}}\in\mathbf{H}^1_n(\Omega)\,.\end{aligned} \tag{3.2}$$

with the initial condition $\psi_0\in\mathcal{H}^1(\Omega)$ and $\mathbf{A}_0\in\mathbf{H}^1_n(\operatorname{div};\Omega)$. The initial conditions make sense for functions in $\mathcal{V}\times\mathbf{V}$ that satisfy the weak forms. For convenience, we assume that the applied field $\mathbf{H}\in\mathbf{H}^1(\Omega)$.

To study the existence and uniqueness of the solutions of the above system, we first consider the following modified problem:

Find $(\psi^\epsilon, \mathbf{A}^\epsilon)\in\mathcal{V}\times\mathbf{V}$ such that

$$\begin{aligned}&\eta\frac{d}{dt}(\psi^\epsilon,\tilde{\psi}) + \left(\left[-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon\right], \left[-\frac{i}{\kappa}\nabla\tilde{\psi} - \mathbf{A}^\epsilon\tilde{\psi}\right]\right)\\ &\quad + ([|\psi^\epsilon|^2 - 1]\psi^\epsilon, \tilde{\psi}) = 0 \quad \forall\, \tilde{\psi}\in\mathcal{H}^1(\Omega)\,;\end{aligned} \tag{3.1$_\epsilon$}$$

$$(3.2_\epsilon)\qquad \begin{aligned}&\frac{d}{dt}(\mathbf{A}^\epsilon, \tilde{\mathbf{A}}) + (\operatorname{curl}\mathbf{A}^\epsilon, \operatorname{curl}\tilde{\mathbf{A}}) + \epsilon(\operatorname{div}\mathbf{A}^\epsilon, \operatorname{div}\tilde{\mathbf{A}}) + (|\psi^\epsilon|^2\mathbf{A}^\epsilon, \tilde{\mathbf{A}})\\ &\quad + \Re\{(\frac{i}{\kappa}\nabla\psi^\epsilon, \psi^\epsilon\tilde{\mathbf{A}})\} = (\mathbf{H}, \operatorname{curl}\tilde{\mathbf{A}}) \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)\end{aligned}$$

and the initial conditions are the same as the original equations, i.e., they are independent of ϵ.

Here, $\epsilon > 0$ is an arbitrary parameter. Note that the above modified system reduces to the original system (3.1)-(3.2) when $\epsilon = 0$. Our idea is to use the modified systems and their Galerkin approximations as intermediate steps. This set of modified problems also have quite interesting dynamics. It can be shown that the ω-limit set with $\epsilon > 0$ is the solution set of the steady state G-L equations in the Coulomb gauge.

3.2. Existence and uniqueness of modified problems.

The existence and uniqueness of solutions are proved using Galerkin finite dimensional approximations and compactness arguments, a standard technique for nonlinear evolution equations, see Temam[28] for a similar account for the Navier-Stokes equations. A unique feature in our discussion is that we apply such a technique to the set of modified problems first. Then, we pass to the limit $\epsilon \to 0$.

Let $\mathbf{\Lambda}_n$ and $\mathcal{Z}_n$ be n-dimensional subspaces of $\mathbf{H}^1_n(\Omega)$ and $\mathcal{H}^1(\Omega)$ respectively such that

$$\bigcup \mathbf{\Lambda}_n \quad \text{is dense in } \mathbf{H}^1_n(\Omega), \quad \text{and} \quad \bigcup \mathcal{Z}_n \quad \text{is dense in } \mathcal{H}^1(\Omega)\,.$$

The standard Galerkin finite dimensional approximation may be given by:

Find $(\psi^\epsilon_n(t), \mathbf{A}^\epsilon_n(t)) \in \mathcal{Z}_n \times \mathbf{\Lambda}_n$ such that

$$(3.3)\qquad (\nabla\psi^\epsilon_n(0), \nabla\tilde{\psi}_n) + (\psi^\epsilon_n(0), \tilde{\psi}_n) = (\nabla\psi(0), \nabla\tilde{\psi}_n) + (\psi(0), \tilde{\psi}_n) \quad \forall\tilde{\psi}_n \in \mathcal{Z}_n\,,$$

$$(3.4)\qquad (\nabla\mathbf{A}^\epsilon_n(0), \nabla\tilde{\mathbf{A}}_n) + (\mathbf{A}^\epsilon_n(0), \tilde{\mathbf{A}}_n) = (\nabla\mathbf{A}(0), \nabla\tilde{\mathbf{A}}_n) + (\mathbf{A}(0), \tilde{\mathbf{A}}_n) \quad \forall\tilde{\mathbf{A}}_n \in \mathbf{\Lambda}_n\,,$$

and

$$(3.5)\qquad \begin{aligned}&\eta\frac{d}{dt}(\psi^\epsilon_n, \tilde{\psi}_n) + ([-\frac{i}{\kappa}\nabla\psi^\epsilon_n - \mathbf{A}^\epsilon_n\psi^\epsilon_n], [-\frac{i}{\kappa}\nabla\tilde{\psi}_n - \mathbf{A}^\epsilon_n\tilde{\psi}_n])\\ &\quad + ([|\psi^\epsilon_n|^2 - 1]\psi^\epsilon_n, \tilde{\psi}_n) = 0 \quad \forall\,\tilde{\psi}_n \in \mathcal{Z}_n\,;\end{aligned}$$

$$(3.6)\qquad \begin{aligned}&\frac{d}{dt}(\mathbf{A}^\epsilon_n, \tilde{\mathbf{A}}_n) + (\operatorname{curl}\mathbf{A}^\epsilon_n, \operatorname{curl}\tilde{\mathbf{A}}_n) + \epsilon(\operatorname{div}\mathbf{A}^\epsilon_n, \operatorname{div}\tilde{\mathbf{A}}_n) + (|\psi^\epsilon_n|^2\mathbf{A}^\epsilon_n, \tilde{\mathbf{A}}_n)\\ &\quad + \Re\{(\frac{i}{\kappa}\nabla\psi^\epsilon_n, \psi^\epsilon_n\tilde{\mathbf{A}}_n)\} = (\mathbf{H}, \operatorname{curl}\tilde{\mathbf{A}}_n) \quad \forall\,\tilde{\mathbf{A}}_n \in \mathbf{\Lambda}_n\,.\end{aligned}$$

Using local existence and uniqueness results on ODEs, we first have that

LEMMA 3.1. *Given $\epsilon > 0$, for any $n > 0$, there exists a unique solution to the above system in $[0, T_n]$ for some $T_n > 0$.*

Next, we derive some energy type estimates to show the solution to the ODEs is global.
Let us define

$$(3.7)\qquad \begin{aligned}\mathcal{G}_\epsilon(\psi, \mathbf{A}) &= \mathcal{G}(\psi, \mathbf{A}) + \epsilon\int_\Omega |\operatorname{div}\mathbf{A}|^2\, d\Omega\\ &= \int_\Omega \left[|\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\operatorname{curl}\mathbf{A} - \mathbf{H}|^2 + \epsilon|\operatorname{div}\mathbf{A}|^2\right] d\Omega\,.\end{aligned}$$

Let δ denote the first variation of the functional. Then, we get

LEMMA 3.2. *For any $t > 0$, if the solution $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ exists, then*

$$\frac{1}{2}\frac{d}{dt}\mathcal{G}_\epsilon(\psi_n^\epsilon, \mathbf{A}_n^\epsilon) + (\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}) + \eta(\frac{\partial \psi_n^\epsilon}{\partial t}, \frac{\partial \psi_n^\epsilon}{\partial t}) = 0\,. \tag{3.8}$$

Proof. Let $\tilde{\psi}_n = \frac{\partial \psi_n^\epsilon}{\partial t}$ and $\tilde{\mathbf{A}}_n = \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}$ in the weak form, and add the resulting equations together. Then

$$\begin{aligned}\eta(\frac{\partial \psi_n^\epsilon}{\partial t}, \frac{\partial \psi_n^\epsilon}{\partial t}) + (\frac{\partial \mathbf{A}_n^\epsilon}{\partial t}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t}) &= -\Re\{(\frac{\delta \mathcal{G}_\epsilon}{\delta \psi_n^\epsilon}, \frac{\partial \psi_n^\epsilon}{\partial t}) + (\frac{\delta \mathcal{G}_\epsilon}{\delta \mathbf{A}_n^\epsilon}, \frac{\partial \mathbf{A}_n^\epsilon}{\partial t})\} \\ &= -\frac{1}{2}\frac{d}{dt}\mathcal{G}_\epsilon(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)\,.\end{aligned}$$

□

Using the above estimates, we get

COROLLARY 3.3. *Given $\epsilon > 0$, for any given $T > 0$, there exists a unique solution $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ for the ODE system (3.3)-(3.6). Moreover, $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ is uniformly bounded in $\mathcal{V} \times \mathbf{V}$.*

We now quote a special case of the compactness lemma of Lions[29],

LEMMA 3.4. (Lions) *Let B be a Banach space and B_i, $i = 0, 1$, Hilbert spaces. Suppose that $B_0 \hookrightarrow\hookrightarrow B$, i.e., the imbedding is compact, and suppose that the imbedding from B to B_1 is continuous, then*

$$L^p(0,T;B_0) \cap W^{1,q}(0,T;B_1) \hookrightarrow\hookrightarrow L^p(0,T;B) \quad \forall\, 1 < p,q < \infty\,.$$

Consequently, we may prove that:

COROLLARY 3.5. *Given $\epsilon > 0$, $T > 0$, there exists a subsequence $\{(\psi_{n_k}^\epsilon, \mathbf{A}_{n_k}^\epsilon)\}$, which converges weakly (or weakly *) in*

$$\left[\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))\right] \times \left[\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))\right]$$

and strongly in

$$\mathcal{L}^p(0,T;\mathcal{L}^q(\Omega)) \times \mathbf{L}^p(0,T;\mathbf{L}^q(\Omega))$$

*as $n_k \to +\infty$. Here, $p \in (1,\infty)$ and $q \in (1,\infty)$ for $d = 2$ and $q \in (1,6)$ for $d = 3$. In particular, the subsequence converges weakly (or weakly *) in $\mathcal{V} \times \mathbf{V}$ and strongly in $\mathcal{S} \times \mathbf{S}$.*

Passing to the limit $n_k \to \infty$ and using Corollary 3.5, we get

THEOREM 3.6. *Given $\epsilon > 0$, $T > 0$, there exists a solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ in $\mathcal{V} \times \mathbf{V}$ to the system $(3.1_\epsilon) - (3.2_\epsilon)$ which is the weak (or weak *) limit of the subsequence $\{(\psi_{n_k}^\epsilon, \mathbf{A}_{n_k}^\epsilon)\}$, $n_k \to +\infty$. Moreover, any solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ of $(3.1_\epsilon) - (3.2_\epsilon)$ satisfies for $t \in [0,T]$,*

$$\mathcal{G}_\epsilon(\psi^\epsilon(t), \mathbf{A}^\epsilon(t)) + 2\int_0^t \left[(\frac{\partial \mathbf{A}^\epsilon}{\partial t}(\tau), \frac{\partial \mathbf{A}^\epsilon}{\partial t}(\tau)) + \eta(\frac{\partial \psi^\epsilon}{\partial t}(\tau), \frac{\partial \psi^\epsilon}{\partial t}(\tau))\right] d\tau = \mathcal{G}(\psi_0, \mathbf{A}_0)\,. \tag{3.9}$$

We omit the proof of the weak limit being the solution of $(3.1_\epsilon) - (3.2_\epsilon)$ since the technique is standard. Let us mention that the estimate (3.9) is obtained like in Lemma 3.2. We now prove the upper bound on the magnitude of the order parameter.

LEMMA 3.7. *If $|\psi_0(\mathbf{x})| \le 1$, a.e. in Ω, then $|\psi^\epsilon(\mathbf{x},t)| \le 1$, a.e. in $\Omega \times [0,T]$.*

Proof. Set test function $\tilde{\psi} = (|\psi^\epsilon| - 1)_+ f$ where $f = \psi/|\psi^\epsilon|$ and where $q_+ = q$ if $q \geq 0$ and $q_+ = 0$ if $q < 0$. Then, when $|\psi^\epsilon| > 1$,

$$\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}^\epsilon\tilde{\psi}^* = \frac{i}{\kappa}f^*\nabla|\psi^\epsilon| + (|\psi^\epsilon| - 1)(\frac{i}{\kappa}\nabla f^* - \mathbf{A}^\epsilon f^*),$$

and

$$-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon = -\frac{i}{\kappa}f\nabla|\psi^\epsilon| + |\psi^\epsilon|(-\frac{i}{\kappa}\nabla f - \mathbf{A}^\epsilon f),$$

so that

$$\Re\left\{(-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon)\cdot(\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}^\epsilon\tilde{\psi}^*)\right\} = (\nabla|\psi^\epsilon|)^2 + |\psi^\epsilon|(|\psi^\epsilon| - 1)\left|-\frac{i}{\kappa}\nabla f - \mathbf{A}^\epsilon f\right|^2.$$

Since

$$\Re\left\{\int_\Omega [\eta\frac{\partial\psi^\epsilon}{\partial t}\cdot\tilde{\psi}^* + (-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon)\cdot(\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}^\epsilon\tilde{\psi}^*) + (|\psi^\epsilon|^2 - 1)\psi^\epsilon\tilde{\psi}^*]\,d\Omega\right\} = 0,$$

we have that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \eta(|\psi^\epsilon| - 1)^+|\psi^\epsilon|\,d\Omega\\ &= -\int_{\Omega\cap\{|\psi^\epsilon|>1\}}\left[(\nabla|\psi|)^2 + |\psi|(|\psi| - 1)\left|-\frac{i}{\kappa}\nabla f - \mathbf{A}f\right|^2\right]d\Omega\\ &\quad -\int_{\Omega\cap\{|\psi^\epsilon|>1\}}\left[|\psi|(|\psi| + 1)(|\psi| - 1)^2\right]d\Omega \leq 0.\end{aligned}$$

By the assumption on the initial condition

$$\int_\Omega (|\psi_0| - 1)^+|\psi_0|\,d\Omega = 0.$$

So,

$$\int_\Omega (|\psi^\epsilon(t)| - 1)^+|\psi^\epsilon(t)|\,d\Omega = 0.$$

This gives the lemma. $\square$

Using the above bounds on the norms of the solution, one immediately get the uniqueness of the solution for $(3.1_\epsilon) - (3.2_\epsilon)$. Again, the technique is similar to that used in Temam[28] for the Navier-Stokes equations.

THEOREM 3.8. *Given $\epsilon > 0$, $T > 0$, the solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ in $\mathcal{V}\times\mathbf{V}$ to the system $(3.1_\epsilon) - (3.2_\epsilon)$ is unique. Moreover, there exists a constant $C_\epsilon > 0$, such that the solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ satisfies*

$$||\psi^\epsilon||_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} \leq C_\epsilon,$$

$$||\mathrm{div}\,\mathbf{A}^\epsilon||_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C_\epsilon,$$

and

$$||\mathrm{curl}\,\mathbf{A}^\epsilon||_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C_\epsilon.$$

Estimates in the last theorem follow from the H^2 regularity result for the Possion's equation with Neumann boundary condition and a partial regularity result for the system

$$-\epsilon\nabla\mathrm{div}\,\mathbf{A} + \mathrm{curl}\,(\mathrm{curl}\,\mathbf{A} - \mathbf{H}) = \mathbf{g} \in \mathbf{L}^2(\Omega)$$

with boundary conditions (2.22) and (2.23) which gives the H^1 bounds of $\operatorname{div}\mathbf{A}$ and $\operatorname{curl}\mathbf{A}$. The latter can be obtained, for example, using the results of[6] on the decomposition of vector field in $\mathbf{L}^2(\Omega)$.

With the uniqueness, we also see that

COROLLARY 3.9. *Given $\epsilon > 0$, $T > 0$, the sequence $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ converges weakly (or weakly *) in $\mathcal{V} \times \mathbf{V}$ (and therefore, converges almost everywhere in $\Omega \times [0,T]$) to the unique solution of the system $(3.1_\epsilon) - (3.2_\epsilon)$.*

Comparing the weak forms $(3.1_\epsilon) - (3.2_\epsilon)$ and (3.5)-(3.6) and using the weak convergence in $\mathcal{V} \times \mathbf{V}$ and the strong convergence in

$$\mathcal{L}^p(0,T;\mathcal{L}^q(\Omega)) \times \mathbf{L}^p(0,T;\mathbf{L}^q(\Omega))$$

where $1 < q < \infty$ for $d = 2$ and $1 < q < 6$ for $d = 3$, one may also get the convergence of the approximate solutions in the strong topology of $\mathcal{V} \times \mathbf{V}$. This is a quite useful when numerical approximations of (3.1)-(3.2) are considered.

PROPOSITION 3.10. *Given $\epsilon > 0$, $T > 0$, the sequence $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ converges strongly in*

$$\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$$

to the unique solution of the system $(3.1_\epsilon) - (3.2_\epsilon)$.

Proof:. We only need the uniform boundedness and the weak (or weak *) convergence in $\mathcal{V} \times \mathbf{V}$ and the strong convergence in $\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega)) \times \mathbf{L}^4(0,T;\mathbf{L}^4(\Omega))$. Take $\tilde{\psi} = \psi^\epsilon$ in equation (3.1_ϵ) and $\tilde{\psi}_n = \psi_n^\epsilon$ in (3.5) respectively and integrate both equations over the time interval $[0,T]$. The proof of the corollary rests on comparing each term in the resulting equations. Note that

$$\lim_{n\to\infty} \int_0^T (\frac{\partial \psi_n^\epsilon}{\partial t} - \frac{\partial \psi^\epsilon}{\partial t}, \psi^\epsilon)\, dt = 0$$

by weak convergence. By the uniform bound (in n) and the strong convergence,

$$\lim_{n\to\infty} |\int_0^T (\frac{\partial \psi_n^\epsilon}{\partial t}, \psi_n^\epsilon - \psi^\epsilon)\, dt| \le \lim_{n\to\infty} [\|\frac{\partial \psi_n^\epsilon}{\partial t}\|_{\mathcal{S}} \|\psi_n^\epsilon - \psi^\epsilon\|_{\mathcal{S}}] = 0$$

So, we get

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \left[\eta(\frac{\partial \psi_n^\epsilon}{\partial t}, \psi_n^\epsilon) - \eta(\frac{\partial \psi^\epsilon}{\partial t}, \psi^\epsilon)\right] dt \\ &= \lim_{n\to\infty} \int_0^T \eta\left[(\frac{\partial \psi_n^\epsilon}{\partial t}, \psi_n^\epsilon - \psi^\epsilon) + (\frac{\partial \psi_n^\epsilon}{\partial t} - \frac{\partial \psi^\epsilon}{\partial t}, \psi^\epsilon)\right] dt = 0\,. \end{aligned}$$

Also,

$$\begin{aligned} |\int_0^T \int_\Omega (|\psi_n^\epsilon|^4 - |\psi^\epsilon|^4)\, d\Omega\, dt| &\le 2\int_0^T \int_\Omega \left[(|\psi^\epsilon|^3 + |\psi_n^\epsilon|^3) | |\psi_n^\epsilon| - |\psi^\epsilon| |\right] d\Omega\, dt \\ &\le 2\left[\int_0^T \int_\Omega (|\psi^\epsilon|^3 + |\psi_n^\epsilon|^3)^2\, d\Omega\, dt\right]^{\frac{1}{2}} \cdot \left[\int_0^T \int_\Omega |\psi_n^\epsilon - \psi^\epsilon|^2 d\Omega\, dt\right]^{\frac{1}{2}} \end{aligned}$$

implies

$$\lim_{n\to\infty} \int_0^T (|\psi_n^\epsilon|^2 \psi_n^\epsilon, \psi_n^\epsilon) dt = \int_0^T (|\psi^\epsilon|^2 \psi^\epsilon, \psi^\epsilon) dt\,.$$

Similarly, by the uniform bound of $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ in $\mathcal{L}^6(0,T;\mathcal{L}^6(\Omega)) \times \mathbf{L}^6(0,T;\mathbf{L}^6(\Omega))$ and the strong convergence in $\mathcal{S} \times \mathbf{S}$, one may use triangle inequalities to verify

$$\lim_{n\to\infty} \int_0^T \int_\Omega (|\psi_n^\epsilon|^2 |\mathbf{A}_n^\epsilon|^2 - |\psi^\epsilon|^2 |\mathbf{A}^\epsilon|^2)\, d\Omega\, dt = 0.$$

Following from strong convergence in $\mathcal{S}$, we also have

$$\lim_{n\to\infty}\int_0^T (\psi_n^\epsilon, \psi_n^\epsilon)\, dt = \int_0^T (\psi^\epsilon, \psi^\epsilon)\, dt$$

Now, notice that

$$\begin{aligned}
&\Big|\int_0^T \big[(\frac{i}{\kappa}\nabla\psi_n^\epsilon, \psi_n^\epsilon \mathbf{A}_n^\epsilon) - (\frac{i}{\kappa}\nabla\psi^\epsilon, \psi^\epsilon\mathbf{A}^\epsilon)\big]\, dt\Big| \le \Big|\int_0^T (\frac{i}{\kappa}\nabla\psi_n^\epsilon, \psi_n^\epsilon[\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon])\, dt\Big| \\
&\quad + \Big|\int_0^T ((\frac{i}{\kappa}\nabla\psi_n^\epsilon, [\psi_n^\epsilon - \psi^\epsilon]\mathbf{A}^\epsilon)\, dt\Big| + \Big|\int_0^T ((\frac{i}{\kappa}[\nabla\psi_n^\epsilon - \nabla\psi^\epsilon], \psi^\epsilon\mathbf{A}^\epsilon)\, dt\Big| \\
&\quad \le \|\frac{i}{\kappa}\nabla\psi_n^\epsilon\|_{\mathcal{S}} \cdot \|\psi^\epsilon\|_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} \cdot \|\mathbf{A}_n^\epsilon - \mathbf{A}^\epsilon\|_{\mathbf{L}^4(0,T;\mathbf{L}^4(\Omega))} \\
&\quad + \|\frac{i}{\kappa}\nabla\psi_n^\epsilon\|_{\mathcal{S}} \cdot \|\psi_n^\epsilon - \psi^\epsilon\|_{\mathcal{L}^4(0,T;\mathcal{L}^4(\Omega))} \cdot \|\mathbf{A}^\epsilon\|_{\mathbf{L}^4(0,T;\mathbf{L}^4(\Omega))} \\
&\quad + \Big|\int_0^T ((\frac{i}{\kappa}[\nabla\psi_n^\epsilon - \nabla\psi^\epsilon], \psi^\epsilon\mathbf{A}^\epsilon) dt\Big|
\end{aligned}$$

we get

$$\lim_{n\to\infty}\int_0^T (\frac{i}{\kappa}\nabla\psi_n^\epsilon, \psi_n^\epsilon\mathbf{A}_n^\epsilon) dt = \int_0^T (\frac{i}{\kappa}\nabla\psi^\epsilon, \psi^\epsilon\mathbf{A}^\epsilon)] dt\ .$$

Therefore, comparing the only terms left in the corresponding weak forms, we have

$$\lim_{n\to\infty}\int_0^T (\frac{i}{\kappa}\nabla\psi_n^\epsilon, \frac{i}{\kappa}\nabla\psi_n^\epsilon) dt = \int_0^T (\frac{i}{\kappa}\nabla\psi^\epsilon, \frac{i}{\kappa}\nabla\psi^\epsilon) dt\ .$$

This gives the desired strong convergence of $\{\psi_n^\epsilon\}$ to ψ^ϵ in $\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))$. Similarly, one can show the strong convergence of $\{\mathbf{A}_n^\epsilon\}$ to $\mathbf{A}^\epsilon$ in $\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$ as $n \to +\infty$. □

3.3. Passing to the limit.

We now look at the process of passing to limit $\epsilon \to 0$ in order to obtain the existence and uniqueness of the strong solution for the original time dependent Ginzburg-Landau equations. At the moment, the bounds given in theorem 3.6 are not enough, since the bound on the div $\mathbf{A}^\epsilon$ term is not uniform in ϵ. To get a uniform bound, we consider

LEMMA 3.11. *Given $\epsilon > 0$, $T > 0$, the solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ in $\mathcal{V}\times\mathbf{V}$ to the system (3.1_ϵ)–(3.2_ϵ) satisfies for $t \in (0,T)$,*

$$\frac{1}{2}(\operatorname{div}\mathbf{A}^\epsilon(t), \operatorname{div}\mathbf{A}^\epsilon(t)) + \int_0^t \big[\epsilon(\nabla\operatorname{div}\mathbf{A}^\epsilon, \nabla\operatorname{div}\mathbf{A}^\epsilon) + \eta\Re\{(\frac{\partial\psi^\epsilon}{\partial t}, i\kappa\psi^\epsilon\operatorname{div}\mathbf{A}^\epsilon)\}\big]\, dt = 0\ . \tag{3.10}$$

Proof:. By the estimates in theorem 3.8 and the point wise bound of ψ^ϵ, we may take $\tilde{\psi} = i\kappa\psi^\epsilon\operatorname{div}\mathbf{A}^\epsilon$ in (3.1_ϵ) and integrate over $[0,t]$, note that

$$\begin{aligned}
&\Re\big[-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon\big] \cdot \big[\frac{i}{\kappa}\nabla(i\kappa(\psi^\epsilon)^*\operatorname{div}\mathbf{A}^\epsilon) - \mathbf{A}^\epsilon(-i\kappa(\psi^\epsilon)^*\operatorname{div}\mathbf{A}^\epsilon)\big] \\
&\quad = \Re\Big[\Big(-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon\Big) \cdot (\psi^\epsilon)^*\nabla\operatorname{div}\mathbf{A}^\epsilon\Big]\ ,
\end{aligned}$$

and

$$\Re\big[(|\psi^\epsilon|^2 - 1)\psi^\epsilon \cdot (-i\kappa(\psi^\epsilon)^*\operatorname{div}\mathbf{A})\big] = 0\ .$$

We get

$$\int_0^t \Re\big[\eta(\frac{\partial\psi^\epsilon}{\partial t}, i\kappa\psi^\epsilon \mathrm{div}\,\mathbf{A}^\epsilon) + (-\frac{i}{\kappa}\nabla\psi^\epsilon - \mathbf{A}^\epsilon\psi^\epsilon\ ,\ \psi^\epsilon\nabla \mathrm{div}\,\mathbf{A}^\epsilon)\big]\ dt = 0 \tag{3.11}$$

where the condition that at $t = 0$, $\mathbf{A}^\epsilon = \mathbf{A}_0 \in \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$ has been used. On the other hand, we may multiply the differential equation for $\mathbf{A}^\epsilon$ by $-\nabla\mathrm{div}\,\mathbf{A}^\epsilon$ and integrate over $\Omega\times[0,t]$. Since for $\mathrm{div}\,\mathbf{A}^\epsilon \in L^2(0,t;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))$,

$$\frac{d}{dt}(\mathrm{div}\,\mathbf{A}^\epsilon(t), \mathrm{div}\,\mathbf{A}^\epsilon(t)) = 2(\mathrm{div}\,\mathbf{A}^\epsilon(t), \frac{d}{dt}\mathrm{div}\,\mathbf{A}^\epsilon(t))$$

(see lemma 1.2 in Chap.III of Temam[28]), we get that

$$\begin{aligned}&\frac{1}{2}(\mathrm{div}\,\mathbf{A}^\epsilon(t), \mathrm{div}\,\mathbf{A}^\epsilon(t)) + \int_0^t \epsilon(\nabla\mathrm{div}\,\mathbf{A}^\epsilon\ ,\ \nabla\mathrm{div}\,\mathbf{A}^\epsilon)\ dt\\ &\quad + \int_0^t \Re\big[(\frac{i}{\kappa}\nabla\psi^\epsilon + \mathbf{A}^\epsilon\psi^\epsilon, \psi^\epsilon\nabla\mathrm{div}\,\mathbf{A}^\epsilon)\big]\ dt = 0\ .\end{aligned} \tag{3.12}$$

Equation (3.10) then follows when we add the equations (3.11), (3.12) together. □

COROLLARY 3.12. *There exists a constant $c>0$, independent of ϵ and $T>0$, such that*

$$\|\mathrm{div}\,\mathbf{A}^\epsilon(t)\|_0^2 + \epsilon\int_0^t \|\nabla\mathrm{div}\,\mathbf{A}^\epsilon(\tau)\|_0^2\ d\tau \le c\ , \quad \text{for } t\in(0,T)\ . \tag{3.13}$$

Hence, $(\psi^\epsilon, \mathbf{A}^\epsilon)$ is uniformly bounded in $\mathcal{V}\times\mathbf{V}$ (independent of ϵ).

Proof:. By lemma 3.7 and the equation (3.10) in lemma 3.11, we get

$$\begin{aligned}&\frac{1}{2}\|\mathrm{div}\,\mathbf{A}^\epsilon(t)\|_0^2 + \int_0^t \big[\epsilon\|\nabla\mathrm{div}\,\mathbf{A}^\epsilon\|_0^2\big]\ dt\\ &\qquad \le\ \eta\kappa\int_0^t \big[\|\frac{\partial\psi^\epsilon}{\partial t}\|_0\ \cdot\ \|\mathrm{div}\,\mathbf{A}^\epsilon\|_0\big]\ dt\ .\end{aligned} \tag{3.14}$$

Let $u(t) = \|\mathrm{div}\,\mathbf{A}^\epsilon(t)\|_0^2$, then

$$u(t) \le 2\eta\kappa\|\frac{\partial\psi^\epsilon}{\partial t}\|_{\mathcal{S}}\big[\int_0^t u(\tau)\ d\tau\big]^{1/2}\ .$$

Solving the differential inequality, we have

$$\int_0^t u(\tau)\ d\tau \le (\eta\kappa\|\frac{\partial\psi^\epsilon}{\partial t}\|_{\mathcal{S}})^2\ .$$

So,

$$u(t) \le\ 2(\eta\kappa\|\frac{\partial\psi^\epsilon}{\partial t}\|_{\mathcal{S}})^2\ ,\ \forall\, t\in[0,T]\ .$$

Using the uniform bound on $\|\frac{\partial\psi^\epsilon}{\partial t}\|_{\mathcal{S}}$ and (3.14), we see that there exists a constant $c>0$, independent of ϵ and T, such that (3.13) holds. Combining with the estimates in Theorem 3.6, we get the uniform boundedness of $(\psi^\epsilon, \mathbf{A}^\epsilon)$ in $\mathcal{V}\times\mathbf{V}$. □

Now, Lions' compactness lemma may be used again, only this time, we let $\epsilon\to 0$.

THEOREM 3.13. *Given $T > 0$, there exists a solution $(\psi, \mathbf{A})$ in $\mathcal{V} \times \mathbf{V}$ to the system* $(3.1)-(3.2)$ *which is the weak (or weak *) limit of a subsequence* $\{(\psi^{\epsilon_k}, \mathbf{A}^{\epsilon_k})\}$, $\epsilon_k \to 0$. *Moreover, any solution $(\psi, \mathbf{A})$ of* $(3.1)-(3.2)$ *satisfies*

$$\mathcal{G}(\psi(t), \mathbf{A}(t)) + 2\int_0^t \left[\|\frac{\partial \mathbf{A}}{\partial t}(\tau)\|_0^2 + \eta \|\frac{\partial \psi}{\partial t}(\tau)\|_0^2 \right] d\tau = \mathcal{G}(\psi_0, \mathbf{A}_0), \quad \forall\, t \in (0,T), \tag{3.15}$$

and there exists a constant $c > 0$, only depend on the initial condition, such that

$$\|\mathrm{div}\, \mathbf{A}(t)\|_0^2 \le c \ , \ \forall\, t \in (0,T) \ a.e..$$

The last estimate follows from the weak * convergence while equations (3.15) may be proved as in Lemma 3.2. Also, we have the following estimates on the solution. Again, we assume that $|\psi_0(\mathbf{x})| \le 1$, *a.e.* in Ω.

LEMMA 3.14. *Given $T > 0$, we have*

1). $|\psi(\mathbf{x}, t)| \le 1$, a.e. in $\Omega \times [0, T]$.

2). There exists a constant $C > 0$, only depending on the initial data and T, such that the solution $(\psi, \mathbf{A})$ of $(3.1)-(3.2)$ *satisfies*

$$\|\psi\|_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} \le c \ ,$$

$$\|\mathrm{curl}\, \mathbf{A}\|_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))} \le c \ ,$$

and

$$\|\mathrm{div}\, \frac{\partial \mathbf{A}}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \le c \ .$$

The bounds obtained for the solutions again implies the uniqueness as well as a result on the continuous dependence of the solution on the initial data.

THEOREM 3.15. *Given $T > 0$, let $(\phi_1, \mathbf{A}_1)$ and $(\phi_2, \mathbf{A}_2)$ be any pair of solutions of* $(3.1)-(3.2)$ *in $\mathcal{V} \times \mathbf{V}$ with initial data $(\phi_{01}, \mathbf{A}_{01})$ and $(\phi_{02}, \mathbf{A}_{02})$, respectively. There exists a constant, independent of t, such that*

$$\begin{aligned} &\|\phi_1(\cdot,t) - \phi_2(\cdot,t)\|_0 + \|\mathbf{A}_1(\cdot,t) - \mathbf{A}_2(\cdot,t)\|_0 \\ &\qquad \le c(\|\phi_{10} - \phi_{20}\|_0 + \|\mathbf{A}_{10} - \mathbf{A}_{20}\|_0) \ , \quad \forall\, t \in (0,T). \end{aligned}$$

Consequently, for a given initial condition, the solution of $(3.1)-(3.2)$ *in $\mathcal{V} \times \mathbf{V}$ is unique.*

Therefore, we have obtained the existence and uniqueness of the strong solution of the original time-dependent Ginzburg-Landau equations in the zero-electric potential gauge. The existence and uniqueness of solutions in other gauges may be obtained through gauge transformations. For example, if $(\phi, \mathbf{A})$ is a solution of $(3.1)-(3.2)$, we may use the transformation G_χ to obtain a solution of $(2.5)-(2.10)$ in the Coulomb gauge where χ satisfies

$$\Delta \chi = -\mathrm{div}\, \mathbf{A} \quad \text{in } \Omega \ ,$$

and

$$\nabla \chi \cdot \mathbf{n} = -\mathbf{A} \cdot \mathbf{n} \quad \text{on } \Gamma \ .$$

with the average of χ being zero. From the uniqueness of the strong solution, we also see that

COROLLARY 3.16. *Given $T > 0$, the sequence $(\psi^\epsilon, \mathbf{A}^\epsilon)$, which consists of solutions of the modified problems* $(3.1_\epsilon)-(3.1_\epsilon)$, *converges weakly (or weakly *) in $\mathcal{V} \times \mathbf{V}$ (and therefore, converges almost everywhere in $\Omega \times [0,T]$) to the unique solution of the system* $(3.1)-(3.2)$.

It is also possible to get other higher order estimates and estimates that are uniform in T, which may be used to study the asymptotic behavior of the solution. For example, we have

COROLLARY 3.17. *The ω-limit set of the solution of the problems $(3.1_\epsilon) - (3.1_\epsilon)$ for $\epsilon > 0$ is the solution set of the steady state solutions of $(2.25 - 2.26)$ in the Coulomb gauge.*

4. APPROXIMATIONS.

Galerkin like approximation has been considered for the modified problems earlier. Approximations are sought in $\mathbf{A}_n$ and $\mathcal{Z}_n$, the n-dimensional subspaces of $\mathbf{H}^1_n(\Omega)$ and $\mathcal{H}^1(\Omega)$ respectively, which are used in the last section. These spaces may be constructed easily, for example, using piecewise continuous finite elements. We will report on the construction and the implementation along with numerical studies of the time-dependent Ginzburg-Landau equations in Du[10]. Here, we merely state a consequence of the results obtained in the previous sections. In fact, using the convergence of $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ to $(\psi^\epsilon, \mathbf{A}^\epsilon)$ as $n \to \infty$ and the convergence of $(\psi^\epsilon, \mathbf{A}^\epsilon)$ to $(\psi, \mathbf{A})$, one can easily get the following:

THEOREM 4.1. *Given $T > 0$, if $\psi_0 \in \mathcal{H}^1(\Omega)$, $|\psi_0(x)| \le 1, a.e.$ and $\mathbf{A}_0 \in \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$, then for $\epsilon \ge 0$., $\{(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)\}$ converges weakly in*

$$\left[\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))\right] \times \left[\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))\right]$$

to the unique solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ of $(3.1_\epsilon) - (3.2_\epsilon)$ as $n \to \infty$. In addition, for $\epsilon > 0$, $(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)$ converges strongly in $\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$ to the solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ of $(3.1_\epsilon) - (3.2_\epsilon)$ as $n \to \infty$.

Proof:. The second part follows directly from Proposition 3.10. In addition, we get that the sequence $\{(\psi_n^\epsilon, \mathbf{A}_n^\epsilon)\}$ is uniformly bounded in

$$\left[\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))\right] \times \left[\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))\right]$$

while the bounds are independent of ϵ by theorem 3.13. Hence, the weak convergence follows. □

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