

## OPTIMIZATION BASED NONOVERLAPPING DOMAIN DECOMPOSITION ALGORITHMS AND THEIR CONVERGENCE\*

QIANG DU†

**Abstract.** We present some optimization based domain decomposition algorithms with nonoverlapping subdomains. A number of existing algorithms, as well as new algorithms, may be derived using this approach. Both continuous problems and their finite element discretizations are considered. Convergence properties are examined.

**Key words.** domain decomposition, optimization, finite element, parallel computation

**AMS subject classifications.** 65N55, 65N30, 65Y10, 35J20, 65K10

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**1. Introduction.** Domain decomposition methods for the numerical solution of partial differential equations have been extensively studied [8, 9, 10, 18, 19, 24]. They have been applied to problems ranging from linear elliptic equations, parabolic equations [7, 13], to systems of nonlinear equations [21]. Domain decomposition or substructuring is also an effective approach for the construction of preconditioners [1, 3, 4, 5, 6, 11, 14, 30, 32]. A domain may be partitioned into overlapping subdomains or nonoverlapping subdomains. The selection of subdomains may be based on considerations of available computing resources and/or the geometry of the underlying physical problems. The latter is, in particular, applicable to complex systems which consist of possibly different governing equations in different physical subdomains. Overlapping domain decompositions become harder to implement in such a setting and the nonoverlapping domain decompositions may be more directly applicable. The algorithms we discuss in this paper are most relevant for the latter cases. We refer to these algorithms as optimization based nonoverlapping domain decomposition algorithms [16]. The main motivation is to devise feasible algorithms for solving large scale problems in multidisciplinary simulations, where the decomposition algorithms are merely ways to incorporate existing numerical softwares already developed for individual subdomain problems into a viable code for the coupled systems.

For a simple illustration, we consider the following model problem:

$$(1.1) \quad -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma,$$

where  $\Omega$  is a bounded, simply connected domain in  $\mathbf{R}^n$  with Lipschitz boundary  $\Gamma$ . Assume that  $\Omega$  is partitioned into two simply connected nonoverlapping *subdomains*  $\Omega_1$  and  $\Omega_2$  so that  $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ . The *interface* between the two subdomains is denoted by  $\Gamma_0$  so that  $\Gamma_0 = \bar{\Omega}_1 \cap \bar{\Omega}_2$ . Let  $\Gamma_1 = \bar{\Omega}_1 \cap \Gamma$  and  $\Gamma_2 = \bar{\Omega}_2 \cap \Gamma$ . (See Figure 1 for a two dimensional view.) Regularity conditions on the interface  $\Gamma_0$  will be assumed later.

Let a pair of functions  $u_1, u_2$  satisfy the given equations in the subdomains

$$(1.2) \quad -\Delta u_1 = f \quad \text{in } \Omega_1, \quad u_1 = 0 \quad \text{on } \Gamma_1,$$

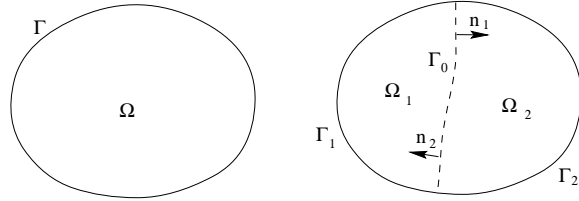
$$(1.3) \quad -\Delta u_2 = f \quad \text{in } \Omega_2, \quad u_2 = 0 \quad \text{on } \Gamma_2.$$

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†Department of Mathematics, Iowa State University, Ames, IA 50011 (qdu@iastate.edu).

FIG. 1.1. The domain  $\Omega$  and two nonoverlapping subdomains.

To piece  $u_1, u_2$  together as the solution of (1.1), we impose the *interface conditions*

$$(1.4) \quad u_1 = u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \quad \text{on } \Gamma_0.$$

Here,  $n_i$ 's are the unit outward normals of  $\Omega_i$ 's on  $\Gamma_0$  (hence,  $n_1 = -n_2$ ). Various energy (cost, objective) functionals (of  $u_1, u_2$  and other auxiliary variables) are defined so that their minimizers correspond to those subdomain solutions that satisfy (1.4). With different functionals and solution strategies, various domain decomposition algorithms are obtained: some have been studied in the literature but are now viewed from a different perspective while others are completely new. For example, in [16], new algorithms based on the gradient descent methods for solving the resulting optimization problems were considered. In this paper, we illustrate that the optimization based framework is intimately related to many existing studies of nonoverlapping domain decomposition [32]. In fact, with special choices of the functionals and the solution strategies for the optimization (such as the approach of alternating variables or alternating directions), various well-known nonoverlapping algorithms can be derived from this framework. The framework can also be extended naturally to nonlinear problems. Related works including recent numerical experiments have been done in [17, 23] for the nonlinear Navier–Stokes equations and for problems related to fluid-elastic structure interaction.

The paper is organized as follows. We briefly formulate an optimization problem in section 2. The domain decomposition algorithms for the continuous problem are studied in section 3. The discrete algorithms and their convergence properties are presented in section 4. Parallel algorithms are considered in section 5. Some final comments are given in section 6.

**2. Formulation of the optimization problem.** For completeness, let us introduce a few auxiliary variables and some technical results. Let  $H^m(\Omega)$ ,  $H_0^m(\Omega)$ , and  $H^s(\Gamma)$  denote the standard Sobolev spaces and the trace spaces. For  $i = 1, 2$ , also let  $H_{0,i}^1(\Omega_i) = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \Gamma_i\}$  and  $H_{00}^{1/2}(\Gamma_0) = \{v|_{\Gamma_0} : v \in H_0^1(\Omega)\}$ . With reasonable assumptions on the boundary, it is known by the interpolation theory that  $H_{00}^{1/2}(\Gamma_0) = [H_0^1(\Gamma_0), L^2(\Gamma_0)]_{1/2}$ .

Given  $(\lambda, g)$  in function spaces to be specified later, let us assume that the following equations are well-posed, i.e., under proper compatibility conditions: for  $i = 1, 2$ ,

$$(2.1) \quad \begin{cases} -\Delta u_i = f & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \Gamma_i \quad \text{and} \quad u_i = \lambda & \text{on } \Gamma_0, \\ \frac{\partial u_i}{\partial n_i} = (-1)^{i+1}g & \text{on } \Gamma_0. \end{cases}$$

For the solutions of (1.4), an energy (cost, objective) functional  $\mathcal{J}$  may be defined for variables  $(u_1, u_2, \lambda, g)$  subject to (1.2)–(1.3) [16]. In light of the interface conditions (1.4), it is natural to have  $\mathcal{J}$  satisfy the following assumption.

ASSUMPTION 1.  $u$  solves (1.1) if and only if  $\{u|_{\Omega_1}, u|_{\Omega_2}, u|_{\Gamma_0}, \frac{\partial u}{\partial n_1}|_{\Gamma_0}\}$  minimizes the functional  $\mathcal{J}(u_1, u_2, \lambda, g)$  subject to (1.2)–(1.3).

The functional  $\mathcal{J}$  and the function spaces may take various forms [16] such as

$$(2.2) \quad \mathcal{J}^\delta(u_1, u_2, \lambda, g) := \mathcal{J}_1^\delta(u_1, \lambda, g) + \mathcal{J}_2^\delta(u_2, \lambda, g),$$

where  $\delta$  is some positive constant and

$$(2.3) \quad \mathcal{J}_i^\delta(u_i, \lambda, g) = \frac{1}{2} \left\{ \delta \|u_i - \lambda\|_{X_i}^2 + \left\| \frac{\partial u_i}{\partial n_i} + (-1)^i g \right\|_{Y_i}^2 \right\}, \quad i = 1, 2,$$

with appropriate Hilbert spaces equipped, respectively, with norms  $\|\cdot\|_{X_i}$  and  $\|\cdot\|_{Y_i}$  ( $i = 1, 2$ ). Possible examples for the pair of function spaces are

- **(E1)**:  $X = H_{00}^{1/2}(\Gamma_0)$ ,  $Y = H^{-1/2}(\Gamma_0)$ ;
- **(E2)**:  $X = L^2(\Gamma_0)$ ,  $Y = H^{-1/2}(\Gamma_0)$ ;
- **(E3)**:  $X = H_{00}^{1/2}(\Gamma_0)$ ,  $Y = L^2(\Gamma_0)$ ;
- **(E4)**:  $X = Y = L^2(\Gamma_0)$ .

Due to the compatibility of norms and the regularities of traces, it might falsely appear that only E1 (which stands for the *Example 1*) is the most appropriate choice. Using a simple relaxation strategy for solving the optimization problem, we later show that E1 is related to the well-known D-N alternating methods given in [28], and the approach proposed in [27], as well as the Richardson relaxation of the operator equation (2.7). With the same strategy, E2 and E3 also lead to a well-known method analyzed in [26] that uses a Robin-type boundary condition for subdomain problems (see also [1, 15]). The last example, E4, however, leads to a new algorithm. For more discussion on the use of optimization techniques in domain decompositions, we refer to [16, 20, 22]. It is expected that more new algorithms can be produced with other combinations of norms in the functional.

More generally, the functional  $\mathcal{J}^\delta(u_1, u_2, \lambda, g)$  may use different weights for different norms, and its arguments need not be independently chosen [16]. The convergence of the resulting algorithm is evidently affected by the chosen weights and the choice of independent variables.

Before proceeding further, we let  $\{\hat{u}_i \in H_0^1(\Omega_i)\}$  denote the solution of the subdomain problem

$$(2.4) \quad -\Delta \hat{u}_i = f \quad \text{in } \Omega_i, \quad i = 1, 2.$$

For  $\theta \in H_{00}^{1/2}(\Gamma_0)$ , let the harmonic extensions  $\{w_i = \mathcal{R}_i(\theta)\}_{i=1}^2$  be defined by

$$(2.5) \quad \begin{cases} -\Delta w_i = 0 & \text{in } \Omega_i, \\ w_i = 0 & \text{on } \Gamma_i, \quad w_i = \theta & \text{on } \Gamma_0. \end{cases}$$

We also define the Steklov–Poincaré operators  $\mathcal{S}_i : H_{00}^{1/2}(\Gamma_0) \rightarrow H^{-1/2}(\Gamma_0)$  by

$$(2.6) \quad \mathcal{S}_i(\theta) = \frac{\partial w_i}{\partial n_i} \quad \text{on } \Gamma_0,$$

where  $w_i$ s are the solutions of (2.5) for  $i = 1, 2$ , respectively. The interface conditions (1.4) are equivalent to

$$(2.7) \quad \begin{cases} \beta_1 = \beta_2 = \beta & \text{on } \Gamma_0, \\ \mathcal{S}_1(\beta) + \mathcal{S}_2(\beta) = -\frac{\partial \hat{u}_1}{\partial n_1} - \frac{\partial \hat{u}_2}{\partial n_2} := -\xi & \text{on } \Gamma_0. \end{cases}$$

**3. Domain decomposition algorithms for the continuous problem.** We now study various iterative schemes for the solution of the optimization problem. Generally speaking, we may use any possible numerical algorithms for constrained global/local optimization problems, including gradient-type or conjugate gradient-type methods [17, 22, 29]. Here, we limit ourselves to consideration of an alternating domain approach and later a parallel implementation which reveals the connection of the current investigation with studies of many well-known nonoverlapping domain decomposition methods.

**3.1. Alternating domain iteration.** The idea of an alternating domain is that we obtain the new iterate by searching for the minimum with respect to variables defined in subdomains alternatively. More precisely, we have the following algorithm.

**Alternating domain iteration—sequential version.**

For  $n = 0, 1, 2, \dots$

**Step 1.** For  $n$  even, given  $u_1^n$  satisfying (1.2) with

$$(3.1) \quad \lambda^n = u_1^n \quad \text{and} \quad g^n = \frac{\partial u_1^n}{\partial n_1} \quad \text{on } \Gamma_0,$$

we update  $u_2^{n+1}$  by

$$(3.2) \quad \mathcal{J}_2^\delta(u_2^{n+1}, \lambda^n, g^n) = \min \mathcal{J}_2^\delta(u_2, \lambda^n, g^n)$$

over some function spaces for  $u_2$  subject to (1.3).

**Step 2.** Given  $u_2^{n+1}$  as found in the Step 1 with

$$(3.3) \quad \lambda^{n+1} = u_2^{n+1} \quad \text{and} \quad g^{n+1} = -\frac{\partial u_2^{n+1}}{\partial n_1} \quad \text{on } \Gamma_0,$$

we update  $u_1^{n+2}$  by

$$(3.4) \quad \mathcal{J}_1^\delta(u_1^{n+2}, \lambda^{n+1}, g^{n+1}) = \min \mathcal{J}_1^\delta(u_1, \lambda^{n+1}, g^{n+1})$$

over some function spaces for  $u_1$  subject to (1.2).

The above procedure is referred as the sequential version, simply because the update of the new iterate on one subdomain relies on the new update obtained on the other subdomain, reminiscent of the alternating variable method for solving minimization problems or the Gauss–Seidel approach for the solution of linear systems. A parallel version is given in a later section.

If existing numerical softwares are available for the solution of subdomain problems, they can then be used to find the subdomain solutions in Steps 1 and 2.

**3.2. Orthogonality property.** At a typical iteration step, the optimality of the iteration provides some useful orthogonality properties. In turn, these orthogonality properties imply orthogonality of the errors with proper subspaces.

For example, for any  $v \in H_{0,2}^1(\Omega_2)$  with

$$(3.5) \quad -\Delta v = 0 \quad \text{in } \Omega_2,$$

the optimality condition at Step 1 of a typical iteration gives

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \left( \delta \|u_2^{n+1} + \epsilon v - \lambda^n\|_{X_2}^2 + \left\| \frac{\partial u_2^{n+1}}{\partial n_2} + \epsilon \frac{\partial v}{\partial n_2} + g^n \right\|_{Y_2}^2 \right) = 0$$

for any  $v$  satisfying (3.5). Therefore,

$$(3.6) \quad \delta(u_2^{n+1} - \lambda^n, v)_{X_2} + \left( \frac{\partial u_2^{n+1}}{\partial n_2} + g^n, \frac{\partial v}{\partial n_2} \right)_{Y_2} = 0,$$

where  $\lambda^n$  and  $g^n$  are determined from earlier iterations.

Equation (3.6) is regarded as an orthogonality property of the iterative solutions. A similar property holds for the second step of a typical iteration as well.

**3.3. Convergence property.** First, from the orthogonality property (3.6), we easily see that as  $n \rightarrow \infty$ , if the sequence of solutions is convergent, then the interface conditions are satisfied for the limits, so we have the following theorem.

**THEOREM 3.1.** *Given  $\delta > 0$  and the initial iteration, if the sequence  $\{u_i^n\}$  defined by the alternating domain algorithm is convergent and  $u = \lim_{n \rightarrow \infty} u_i^{2n+i}$ , respectively, in  $\Omega_i$  for  $i = 1, 2$ , then  $u$  is the solution of (2.4) on the whole domain  $\Omega$ .*

If we denote the error by  $e^n = u^n - u$ , where  $u$  is the solution of (2.4) on the whole domain, then the orthogonality condition gives

$$(3.7) \quad \delta(e_2^{n+1} - e_1^n, v)_{X_2} + (\mathcal{S}_2 e_2^{n+1} + \mathcal{S}_1 e_1^n, \mathcal{S}_2 v)_{Y_2} = 0.$$

To discuss conditions on the convergence, let us define some operators  $P_i$  by

$$(3.8) \quad (P_i u, \mathcal{S}_i v)_{Y_i} = (u, v)_{X_i} \quad \forall v \in X_i, \mathcal{S}_i v \in Y_i.$$

For the four examples we had earlier, we have  $P_i = \mathcal{S}_i, I, I$ , and  $\mathcal{S}_i^{-1}$ , respectively. Obviously,  $P_i$  is self-adjoint and positive definite. Then, we obtain

$$(3.9) \quad e_2^{n+1} = (\delta P_2 + \mathcal{S}_2)^{-1}(\delta P_2 - \mathcal{S}_1)e_1^n, \quad e_1^n = (\delta P_1 + \mathcal{S}_1)^{-1}(\delta P_1 - \mathcal{S}_2)e_2^{n-1}.$$

Let  $P = (\delta P_2 + \mathcal{S}_2)^{-1}(\delta P_2 - \mathcal{S}_1)(\delta P_1 + \mathcal{S}_1)^{-1}(\delta P_1 - \mathcal{S}_2)$ ; we have the following theorem.

**THEOREM 3.2.** *The alternating domain algorithm is convergent if and only if*

$$(3.10) \quad \|P^n e^0\| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

in some appropriately chosen norm for any  $e^0$ .  $\square$

For a recent general discussion about estimating the product of nonexpansive maps, we refer to [31]. In the current context, we next check the condition (3.10) for each example considered earlier. We omit the superscripts for the iteration indices occasionally in cases where no confusion is caused.

**3.4. Example one.** This corresponds to  $X = H_{00}^{1/2}(\Gamma_0)$ ,  $Y = H^{-1/2}(\Gamma_0)$ . To define the relevant norms in these spaces, let us first define  $\phi \in H^1(\Omega_2)$  by

$$(3.11) \quad \begin{cases} -\Delta \phi = 0 & \text{in } \Omega_2, \\ \phi = 0 & \text{on } \Gamma_2, \quad \frac{\partial \phi}{\partial n_2} = \frac{\partial u_2}{\partial n_2} + g & \text{on } \Gamma_0; \end{cases}$$

then, on  $Y = H^{-1/2}(\Gamma_0)$ , we use the norm

$$\left\| \frac{\partial u_2}{\partial n_2} + g \right\|_{Y_2}^2 = \int_{\Omega_2} |\nabla \phi|^2 d\Omega.$$

In terms of the Steklov–Poincaré operator, this is equivalent to

$$\|\theta\|_{Y_2}^2 = \int_{\Gamma_0} \theta \mathcal{S}_2^{-1}(\theta) d\Gamma \quad \forall \theta \in H^{-1/2}(\Gamma_0).$$

Next, we define  $\psi \in H^1(\Omega_2)$  by

$$(3.12) \quad \begin{cases} -\Delta\psi = 0 & \text{in } \Omega_2, \\ \psi = 0 & \text{on } \Gamma_2, \quad \psi = u_2 - \lambda & \text{on } \Gamma_0. \end{cases}$$

Then, on  $X = H_{00}^{1/2}(\Gamma_0)$ , we use the norm

$$\|u_2 - \lambda\|_{X_2}^2 = \int_{\Omega_2} |\nabla\psi|^2 d\Omega.$$

In terms of the Steklov–Poincaré operator, this is equivalent to

$$\|\beta\|_{Y_2}^2 = \int_{\Gamma_0} \beta \mathcal{S}_2(\beta) d\Gamma \quad \forall \beta \in H_{00}^{1/2}(\Gamma_0).$$

Comparing with (3.8), we see that  $P_2 = \mathcal{S}_2$ ; then, at the first step of a typical iteration (with  $\lambda = \lambda^n, g = g^n$ ), we update the solution in  $\Omega_2$  by

$$\delta \mathcal{S}_2(u_2 - \lambda) + \frac{\partial u_2}{\partial n_2} + g = 0.$$

Thus,

$$(3.13) \quad \begin{cases} -\Delta u_2 = f & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \Gamma_2, \quad \frac{\partial u_2}{\partial n_2} = -g - \delta \frac{\partial \psi}{\partial n_2} & \text{on } \Gamma_0. \end{cases}$$

Using the solutions  $\hat{u}_i$  ( $i = 1, 2$ ) for the inhomogeneous equations (2.4) and the variable  $\xi$  given in (2.7), we have on the interface  $\Gamma_0$

$$\begin{aligned} \frac{\partial}{\partial n_2}(u_2 - \hat{u}_2) &= -\frac{\partial}{\partial n_1}(\lambda - \hat{u}_1) - \xi - \delta \frac{\partial \psi}{\partial n_2}, \\ \psi &= u_2 - \hat{u}_2 - (\lambda - \hat{u}_1). \end{aligned}$$

Putting back the iteration indices, defining  $\beta^{n+1} = u_2^{n+1} - \hat{u}_2$  and  $\beta^n = u_1^n - \hat{u}_1$ , the above equations may then be expressed in terms of the Steklov–Poincaré operators

$$(3.14) \quad (1 + \delta)\mathcal{S}_2(\beta^{n+1} - \beta^n) = -\xi - \mathcal{S}(\beta^n) \quad \text{on } \Gamma_0.$$

This is equivalent to a symmetric (with respect to  $\Omega_1$  and  $\Omega_2$ ) version of the so-called Richardson relaxation of the operator equation  $\mathcal{S}(\beta) = -\xi$ :

$$(3.15) \quad \mathcal{S}_i(\beta^{n+1} - \beta^n) = -\alpha(\xi + \mathcal{S}(\beta^n)) \quad \alpha \in (0, 1),$$

where  $i = 1, 2$ , depending on  $n$  being odd or even. The Richardson iteration is also equivalent to the so-called Dirichlet–Neumann alternating method. The iteration given in [27] is also along the same line. In those contexts,  $\alpha$  is viewed as a suitable relaxation parameter which may change from iteration to iteration.

We now consider the convergence using the criterion given earlier. Notice that the operator defined in (3.8) is given by  $P_i = \mathcal{S}_i$  in this case. Since the operator  $\mathcal{S}_1$  is self-adjoint and positive definite, we may define the operator  $\mathcal{S}_1^{1/2}$  in an appropriate subspace  $D = D(\mathcal{S}_1^{1/2})$  of  $L^2(\Gamma_0)$ . Then,

$$P = \mathcal{S}_1^{-1/2} \left[ \frac{\delta^2 + 1}{(\delta + 1)^2} - \frac{\delta}{(\delta + 1)^2} (T + T^{-1}) \right] \mathcal{S}_1^{1/2} := \mathcal{S}_1^{-1/2} \hat{P} \mathcal{S}_1^{1/2}$$

for some positive definite self-adjoint operator  $T = \mathcal{S}_1^{1/2} \mathcal{S}_2^{-1} \mathcal{S}_1^{1/2}$ . By the properties of the Steklov–Poincaré operators, both  $T$  and  $T^{-1}$  are bounded, self-adjoint, positive definite operators from  $D$  to  $D$ . Thus, if we have

$$(3.16) \quad \delta + 1/\delta > \rho/2 - 1$$

with  $\rho$  being the spectral radius of  $T + T^{-1}$ , then the self-adjoint operator  $\hat{P}$  has spectral radius strictly less than one. Since  $P^n = \mathcal{S}_1^{-1/2} \hat{P}^n \mathcal{S}_1^{1/2}$ , we thus get the geometric convergence of the algorithm corresponding to E1.

**COROLLARY 3.3.** *For any positive constant  $\delta$  satisfying (3.16), the alternating domain algorithm (3.1–3.4) is geometrically convergent for any choice of norms in the functional that leads to  $P_i = \mathcal{S}_i$  ( $i = 1, 2$ ).*  $\square$

In terms of the parameter  $\alpha$ , our convergence result given in Corollary 3.3 is valid for  $\alpha$  in  $(0, \theta_1) \cup (1 - \theta_1, 1)$  for some constant  $0 < \theta_1 < 1$ . For the constant parameter case, the existing convergence analysis in the literature for the nonsymmetric implementation of (3.15), that is,  $i = 1$  or  $2$  for all  $n$  in (3.15), requires  $\alpha$  to be in  $(0, \theta)$  for some  $\theta < 1$ .

Let us comment here that if the choices of the norms in the energy functional are slightly changed to yield  $P_1 = P_2 = \mathcal{Q}$ ,  $\mathcal{Q}$  is equivalent to  $\mathcal{S}$  in the sense that there are generic positive constants  $m > 0$ ,  $M > 0$  such that

$$(3.17) \quad m(g, \mathcal{S}g)_{0, \Gamma_0} \leq (g, \mathcal{Q}g)_{0, \Gamma_0} \leq M(g, \mathcal{S}g)_{0, \Gamma_0} \quad \forall g \in H_{00}^{1/2}(\Gamma_0).$$

Here,  $(\cdot, \cdot)_{0, \Gamma_0}$  denotes the standard  $L^2$  inner product on  $\Gamma_0$  and we also use  $\|\cdot\|_{0, \Gamma_0}$  to denote the standard  $L^2$  norm. Then, the resulting algorithm is essentially equivalent to the algorithm studied in [28]. Moreover, by the properties of the operators  $\mathcal{S}_i$  ( $i = 1, 2$ ), there are generic positive constants  $m_i > 0$ ,  $M_i > 0$  satisfying

$$m_i(g, \mathcal{Q}g)_{0, \Gamma_0} \leq (g, \mathcal{S}_i g)_{0, \Gamma_0} \leq M_i(g, \mathcal{Q}g)_{0, \Gamma_0} \quad \forall g \in H_{00}^{1/2}(\Gamma_0)$$

for  $i = 1, 2$ . Therefore, we get for any positive  $\delta$

$$-1 < \frac{\delta - M_i}{\delta + M_i} \leq \frac{(g, (\delta \mathcal{Q} - \mathcal{S}_i)g)_{0, \Gamma_0}}{(g, (\delta \mathcal{Q} + \mathcal{S}_i)g)_{0, \Gamma_0}} \leq \frac{\delta - m_i}{\delta + m_i} < 1 \quad \forall g \in H_{00}^{1/2}(\Gamma_0).$$

That is, the operators  $(\delta \mathcal{Q} - \mathcal{S}_i)(\delta \mathcal{Q} + \mathcal{S}_i)^{-1}$  ( $i = 1, 2$ ) are contractions. Since

$$P^n = (\delta \mathcal{Q} + \mathcal{S}_2)^{-1} [(\delta \mathcal{Q} - \mathcal{S}_1)(\delta \mathcal{Q} + \mathcal{S}_1)^{-1} (\delta \mathcal{Q} - \mathcal{S}_2)(\delta \mathcal{Q} + \mathcal{S}_2)^{-1}]^n (\delta \mathcal{Q} + \mathcal{S}_2),$$

we hence have the following convergence result.

**COROLLARY 3.4.** *For any positive constant  $\delta$ , the alternating domain algorithm (3.1–3.4) is geometrically convergent for any choice of norms in the functional that leads to  $P_1 = P_2 = \mathcal{Q}$ , where  $\mathcal{Q}$  satisfies (3.17) for some positive constants  $m, M$ .*  $\square$

This conclusion and similar analysis have been given in [28].

**3.5. Example two.** In this case,  $X = L^2(\Gamma_0)$ ,  $Y = H^{-1/2}(\Gamma_0)$ . Again, we define  $\phi$  as in (3.11); the functional we consider here takes the form

$$(3.18) \quad \mathcal{J}_2^\delta(u_2) = \frac{1}{2} \left( \delta \int_{\Gamma_0} |u_2 - \lambda|^2 d\Gamma + \int_{\Omega_2} |\nabla \phi|^2 d\Omega \right).$$

In this case,  $P_i = I$ , we get from the orthogonality that

$$\delta(u_2 - \lambda) + \frac{\partial u_2}{\partial n_2} + g = 0.$$

Putting back the iteration indices, we get the alternating Schwarz algorithm with nonoverlapping subdomains, i.e.,

$$(3.19) \quad \begin{cases} -\Delta u_2^{n+1} = f & \text{in } \Omega_2, \\ u_2^{n+1} = 0 & \text{on } \Gamma_2, \\ \frac{\partial u_2^{n+1}}{\partial n_2} + \delta u_2^{n+1} = -\frac{\partial u_1^n}{\partial n_1} + \delta u_1^n & \text{on } \Gamma_0. \end{cases}$$

The convergence of the above iteration has been well-documented by Lions [26], mostly based on energy-type estimates and compactness arguments. Related discussions can also be found in [1] and [15].

To use our convergence criterion, we notice that

$$P^n = (\delta I + \mathcal{S}_2)^{-1} [(\delta I - \mathcal{S}_1)(\delta I + \mathcal{S}_1)^{-1}(\delta I - \mathcal{S}_2)(\delta I + \mathcal{S}_2)^{-1}]^n (\delta I + \mathcal{S}_2).$$

Let  $f_n$  be a sequence defined by  $f_0 = (\delta I + \mathcal{S}_2)e^0$ , and

$$(3.20) \quad (I + \delta \mathcal{S}_2^{-1})f_{2n+1} = (I - \delta \mathcal{S}_2^{-1})f_{2n},$$

$$(3.21) \quad (I + \delta \mathcal{S}_1^{-1})f_{2n+2} = (I - \delta \mathcal{S}_1^{-1})f_{2n+1}.$$

Then,  $P^{n+1}e^0 = (\delta I + \mathcal{S}_2)^{-1}f_{2n+2}$ . Moreover, by making proper regularity assumptions ( $f_k \in L^2(\Gamma_0)$  for all  $k$ ) and taking the  $L^2$  inner product of the above equations with  $f_{2n+1} + f_{2n}$  and  $f_{2n+2} + f_{2n+1}$ , respectively, and summing over  $n$ , we get

$$\begin{aligned} & \|f_{2N+2}\|_{0,\Gamma_0}^2 + \delta \sum_{n=0}^N [(S_1^{-1}(f_{2n+2} + f_{2n+1}), f_{2n+2} + f_{2n+1})_{0,\Gamma_0} \\ & + (S_2^{-1}(f_{2n+1} + f_{2n}), f_{2n+1} + f_{2n})_{0,\Gamma_0}] = \|f_0\|_{0,\Gamma_0}^2 \end{aligned}$$

for any  $N$ . Since both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  and their inverses are positive definite, we thus conclude that as  $n \rightarrow \infty$ ,

$$\|f_{2n+2} + f_{2n+1}\|_{-1/2,\Gamma_0} \rightarrow 0 \quad \text{and} \quad \|f_{2n+1} + f_{2n}\|_{-1/2,\Gamma_0} \rightarrow 0.$$

By definition (3.20), we get

$$\|f_n - f_{n-1}\|_{1/2,\Gamma_0} \rightarrow 0,$$

as  $n \rightarrow \infty$ , which implies that as  $n \rightarrow \infty$ ,

$$\|f_n\|_{-1/2,\Gamma_0} \rightarrow 0.$$

In turn, we get

$$\|e_n\|_{1/2,\Gamma_0} \rightarrow 0,$$

that is, we have shown the following convergence result.

**COROLLARY 3.5.** *For any  $\delta > 0$ , the alternating domain algorithm (3.1–3.4) is convergent for any choice of norms in the functional that leads to  $P_1 = P_2 = I$ .  $\square$*

The result obtained here and the analysis are similar to that in [26].

**3.6. Example three.** We now consider the case  $X = H_{00}^{1/2}(\Gamma_0)$ ,  $Y = L^2(\Gamma_0)$ . Higher regularity on the solutions is needed to ensure the traces are well-defined. We again define  $\psi \in H^1(\Omega_2)$  as in (3.12) and use the norm

$$\|u_2 - \lambda\|_{X_2}^2 = \int_{\Omega_2} |\nabla\psi|^2 d\Omega.$$

Then, the functional takes on the form

$$(3.22) \quad \mathcal{J}_2^\delta(u_2) = \frac{1}{2} \left( \delta \int_{\Omega_2} |\nabla\psi|^2 d\Omega + \int_{\Gamma_0} \left| \frac{\partial u_2}{\partial n_2} + g \right|^2 d\Gamma \right).$$

With  $v$  given by (3.5), the orthogonality property (3.6) takes on the form

$$\begin{aligned} 0 &= \delta \int_{\Omega_2} \nabla\psi \cdot \nabla v d\Omega + \int_{\Gamma_0} \left( \frac{\partial u_2}{\partial n_2} + g \right) \frac{\partial v}{\partial n_2} d\Gamma \\ &= \delta \int_{\Gamma_0} \psi \frac{\partial v}{\partial n_2} d\Gamma + \int_{\Gamma_0} \left( \frac{\partial u_2}{\partial n_2} + g \right) \frac{\partial v}{\partial n_2} d\Gamma. \end{aligned}$$

Therefore

$$\delta(u_2 - \lambda) + \frac{\partial u_2}{\partial n_2} + g = 0.$$

This is again equivalent to the alternating Schwarz algorithm given in the earlier discussion. Its convergence analysis is the same as that in Corollary 3.5.

**3.7. Example four.** As the last example, we consider the choice  $X = Y = L^2(\Gamma_0)$ . Solutions are assumed to be regular enough to have the traces properly defined in the corresponding spaces. We let the norms be the same as the standard  $L^2$  norm on  $X = Y$ . In this case, at a typical step, we need to minimize

$$(3.23) \quad \mathcal{J}_2^\delta(u_2) = \frac{1}{2} \left( \delta \int_{\Gamma_0} |u_2 - \lambda|^2 d\Gamma + \int_{\Gamma_0} \left| \frac{\partial u_2}{\partial n_2} + g \right|^2 d\Gamma \right).$$

Then, for any  $v \in H_{0,2}^1(\Omega_2)$  satisfying (3.5), the orthogonality property (3.6) implies

$$\delta \int_{\Gamma_0} (u_2 - \lambda)v d\Gamma + \int_{\Gamma_0} \left( \frac{\partial u_2}{\partial n_2} + g \right) \frac{\partial v}{\partial n_2} d\Gamma = 0.$$

Define  $\theta \in H^1(\Omega_2)$  by

$$\begin{aligned} -\Delta\theta &= 0 && \text{in } \Omega_2, \\ \theta &= 0 && \text{on } \Gamma_2, \\ \frac{\partial\theta}{\partial n_2} &= u_2 - \lambda && \text{on } \Gamma_0. \end{aligned}$$

By Green's theorem, we get

$$\delta \int_{\Gamma_0} (u_2 - \lambda)v d\Gamma = \delta \int_{\Gamma_0} v \frac{\partial\theta}{\partial n_2} d\Gamma = \delta \int_{\Gamma_0} \theta \frac{\partial v}{\partial n_2} d\Gamma.$$

Therefore, we get that

$$\delta\theta + \frac{\partial u_2}{\partial n_2} + g = 0 \quad \text{on } \Gamma_0.$$

Thus, we have the following optimality systems:

$$(3.24) \quad \begin{cases} -\Delta u_2 = f & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \Gamma_2, \\ \frac{\partial u_2}{\partial n_2} = -g - \delta\theta & \text{on } \Gamma_0, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\theta = 0 & \text{in } \Omega_2, \\ \theta = 0 & \text{on } \Gamma_2, \\ \frac{\partial\theta}{\partial n_2} = u_2 - \lambda & \text{on } \Gamma_0. \end{cases}$$

Using (2.4) and the Steklov–Poincaré operator, we get

$$(\delta\mathcal{S}_2^{-1} + \mathcal{S}_2)(\beta^{n+1} - \beta^n) = -\xi - \mathcal{S}(\beta^n) \quad \text{on } \Gamma_0$$

for  $\beta^{n+1} = u_2^{n+1} - \hat{u}_2$ ,  $\beta^n = u_1^n - \hat{u}_1$ , with  $\hat{u}_1, \hat{u}_2$  given by (2.4) and  $\xi$  given by (2.7). Such an iteration may be viewed as a nonstandard relaxation scheme for the equation  $\mathcal{S}(\beta) = -\xi$ . There seems to be no existing works related to this iteration in the literature. Though a comparable energy functional is used in [22], the algorithms given there are of gradient type (steepest descent or conjugate gradient).

Using previous notation, to check our convergence criterion, we notice that

$$(3.25) \quad (\delta\mathcal{S}_2^{-1} + \mathcal{S}_2)e_{2n+1} = (\delta\mathcal{S}_2^{-1} - \mathcal{S}_1)e_{2n},$$

$$(3.26) \quad (\delta\mathcal{S}_1^{-1} + \mathcal{S}_1)e_{2n+2} = (\delta\mathcal{S}_1^{-1} - \mathcal{S}_2)e_{2n+1}.$$

Again, we make the regularity assumption that  $\mathcal{S}_i e_{2n+1+i} \in L^2(\Gamma_0)$  for all  $n$  and  $i = 1, 2$ . Let the operators  $\mathcal{B}_i$  ( $i = 1, 2$ ) be defined by

$$\mathcal{B}_i = (I + \delta^{-1}\mathcal{S}_i^2)^{-1/2}.$$

One may easily verify that  $\mathcal{B}_i$  are self-adjoint linear operators from  $H^{-1/2}(\Gamma_0)$  to  $H_{00}^{1/2}(\Gamma_0)$ . We may similarly define  $\mathcal{B}_i^{-1}$  for  $i = 1, 2$ . Moreover, we have the following lemma.

LEMMA 3.6. *There is some constant  $c > 0$  such that*

$$\|\mathcal{B}_i g\|_{1/2, \Gamma_0} \leq c \|g\|_{-1/2, \Gamma_0}$$

for any  $g \in H^{-1/2}(\Gamma_0)$ .

*Proof.* Let  $f = \mathcal{B}_i g$ . By norm equivalence, we have

$$\begin{aligned} \|g\|_{-1/2, \Gamma_0}^2 &\geq c(g, \mathcal{S}_i^{-1}g)_{0, \Gamma_0} = c(f, \mathcal{S}_i^{-1}\mathcal{B}_i^{-2}f)_{0, \Gamma_0} = c\left(f, \left(\mathcal{S}_i^{-1} + \frac{1}{\delta}\mathcal{S}_i\right)f\right)_{0, \Gamma_0} \\ &\geq c\|f\|_{-1/2, \Gamma_0}^2 + \frac{c}{\delta}\|f\|_{1/2, \Gamma_0}^2 \geq \frac{c}{\delta}\|f\|_{1/2, \Gamma_0}^2 \end{aligned}$$

for some generic constant  $c > 0$ . Since  $\|f\|_{1/2, \Gamma_0} = \|\mathcal{B}_i g\|_{1/2, \Gamma_0}$ , we thus get the lemma.  $\square$

Now, with the operators  $\mathcal{B}_1, \mathcal{B}_2$ , we get from (3.25) and (3.26) that

$$\begin{aligned}
 \mathcal{S}_1^{-1} \mathcal{B}_1^{-2} e_{2n+2} &= \mathcal{S}_1^{-1} \left( I - \frac{1}{\delta} \mathcal{S}_1 \mathcal{S}_2 \right) e_{2n+1} \\
 &= \mathcal{S}_1^{-1} \left( I - \frac{1}{\delta} \mathcal{S}_1 \mathcal{S}_2 \right) \mathcal{B}_2^2 \left( I - \frac{1}{\delta} \mathcal{S}_2 \mathcal{S}_1 \right) e_{2n} \\
 &= \mathcal{S}_1^{-1} \left( \mathcal{B}_2^2 - \frac{1}{\delta} (\mathcal{S}_1 \mathcal{S}_2 \mathcal{B}_2^2 + \mathcal{S}_2 \mathcal{B}_2^2 \mathcal{S}_1) + \frac{1}{\delta^2} \mathcal{S}_1 \mathcal{S}_2 \mathcal{B}_2^2 \mathcal{S}_2 \mathcal{S}_1 \right) e_{2n} \\
 &= \mathcal{S}_1^{-1} \left( \mathcal{B}_2^2 + \frac{1}{\delta} \mathcal{S}_2 \mathcal{B}_2^2 \mathcal{S}_2 + \frac{1}{\delta^2} \mathcal{S}_1 \mathcal{S}_2 \mathcal{B}_2^2 \mathcal{S}_2 \mathcal{S}_1 + \frac{1}{\delta} \mathcal{S}_1 \mathcal{B}_2^2 \mathcal{S}_1 - \frac{1}{\delta} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} \right) e_{2n} \\
 &= \mathcal{S}_1^{-1} \left( I + \frac{1}{\delta} \mathcal{S}_1^2 - \frac{1}{\delta} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} \right) e_{2n} = \mathcal{S}_1^{-1} \left( \mathcal{B}_1^{-2} - \frac{1}{\delta} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} \right) e_{2n}.
 \end{aligned}$$

That is, equivalently,

$$(3.27) \quad \mathcal{S}_1^{-1} \left( 2\mathcal{B}_1^{-2} - \frac{1}{\delta} \mathcal{B}_2^2 \mathcal{S} \right) (e_{2n+2} - e_{2n}) + \frac{1}{\delta} \mathcal{S}_1^{-1} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} (e_{2n+2} + e_{2n}) = 0.$$

Taking the  $L^2(\Gamma_0)$  inner product of (3.27) with  $\mathcal{S}_1(e_{2n+2} + e_{2n})$ , we have

$$\begin{aligned}
 &\left( \left( 2\mathcal{B}_1^{-2} - \frac{1}{\delta} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} \right) e_{2n+2}, e_{2n+2} \right)_{0, \Gamma_0} + \frac{1}{\delta} \|\mathcal{B}_2 \mathcal{S} (e_{2n+2} + e_{2n})\|_{0, \Gamma_0}^2 \\
 &= \left( \left( 2\mathcal{B}_1^{-2} - \frac{1}{\delta} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} \right) e_{2n}, e_{2n} \right)_{0, \Gamma_0}.
 \end{aligned}$$

Summing over  $n$ , we get

$$\begin{aligned}
 &((2\mathcal{B}_1^{-2} - \delta^{-1} \mathcal{S} \mathcal{B}_2^2 \mathcal{S}) e_{2N+2}, e_{2N+2})_{0, \Gamma_0} + \delta^{-1} \sum_{n=0}^N \|\mathcal{B}_2 \mathcal{S} (e_{2n+2} + e_{2n})\|_{0, \Gamma_0}^2 \\
 (3.28) \quad &= ((2\mathcal{B}_1^{-2} - \delta^{-1} \mathcal{S} \mathcal{B}_2^2 \mathcal{S}) e_0, e_0)_{0, \Gamma_0}
 \end{aligned}$$

for any  $N > 0$ . Now, from the earlier derivation,

$$\mathcal{B}_1^{-2} - \delta^{-1} \mathcal{S} \mathcal{B}_2^2 \mathcal{S} = (I - \delta^{-1} \mathcal{S}_1 \mathcal{S}_2) \mathcal{B}_2^2 (I - \delta^{-1} \mathcal{S}_2 \mathcal{S}_1)$$

is positive semidefinite; it follows from the regularity assumptions that

$$\sum_{n=0}^N \|\mathcal{B}_2 \mathcal{S} (e_{2n+2} + e_{2n})\|_{0, \Gamma_0}^2$$

is uniformly bounded as  $N \rightarrow \infty$ . Therefore, we get  $\mathcal{B}_2 \mathcal{S} (e_{2n+2} + e_{2n}) \rightarrow 0$  in  $L^2(\Gamma_0)$ .

Back to (3.27), this in turn implies

$$\mathcal{B}_1^{-1} (e_{2n+2} - e_{2n}) \rightarrow 0, \quad \text{and} \quad \mathcal{B}_2 \mathcal{S} (e_{2n+2} - e_{2n}) \rightarrow 0 \quad \text{in} \quad L^2(\Gamma_0).$$

Consequently,

$$(3.29) \quad \mathcal{B}_2 \mathcal{S} e_{2n} \rightarrow 0 \quad \text{in} \quad L^2(\Gamma_0).$$

On the other hand, (3.28) also gives a uniform bound of

$$\|\mathcal{B}_1^{-1} e_{2n}\|_{0, \Gamma_0}^2 = \|e_{2n}\|_{0, \Gamma_0}^2 + \delta^{-1} \|\mathcal{S}_1 e_{2n}\|_{0, \Gamma_0}^2.$$

Therefore, from properties of  $\mathcal{S}_1$  and compactness results on the Sobolev spaces, we may get a subsequence  $\{n_k\}$  such that

$$e_{2n_k} \rightarrow v, \quad \text{in } H_{00}^{1/2}(\Gamma_0), \quad \text{as } n_k \rightarrow \infty.$$

Then using properties of  $\mathcal{S}$  and the property of  $\mathcal{B}_2$  given in Lemma 3.6, we have

$$\mathcal{B}_2 \mathcal{S} e_{2n_k} \rightarrow \mathcal{B}_2 \mathcal{S} v, \quad \text{in } H_{00}^{1/2}(\Gamma_0), \quad \text{as } n_k \rightarrow \infty.$$

Therefore, by (3.29), we have  $\mathcal{B}_2 \mathcal{S} v = 0$  which implies that  $v = 0$ . Since the limit of the subsequence is independent of the choice of the subsequence, we thus get

$$e_{2n} \rightarrow 0, \quad \text{in } H_{00}^{1/2}(\Gamma_0), \quad \text{as } n \rightarrow \infty.$$

Similar discussion gives  $e_{2n+1} \rightarrow 0$  in  $H_{00}^{1/2}(\Gamma_0)$ . We thus have the following convergence result.

**COROLLARY 3.7.** *For any  $\delta > 0$ , the alternating domain algorithm (3.1–3.4) is convergent for any choice of norms in the functional that leads to  $\{P_i = \mathcal{S}_i^{-1}\}_1^2$ .  $\square$*

The iteration is new and it corresponds to choice  $P_i = \mathcal{S}_i^{-1}$  ( $i = 1, 2$ ).

**4. Domain decomposition algorithms for the discrete problem.** Let  $V^h$  be a finite element subspace of  $H_0^1(\Omega)$  with respect to a regular triangulation  $\tau$  of the domain  $\Omega$  [12], and  $\Gamma_0$  consists of edges of the triangles in  $\tau$ . Let  $V_i^h = V^h \cap H_0^1(\Omega_i)$  ( $i = 1, 2$ ) and  $V^h = V_1^h + V_2^h + V_0^h$ . Let  $\Lambda^h = \{v^h|_{\Gamma_0} \mid v^h \in V_0^h\}$  be a discretization of  $H_{00}^{1/2}(\Gamma_0)$  with the property that the restriction map from  $V_0^h$  to  $\Lambda^h$  defined by  $v^h \rightarrow v^h|_{\Gamma_0}$  is one to one.

We now define the following discrete harmonic function spaces: for  $i = 1, 2$ ,

$$Z_i^h = \left\{ w^h|_{\Omega_i} \mid w^h \in V_i^h + V_0^h, \int_{\Omega_i} \nabla w^h \nabla v^h d\Omega = 0 \quad \forall v^h \in V_i^h \right\}.$$

For any  $\lambda^h \in \Lambda^h$ , since the restriction map is one to one, the discrete harmonic extension  $\mathcal{R}_i^h(\lambda^h)$  is an element in  $Z_i^h$  that satisfies  $\mathcal{R}_i^h(\lambda^h)|_{\Gamma_0} = \lambda^h$ . The discrete Steklov–Poincaré operators  $\mathcal{S}_i^h : \Lambda^h \rightarrow \Lambda^h$  (or  $V_0^h \rightarrow V_0^h$  via the restriction map) are defined by

$$\int_{\Gamma_0} \mathcal{S}_i^h(w^h)v^h d\Gamma = \int_{\Omega_i} \nabla \mathcal{R}_i^h(w^h) \nabla v^h d\Omega \quad \forall v^h \in V_0^h$$

or, equivalently,

$$\int_{\Gamma_0} \mathcal{S}_i^h(w^h)v^h d\Gamma = \int_{\Omega_i} \nabla \mathcal{R}_i^h(w^h) \nabla v^h d\Omega \quad \forall v^h \in V^h.$$

It is easy to check that both  $\mathcal{S}_1^h$  and  $\mathcal{S}_2^h$  are well-defined symmetric positive definite operators. We let  $\mathcal{S}^h = \mathcal{S}_1^h + \mathcal{S}_2^h$ , which is also symmetric and positive definite. Then, we may define various inner products on  $\Lambda^h$ : for  $k = 0, 1$  and  $-1$ ,

$$(4.1) \quad (\lambda^h, \theta^h)_{X_{i,h}} = \int_{\Gamma_0} \lambda^h (\mathcal{S}_i^h)^k (\theta^h) d\Gamma \quad \forall \lambda^h, \theta^h \in \Lambda^h,$$

and the induced norm is denoted by  $\|\cdot\|_{X_{i,h}}$ . Notice that for  $k = 0$ , we have the standard  $L^2$  inner product and norm.

Next, the discretization of the problems in the subdomains leads to equations for the discrete solutions  $u_i^h \in V_i^h + V_0^h$  ( $i = 1, 2$ ) which satisfy

$$\int_{\Omega_i} \nabla u_i^h \nabla v^h d\Omega = \int_{\Omega_i} f v^h d\Omega \quad \forall v^h \in V_i^h.$$

For such solutions, we define the operators  $\{\frac{\partial^h}{\partial n_i}, i = 1, 2\}$  by

$$\int_{\Gamma_0} \frac{\partial^h}{\partial n_i} (u_i^h) v^h d\Gamma = \int_{\Omega_i} \nabla u_i^h \nabla v^h d\Omega - \int_{\Omega_i} f v^h d\Omega \quad \forall v^h \in V_0^h.$$

**4.1. Discrete algorithms and orthogonality property.** With the above definitions, the discrete analogue of the iterations (3.1–3.4) we have studied before can be defined. At a typical step, such as Step 1 in the algorithm in section 3.1, they may come as the solution of the following discrete minimization problem.

**Discrete sequential alternating domain algorithms.**

For  $n = 0, 1, 2, \dots$

**Step 1.** Find  $u_{2,n+1}^h \in V_2^h + V_0^h$  as the minimizer of

$$\mathbf{Min} \quad \delta \|u_2^h - u_{1,n}^h\|_{X_{2,h}}^2 + \left\| \frac{\partial^h}{\partial n_2} u_2^h + \frac{\partial^h}{\partial n_1} u_{1,n}^h \right\|_{Y_{2,h}}^2$$

subject to  $u_2^h \in V_2^h + V_0^h$  with

$$(4.2) \quad \int_{\Omega_2} \nabla u_2^h \nabla w^h d\Omega = \int_{\Omega_2} f w^h d\Omega \quad \forall w^h \in V_2^h.$$

**Step 2.** Find  $u_{1,n+2}^h \in V_1^h + V_0^h$  as the minimizer of

$$\mathbf{Min} \quad \delta \|u_1^h - u_{2,n+1}^h\|_{X_{1,h}}^2 + \left\| \frac{\partial^h}{\partial n_1} u_1^h + \frac{\partial^h}{\partial n_2} u_{2,n+1}^h \right\|_{Y_{1,h}}^2$$

subject to  $u_1^h \in V_1^h + V_0^h$  with

$$(4.3) \quad \int_{\Omega_1} \nabla u_1^h \nabla w^h d\Omega = \int_{\Omega_1} f w^h d\Omega \quad \forall w^h \in V_1^h.$$

Here, the discrete domain decomposition algorithms again depend on the choices of norms  $\|\cdot\|_{X_{i,h}}$  and  $\|\cdot\|_{Y_{i,h}}$ , both of which may be given by (4.1). One may also get the following corresponding discrete orthogonality property: for any  $v^h \in Z_2^h$ ,

$$(4.4) \quad \delta (u_{2,n+1}^h - u_{1,n}^h, v^h)_{X_{2,h}} + \left( \frac{\partial^h}{\partial n_2} u_{2,n+1}^h + \frac{\partial^h}{\partial n_1} u_{1,n}^h, \frac{\partial^h v^h}{\partial n_2} \right)_{Y_{2,h}} = 0.$$

Of course, the discrete inner products and norms one can choose are not limited to those defined by (4.1). They can be chosen from other discrete mesh dependent inner products, and mesh dependent weights may also be used; see the recent work [2] for an application of discrete mesh dependent norms in solving the first order elliptic systems by the optimization approach.

**4.2. Examples of the discrete domain decomposition algorithms.** The energy functional for the continuous problem may contain the norms in fractional Sobolev spaces. However, by the orthogonality property (4.4), we may implement the discrete domain decomposition algorithms without computing any fractional derivatives. Naturally, there is more freedom in constructing the discrete norms as the mesh dependent norms and weights may also be utilized. Instead of making a general discussion here, we examine a few specific cases.

As an example, the alternating Schwarz iteration given by E2 may be implemented by letting  $\|\cdot\|_{X_{2,h}}$  be the standard  $L^2$  norm and  $\|\cdot\|_{Y_{2,h}}$  be defined by

$$\|g^h\|_{Y_{2,h}}^2 = (g^h, (\mathcal{S}_2^h)^{-1}g^h)_{0,\Gamma_0} \quad \forall g^h \in \Lambda^h,$$

where  $(\cdot, \cdot)$  is the standard  $L^2$  inner product on  $\Gamma_0$ .

It follows that the discrete analogue of E2 is given by the following algorithm.

**Discrete sequential decomposition algorithms—Example 1.**

For  $n = 0, 1, 2, \dots$

**Step 1.** Given  $u_{1,2n}^h \in V_1^h + V_0^h$  satisfying

$$(4.5) \quad \int_{\Omega_1} \nabla u_{1,2n}^h \nabla w^h d\Omega = \int_{\Omega_1} f w^h d\Omega \quad \forall w^h \in V_1^h,$$

find  $u_{2,2n+1}^h \in V_2^h + V_0^h$  such that for any  $v^h \in V_2^h + V_0^h$ ,

$$(4.6) \quad \int_{\Omega_2} \nabla u_{2,2n+1}^h \nabla v^h d\Omega + \delta \int_{\Gamma_0} u_{2,2n+1}^h v^h d\Gamma = \int_{\Omega} f v^h d\Omega \\ - \int_{\Omega_1} \nabla u_{1,2n}^h \nabla v^h d\Omega + \delta \int_{\Gamma_0} u_{1,2n}^h v^h d\Gamma.$$

**Step 2.** For the  $u_{1,2n+1}^h$  found in Step 1, find  $u_{1,2n+2}^h \in V_1^h + V_0^h$  such that for any  $v^h \in V_1^h + V_0^h$ ,

$$(4.7) \quad \int_{\Omega_1} \nabla u_{2,2n+2}^h \nabla v^h d\Omega + \delta \int_{\Gamma_0} u_{2,2n+2}^h v^h d\Gamma = \int_{\Omega} f v^h d\Omega \\ - \int_{\Omega_2} \nabla u_{1,2n+1}^h \nabla v^h d\Omega + \delta \int_{\Gamma_0} u_{1,2n+1}^h v^h d\Gamma.$$

In the above implementation, we note that the computation of terms like

$$\int_{\Omega} f v^h d\Omega - \int_{\Omega_1} \nabla u_{1,2n}^h \nabla v^h d\Omega$$

is required for elements  $v^h$  that are in  $V_0^h$ . Similar to the idea in [27], this serves to communicate the Neumann data on the interface even though the normal derivatives (or flux variables) are not explicitly formed. A more efficient implementation of a similar parallel algorithm is presented in a later section.

If the above iteration is convergent, and we denote

$$(4.8) \quad u_*^h |_{\Omega_i} = \lim_{n \rightarrow \infty} u_{i,2n+1-i}^h$$

for  $i = 1, 2$ , then by the continuity on the interface, we have  $u_*^h \in V^h$ . Moreover,

$$(4.9) \quad \int_{\Omega_2} \nabla u_*^h \nabla w^h d\Omega + \int_{\Omega_1} \nabla u_*^h \nabla w^h d\Omega = \int_{\Omega} f w^h d\Omega \quad \forall w^h \in V^h.$$

Thus, we see that the limit is the standard Galerkin finite element approximation of the equation in the whole domain.

Similarly, the discretization of the algorithm in section 3.4 (symmetric version of the alternating D-N iteration or the Richardson relaxation) can be derived as follows.

**Discrete sequential decomposition algorithms—Example 2.**

For  $n = 0, 1, 2, \dots$

**Step 0.** For  $i = 1$ , given

$$(4.10) \quad \int_{\Omega_2} \nabla u_{2,2n-1}^h \nabla w^h d\Omega = \int_{\Omega_2} f w^h d\Omega \quad \forall w^h \in V_2^h,$$

find  $u_{i,2n-1+i}^h, \theta^h \in V_i^h + V_0^h$  with  $\theta^h|_{\Gamma_0} = (u_{i,2n-1+i}^h - u_{2n-2+i,n}^h)|_{\Gamma_0}$  such that

$$(4.11) \quad \begin{aligned} & \int_{\Omega_i} \nabla u_{i,2n-1+i}^h \nabla v^h d\Omega + \delta \int_{\Omega_i} \nabla \theta^h \nabla v^h d\Gamma \\ & = \int_{\Omega} f v^h d\Omega - \int_{\Omega_{3-i}} \nabla u_{3-i,2n-2+i}^h \nabla v^h d\Omega \quad \forall v^h \in V_i^h + V_0^h, \end{aligned}$$

$$(4.12) \quad \int_{\Omega_i} \nabla \theta^h \nabla w^h d\Gamma = 0 \quad \forall w^h \in V_i^h.$$

**Step 1.** Given  $u_{1,2n}^h$  obtained in Step 0, repeat the procedure in Step 0 with  $i = 2$ .

If in Step 0 we define  $\beta^h = u_{1,2n}^h + \delta \theta^h$ , then the above iteration is equivalent to the following: find  $\beta^h \in V_1^h + V_0^h$  such that for  $u_{2,2n-1}^h$  satisfying (4.10), we have

$$(4.13) \quad \int_{\Omega_1} \nabla \beta^h \nabla v^h d\Gamma = \int_{\Omega} f v^h d\Omega - \int_{\Omega_2} \nabla u_{2,2n-1}^h \nabla v^h d\Omega \quad \forall v^h \in V_1^h + V_0^h,$$

$$(4.14) \quad u_{1,2n}^h|_{\Gamma_0} = \alpha u_{2,2n-1}^h|_{\Gamma_0} + (1 - \alpha) \beta^h|_{\Gamma_0},$$

$$(4.15) \quad \int_{\Omega_1} \nabla u_{2,2n}^h \nabla w^h d\Omega = \int_{\Omega_1} f w^h d\Omega \quad \forall w^h \in V_1^h,$$

where  $\alpha = \delta/(\delta + 1) \in (0, 1)$ . This is equivalent to the symmetric implementation (through an alternating choice of subdomains) of an iteration studied in [27].

Again, the limits of the iteration given in (4.13)–(4.15) is the standard Galerkin finite element approximation in the whole domain; that is, we have the following theorem.

**THEOREM 4.1.** *Given  $\delta > 0$  and the initial iteration, if the sequences  $\{u_{i,2n-1+i}^h\}$  ( $i = 1, 2$ ) defined by discrete domain decomposition algorithm examples 1 or 2 are convergent and  $u_*^h$  is defined by (4.8), then  $u_*^h$  is the standard Galerkin finite element approximation in  $V^h$  of (2.4) in the whole domain  $\Omega$ .*

The convergence of the standard Galerkin finite element approximation as  $h \rightarrow 0$  has been well-documented in the literature; see for instance [12]. Observing that the discrete operators and the discrete algorithms depend almost exclusively on the properties of the finite element spaces in the respective subdomains, we may thus generalize them to the case where different meshes and finite element spaces are used in the different subdomains (see [22]) as long as the restriction map can be well-defined. The only complication is that, without assuming  $V^h \subset H_0^1(\Omega)$ , the limit of the iteration may not be the standard Galerkin finite element approximation as claimed in the above theorem.

Notice also that in some of the discrete algorithms, the subdomain solution requires the computation of additional variables (say,  $\beta^h$  for E1). For the discrete implementation of E4, the extra variables are also coupled with the subdomain solutions. On one hand, the extra work may be more easily offset for situations where more than two subdomains are used in the decomposition. On the other hand, there are also variants of the decomposition algorithms based on gradient-type minimization strategies which avoid the solution of coupled systems [22].

**4.3. Convergence of the discrete algorithms.** Similar to the continuous analogue, we let  $e_n^h = u_{1,n}^h - u_*^h$  in  $\Omega_1$  and  $e_{n+1}^h = u_{2,n+1}^h - u_*^h$  in  $\Omega_2$  for even integer  $n$ . Then, from the discrete orthogonality property, we can get

$$(4.16) \quad \delta (e_{n+1}^h - e_n^h, v)_{X_{2,h}} + \left( \frac{\partial^h}{\partial n_2} e_{n+1}^h + \frac{\partial^h}{\partial n_1} e_n^h, \frac{\partial^h v^h}{\partial n_2} \right)_{Y_{2,h}} = 0$$

for any  $v^h \in Z_2^h$ .

We may define some discrete operators  $P_i^h$ , as in the continuous case, by

$$(4.17) \quad (P_i^h u^h, \mathcal{S}_i^h v^h)_{Y_{i,h}} = (u^h, v^h)_{X_{i,h}} \quad \forall v^h \in X_{i,h}, \mathcal{S}_i^h v^h \in Y_{i,h}.$$

Then, as in the continuous case, the orthogonality condition implies

$$(4.18) \quad e_{n+1}^h = (\delta P_2^h + \mathcal{S}_2^h)^{-1} (\delta P_2^h - \mathcal{S}_1^h) (\delta P_1^h + \mathcal{S}_1^h)^{-1} (\delta P_1^h - \mathcal{S}_2^h) e_{n-1}^h := P_h e_{n-1}^h.$$

Thus, we get the following theorem.

**THEOREM 4.2.** *The iteration defined by (4.5, 4.6, 4.7) is convergent if and only if*

$$(4.19) \quad \|(P_h)^n e_0^h\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad \square$$

By the equivalence of norms in the finite dimensional setting, the norm used in the above theorem may take any norm.

Again, the limit  $u_*^h$  of the iteration given in the theorem is the standard Galerkin finite element approximation of the solution of (1.1). One may verify, for example, under proper assumptions on the mesh, that the condition (4.19) is satisfied for any positive  $\delta$  for E2, E3, and E4, while some limitation on  $\delta$  may be required for E1. In the cases of E2 and E3, the discrete analogue of (3.17) may be verified with the help of inverse inequalities that are valid for any quasi-uniform mesh (so the constants are mesh dependent). Then, similar to the results in [1, 27], one may show that the convergence is geometric. For E1, the geometric rate can be chosen to be independent of the mesh parameter  $h$ .

For the discrete framework presented in this section, the convergence of the discrete algorithms clearly relies on the properties of the operators  $P_i^h$  defined in (4.17) and its relation with the parameter  $\delta$  and the discrete mesh parameter  $h$ . It is thus of great interest to explore further the use of mesh dependent discrete norms and mesh dependent weights in the discrete formulation in order to get the best possible convergence rate in the actual implementation.

**5. Parallel algorithms.** One of the distinct features of the modern high performance computer is its parallel architecture. We now consider how the sequential version may be modified to allow parallel implementation.

**5.1. The algorithms.** Again, we may consider functionals such as those given by (2.2), or more generally,

$$\begin{aligned}
 &\mathcal{J}^{\alpha,\beta}(u_1, u_2, \lambda, g) := \mathcal{J}^{\alpha_1,\beta_1}(u_1, \lambda, g) + \mathcal{J}^{\alpha_2,\beta_2}(u_2, \lambda, g) \\
 (5.1) \quad &:= \frac{\alpha_1}{2} \|u_1 - \lambda\|_{X_1}^2 + \frac{\beta_1}{2} \left\| \frac{\partial u_1}{\partial n_1} - g \right\|_{Y_1}^2 + \frac{\alpha_2}{2} \|u_2 - \lambda\|_{X_2}^2 + \frac{\beta_2}{2} \left\| \frac{\partial u_2}{\partial n_2} + g \right\|_{Y_2}^2.
 \end{aligned}$$

Keeping in mind that we wish to update the subdomain solutions simultaneously at each iteration, the solution algorithms for minimizing the above functional may be constructed accordingly. For instance, a possible algorithm is given as follows.

**Parallel nonoverlapping domain decomposition algorithms.**

For  $n = 0, 1, 2, \dots$

**Step 0.** Given  $u_i^n$  ( $i = 1, 2$ ), let

$$(5.2) \quad \lambda^{n+1} = \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} u_1^n + \frac{\alpha_2}{\alpha_1 + \alpha_2} u_2^n \right) |_{\Gamma_0},$$

$$(5.3) \quad g^{n+1} = \frac{\beta_1}{\beta_1 + \beta_2} \frac{\partial}{\partial n_1} u_1^n - \frac{\beta_2}{\beta_1 + \beta_2} \frac{\partial}{\partial n_2} u_2^n.$$

**Step 1.** We update  $u_1^{n+1}$  by

$$(5.4) \quad \mathcal{J}^{\alpha_1,\beta_1}(u_1^{n+1}, \lambda^{n+1}, g^{n+1}) = \min \mathcal{J}^{\alpha_1,\beta_1}(u_1, \lambda^{n+1}, g^{n+1})$$

over suitable function spaces for  $u_1$  subject to (1.2).

**Step 2.** We update  $u_2^{n+1}$  by

$$(5.5) \quad \mathcal{J}^{\alpha_2,\beta_2}(u_2^{n+1}, \lambda^{n+1}, g^{n+1}) = \min \mathcal{J}^{\alpha_2,\beta_2}(u_2, \lambda^{n+1}, g^{n+1})$$

over suitable function spaces for  $u_2$  subject to (1.3).

Obviously, if  $\|\cdot\|_{X_1} = \|\cdot\|_{X_2}$ ,  $\|\cdot\|_{Y_1} = \|\cdot\|_{Y_2}$ , then Step 0 can also be viewed as solving a minimization problem: given  $u_i^n$  ( $i = 1, 2$ ), find  $(\lambda^{n+1}, g^{n+1})$  such that

$$\mathcal{J}^{\alpha,\beta}(u_1^n, u_2^n, \lambda^{n+1}, g^{n+1}) = \min \mathcal{J}^{\alpha,\beta}(u_1^n, u_2^n, \lambda, g)$$

over proper function spaces for  $\lambda$  and  $g$ .

For Steps 1 and 2, simultaneous computations are allowed. Since Step 0 is carried out only on the boundary, the computation is minimal compared with the update of  $u_i^n$  in the subdomains  $\Omega_i$ ,  $i = 1, 2$ .

**5.2. Orthogonality property.** Similar to the earlier discussion, we have the orthogonality property for Steps 1 and 2.

$$(5.6) \quad \alpha_i (u_i^{n+1} - \lambda^{n+1}, v)_{X_i} + \beta_i \left( \frac{\partial u_i^{n+1}}{\partial n_i} + (-1)^i g^{n+1}, \frac{\partial v}{\partial n_i} \right)_{Y_i} = 0$$

for any  $v \in H^1_{0,i}(\Omega_i)$  satisfying (3.5). Let  $e_i^n = u_i^n - u$ , where  $u$  is the solution of (1.1); using the operators  $P_i$  ( $i = 1, 2$ ) defined in (3.8), we get

$$\begin{aligned}
 (\alpha_i P_i + \beta_i \mathcal{S}_i) e_i^{n+1} &= \left( \frac{\alpha_i \alpha_1}{\alpha_1 + \alpha_2} P_i + (-1)^{i+1} \frac{\beta_i \beta_1}{\beta_1 + \beta_2} \mathcal{S}_1 \right) e_1^n \\
 (5.7) \quad &+ \left( \frac{\alpha_i \alpha_2}{\alpha_1 + \alpha_2} P_i + (-1)^i \frac{\beta_i \beta_2}{\beta_1 + \beta_2} \mathcal{S}_2 \right) e_2^n
 \end{aligned}$$

for  $i = 1, 2$ . We may again find convergence criterion from the above equation. One also easily sees that if the iterations are convergent and

$$u_i = \lim_{n \rightarrow \infty} u_i^n \quad \text{in } \Omega_i$$

for  $i = 1, 2$ , then  $u_1, u_2$  are the restrictions in the respective subdomains of the solution  $u$  of (2.4) in the whole domain.

**5.3. An example.** We use the example  $X = L^2(\Gamma_0)$ ,  $Y = H^{-1/2}(\Gamma_0)$  as an illustration. Again, we use norms as in E2. Then the orthogonality gives for  $i = 1, 2$ ,

$$\alpha_i(u_i^{n+1} - \lambda^{n+1}) + \beta_i \left( \frac{\partial u_i^{n+1}}{\partial n_i} + (-1)^i g^{n+1} \right) = 0.$$

Taking a combination of these equations, we get

$$\alpha_1 u_1^{n+1} - \alpha_2 u_2^{n+1} - (\alpha_1 - \alpha_2) \lambda^{n+1} + \beta_1 \frac{\partial u_1^{n+1}}{\partial n_1} - \beta_2 \frac{\partial u_2^{n+1}}{\partial n_2} - (\beta_1 + \beta_2) g^{n+1} = 0.$$

Now, we may use conditions on  $\lambda^{n+1}, g^{n+1}$  to get

$$(5.8) \quad g^{n+2} = g^{n+1} + \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} \lambda^{n+1} - \frac{\alpha_1}{\beta_1 + \beta_2} u_1^{n+1} + \frac{\alpha_2}{\beta_1 + \beta_2} u_2^{n+1}.$$

The significance of the above equation lies in the fact that, to form  $g^{n+2}$  in Step 0, there is no need to calculate the normal derivatives of the updated solutions  $u_i^{n+1}$  for  $i = 1, 2$ ; hence, there is less requirement on the regularities of the solutions in the subdomains. Similar adaptation of Lions's nonoverlapping algorithm has been studied in [15], where the evaluation of normal derivatives on the interface may also be eliminated.

For the sake of simplicity, we consider only the special case  $\alpha_1 = \alpha_2 = \alpha$  and  $\beta_1 = \beta_2 = \delta\alpha$  here. The iteration defined by (5.2–5.5) with the above choice of norms is then simplified to the following algorithm.

**New formulation of a parallel decomposition algorithm.**

For  $n = 0, 1, 2, \dots$ , let  $u_i^n \in H_{0,i}^1(\Omega_i)$  be given.

**Step 0.** Let

$$(5.9) \quad \lambda^{n+1} = \frac{1}{2} u_1^n + \frac{1}{2} u_2^n \quad \text{on } \Gamma_0,$$

$$(5.10) \quad g^{n+1} = g^n - \frac{1}{2\delta} u_1^n + \frac{1}{2\delta} u_2^n \quad \text{on } \Gamma_0.$$

**Step 1.** For  $i = 1, 2$ , find  $u_i^{n+1} \in H_{0,i}^1(\Omega_i)$  such that

$$(5.11) \quad -\Delta u_i^{n+1} = f \quad \text{in } \Omega_i, i = 1, 2,$$

$$(5.12) \quad \frac{\partial u_i^{n+1}}{\partial n_i} + \delta u_i^{n+1} = \delta \lambda^{n+1} - (-1)^i g^{n+1} \quad \text{on } \Gamma_0, i = 1, 2.$$

**5.4. The convergence.** For the above algorithm, we let  $e_i^n = (u - u_i^n) |_{\Gamma_0}$ , where  $u$  is the solution of (2.4) on the whole domain, and we define  $\vec{w}^n = (e_1^n, e_2^n)^T$ . Notice that we have  $P_i = I$  for  $i = 1, 2$ ; by (5.7), we get the following relation:

$$(5.13) \quad \begin{pmatrix} \delta I + \mathcal{S}_1 & 0 \\ 0 & \delta I + \mathcal{S}_2 \end{pmatrix} \vec{w}^{n+1} = \frac{1}{2} \begin{pmatrix} \delta I + \mathcal{S}_1 & \delta I - \mathcal{S}_2 \\ \delta I - \mathcal{S}_1 & \delta I + \mathcal{S}_2 \end{pmatrix} \vec{w}^n.$$

In matrix terms, we may write the above as  $\mathcal{A}\bar{w}^{n+1} = \mathcal{B}\bar{w}^n$ , or, equivalently,

$$(5.14) \quad (\mathcal{A} + \mathcal{B})(\bar{w}^{n+1} - \bar{w}^n) + (\mathcal{A} - \mathcal{B})(\bar{w}^{n+1} + \bar{w}^n) = \vec{0}.$$

Now, with proper regularity assumptions that  $(\delta I \pm \mathcal{S}_i)e_j^n \in L^2(\Gamma_0)$ , we may take the inner product of the above equation with  $(\mathcal{A} + \mathcal{B})(\bar{w}^{n+1} + \bar{w}^n)$  to get

$$(5.15) \quad \begin{aligned} & ((\mathcal{A} + \mathcal{B})\bar{w}^{n+1}, \bar{w}^{n+1})_{0,\Gamma_0} - ((\mathcal{A} + \mathcal{B})\bar{w}^n, (\mathcal{A} + \mathcal{B})\bar{w}^n)_{0,\Gamma_0} \\ & + ((\mathcal{A} + \mathcal{B})(\bar{w}^{n+1} + \bar{w}^n), (\mathcal{A} - \mathcal{B})(\bar{w}^{n+1} + \bar{w}^n))_{0,\Gamma_0} = 0. \end{aligned}$$

Notice that  $(\mathcal{A} - \mathcal{B})^T(\mathcal{A} + \mathcal{B}) + (\mathcal{A} + \mathcal{B})^T(\mathcal{A} - \mathcal{B}) = 2\mathcal{A}^T\mathcal{A} - 2\mathcal{B}^T\mathcal{B}$ , and by simple calculation,

$$\mathcal{A}^T\mathcal{A} - \mathcal{B}^T\mathcal{B} = 2\delta \begin{pmatrix} \mathcal{S}_1 & 0 \\ 0 & \mathcal{S}_2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} \delta I + \mathcal{S}_1 & \mathcal{S}_1 - \delta I \\ \mathcal{S}_2 - \delta I & \delta I + \mathcal{S}_2 \end{pmatrix} \begin{pmatrix} \delta I + \mathcal{S}_1 & \mathcal{S}_2 - \delta I \\ \mathcal{S}_1 - \delta I & \delta I + \mathcal{S}_2 \end{pmatrix}.$$

Summing over  $n$  in (5.15), we get

$$\begin{aligned} & \|(\mathcal{A} + \mathcal{B})\bar{w}^{N+1}\|_{0,\Gamma_0}^2 - \|(\mathcal{A} + \mathcal{B})\bar{w}^0\|_{0,\Gamma_0}^2 \\ & + 2\delta \sum_{n=0}^N [(e_1^n + e_1^{n+1}, \mathcal{S}_1(e_1^n + e_1^{n+1}))_{0,\Gamma_0} + (e_2^n + e_2^{n+1}, \mathcal{S}_2(e_2^n + e_2^{n+1}))_{0,\Gamma_0}] \\ & + \frac{1}{4} \sum_{n=0}^N [\|(\delta I + \mathcal{S}_1)(e_1^n + e_1^{n+1}) + (\mathcal{S}_2 - \delta I)(e_2^n + e_2^{n+1})\|_{0,\Gamma_0}^2 \\ & \quad + \|(\mathcal{S}_1 - \delta I)(e_1^n + e_1^{n+1}) + (\delta I + \mathcal{S}_2)(e_2^n + e_2^{n+1})\|_{0,\Gamma_0}^2] = 0. \end{aligned}$$

Thus, as  $n \rightarrow \infty$ , we have for  $i = 1, 2$ ,

$$\begin{aligned} e_i^n + e_i^{n+1} &\rightarrow 0 && \text{in } H_{00}^{1/2}(\Gamma_0), \\ \mathcal{S}_1(e_1^n + e_1^{n+1}) + \mathcal{S}_2(e_2^n + e_2^{n+1}) &\rightarrow 0 && \text{in } L^2(\Gamma_0). \end{aligned}$$

It follows from (5.14) that

$$\begin{aligned} (e_1^{n+1} + e_2^{n+1}) - (e_1^n + e_2^n) &\rightarrow 0 && \text{in } L^2(\Gamma_0), \\ \mathcal{S}_1(e_1^{n+1} - e_1^n) + \mathcal{S}_2(e_2^{n+1} - e_2^n) &\rightarrow 0 && \text{in } L^2(\Gamma_0). \end{aligned}$$

They in turn imply

$$e_1^{n+1} - e_2^{n+1} \rightarrow 0 \quad \text{and} \quad \mathcal{S}_1(e_1^{n+1}) + \mathcal{S}_2(e_2^{n+1}) \rightarrow 0 \quad \text{in } L^2(\Gamma_0).$$

From (5.13), we can also get

$$\|\mathcal{A}\bar{w}^{n+1}\|_{0,\Gamma_0} = \|\mathcal{B}\bar{w}^n\|_{0,\Gamma_0} \leq \|\mathcal{A}\bar{w}^n\|_{0,\Gamma_0}.$$

Thus, we get a uniform bound for  $\mathcal{S}_i e_i^n$  ( $i = 1, 2$ ) in  $L^2(\Gamma_0)$ . Compared to the above convergence result, we actually have, as  $n \rightarrow \infty$ ,

$$\mathcal{S}_i e_i^{n+1} \rightarrow 0 \quad \text{weakly in } L^2(\Gamma_0) \quad (i = 1, 2).$$

Therefore, using the compact imbedding theorem and properties of  $\mathcal{S}_i$ , we have

$$e_i^{n+1} \rightarrow 0 \quad \text{in } H_{00}^{1/2}(\Gamma_0) \quad (i = 1, 2).$$

That is, we have shown the following result.

**THEOREM 5.1.** *The iteration given by (5.9)–(5.12) is convergent for any  $\delta > 0$ .  $\square$*

This conclusion and the algorithm itself appear to be new, though the latter is similar to that discussed in [15]. Similar discussion may be given for other choices of norms in the energy functional used in the algorithm.

**5.5. Discrete parallel algorithm.** A discrete finite element analogue of (5.9)–(5.12) can also be implemented easily as the following algorithm.

**Discrete parallel algorithms (nonoverlapping domain).**

For  $n = 0, 1, \dots$ , given  $u_{i,n}^h \in V_i^h + V_0^h$  and  $g_n^h \in \Lambda^h$ .

**Step 0.** Find  $\lambda_{n+1}^h, g_{n+1}^h \in \Lambda^h$  such that

$$(5.16) \quad \lambda_{n+1}^h = \frac{1}{2}(u_{1,n}^h + u_{2,n}^h),$$

$$(5.17) \quad g_{n+1}^h = g_n^h + \frac{1}{2\delta}(u_{2,n}^h - u_{1,n}^h).$$

**Step 1.** For  $i = 1, 2$ , find  $u_{i,n+1}^h \in V_i^h + V_0^h$  such that

$$(5.18) \quad \int_{\Omega_i} \nabla u_{i,n+1}^h \nabla v^h d\Omega + \delta \int_{\Gamma_0} u_{i,n+1}^h v^h d\Gamma = \int_{\Omega_i} f v^h d\Omega \\ + \int_{\Gamma_0} (\delta \lambda_{n+1}^h - (-1)^i g_{n+1}^h) v^h d\Gamma \quad \forall v^h \in V_i^h + V_0^h.$$

With additional assumptions on the mesh, the convergence property of the discrete algorithm follows along its continuous analogue; we omit the details. Notice that the limit of the iteration is again the standard Galerkin finite element approximation as defined by (4.9). Moreover, there is no need to compute the normal derivatives explicitly in the iteration which allows an efficient implementation.

**6. Generalizations and conclusions.** From the previous sections, we see that some existing as well as new nonoverlapping domain decomposition algorithms can be derived and analyzed using the optimization based framework. As for their generalizations, the two-subdomain decomposition may be extended to the multisubdomain cases with the formulation of a more general optimization problem. It is also interesting to explore the relation between the current optimization based framework with other interpretations of the domain decomposition methods such as substructure preconditioner, subspace corrections, and multilevel, multigrid methods [3, 4, 5, 6, 11, 14, 30]. When the number of subdomains becomes very large, enforcing only continuity-type conditions on the interfaces is not enough to accurately represent the physical problem; one may consider how ideas like *coarse space (grid)* solvers [11] may be incorporated into the optimization based framework. More study will be pursued along these lines.

In addition, the optimization approach may be applied to other second order elliptic problems (both self-adjoint or non-self-adjoint) as well as other interface conditions [21] or more general conditions of the type

$$F \left( u_1 |_{\Gamma_0}, u_2 |_{\Gamma_0}, \left. \frac{\partial u_1}{\partial n_1} \right|_{\Gamma_0}, \left. \frac{\partial u_1}{\partial n_1} \right|_{\Gamma_0} \right) = 0.$$

Generalizations for nonlinear equations have also been considered [17].

Meanwhile, there are also many other approaches to solving the optimization problems, such as the gradient-type algorithms. Methods proposed in [16, 17, 22, 29] can be grouped into this category. There is also a great amount of freedom in the discrete implementation with respect to the choices of the mesh (and subdomain) dependent norms or functionals as well as the dynamic selection of the free parameters to accelerate the convergence.

To conclude, in this paper we have illustrated how the ideas of optimization can be applied to a simple model problem with only two nonoverlapping subdomains. This new perspective may shed light on the study of more general nonoverlapping domain decomposition schemes and lead to more complete analysis for the various generalizations mentioned here.

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