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Quantized vortices in BEC and superconductors

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Quantized vortices is a well-know signature of superfluidity. They have been studied extensively in superfluid Helium, type-II superconductors, Bose Einstein condensates and more recently in Fermi gas. We will discuss some mathematical analysis and numerical simulations of the quantized vortex nucleations in superconductors and Bose-Einstein condensates and discuss their connections and similarities.

1 Quantized Vortices

Quantized vortex, a signatures of superfluidity, is a fascinating subject that has been studied first in liquid Helium and in superconductors, then nearly a decade ago in Bose-Einstein condensates (as part of the 2001 Nobel prize winning works of Cornell, Wieman and Ketterle) and just last couple years in Fermi gas. They serve as good examples of manifestly quantum-mechanical behavior of matter being demonstrated on a macroscopic scale. In a superconductor, vortices may be thought as co-dimension 2 defects where the applied magnetic field penetrates the sample. In BEC, vortices are normal cores where the superfluid condensate component is depleted.

The interactions and dynamics of vortices are important to the understanding and application of superconductivity and superfluidity. Pinning of vortices is considered to be crucial in various technological applications of superconductors. Theoretical works in this exciting research area are highlighted by the 2003 Nobel Prize to Ginzburg, Abrikosov and Leggett who made decisive contributions to the understanding of the quantized vortex state.

2 Ginzburg-Landau Models

The macroscopic model of Ginzburg and Landau [17, 23] has been widely used to study both low-temperature and high-temperature superconductors. Due to its highly nonlinear nature, the complex energy landscape and the exotic dynamic responses of its solution to external conditions, numerical simulations have become valuable tools in order to better understand on the properties of the Ginzburg-Landau (G-L) models and thus providing further theoretical insight into the intriguing superconductivity phenomena.

The development of approximation methods of the Ginzburg-Landau model goes back to the 1950s shortly after the inception of the model [21]. Particularly notable works include the seminal paper by Abrikosov [1] on the vortex state in type-II superconductors based on the linearization of G-L equations near the upper critical field. The systematic studies of the G-L models from the numerical analysis point of view, to our knowledge, have not been seriously developed until [17]. In [17], both rigorous mathematical theory on the well-posedness of the equilibrium G-L models and their physical background were presented, along with the systematic development of finite element approximation methods. Subsequent works can be found in [2, 9, 12, 13]. By now, there is a large body of literature devoted to the mathematical and computational studies of the G-L models, we refer to references given in the recent survey [14] and the lecture notes [15]. In this talk, we present some recent works given in on the quantized vortex dynamics in a superconducting spherical shell [19, 20] which generalized some of the earlier works for thin films [7].

3 Gross-Pitaevskii-Schrodinger equations

Recent BEC experiments provided another avenue to study quantized vortices nucleated by stirring lasers or rotating magnetic traps. Remarkably, the phenomenological properties of quantized vortices have largely been captured by models such as the Gross-Pitaevskii (G-P) equations, which are nonlinear Schrodinger equations with spatially dependent potentials and cubic nonlinearity. The formation of vortices in the G-P equation in the rotating frame can be analyzed in the similar fashion as that for the G-L models, this is first pointed out in [3]. Asymptotic expansions are given there for the energy, the critical angular velocities for the nucleation of vortices and the location of vortices, with respect to a small parameter ϵ. The limit $\epsilon \to 0$ corresponds to the Thomas Fermi regime. The non-dimensionalized energy is similar to the Ginzburg-Landau energy for superconductors in the high-kappa high-field limit [8, 16] and the estimates in [3] rely on techniques developed for this

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latter problem. We also take the advantage of this similarity to develop a numerical algorithm for computing the Bose-Einstein vortices. Numerical results and energy diagrams are presented.

The story of vortices in BEC goes beyond rotating magnetic traps. In several BEC experiments done at MIT, some interesting energy dissipation phenomena were revealed with the stirring of the BEC by a laser. The analysis and simulations in [4] provided quantitatively interpretation of the experimental data and the mechanism of the onset of energy dissipation. It was found that there is always a drag around the laser beam whatever the velocity of the stirrer. At low velocity, there are no vortices and the drag has its origin in the wake behind the laser: this is a particularity of trapped systems since the density gets small in an extended region. The shedding of vortices only starts at a threshold velocity and there is a drastic increase in drag. This critical velocity is lower than the critical velocity computed for the corresponding 2D problem at the center of the cloud.

In more recent works [24, 25], more simulations were carried out to compare the quantized vortex dynamics governed by the Ginzburg-Landau-Schrödinger equation (GLSE) and the reduced dynamical laws. The reduced dynamic laws [22] can be solved explicitly for some special initial data. By directly simulating the GLSE with an efficient and accurate numerical method proposed in [6], some conclusive findings are obtained, and discussions on numerical and theoretical results are made to provide further understanding of vortex interactions in Ginzburg-Landau-Schrodinger equations.

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