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CONVERGENCE ANALYSIS OF A NUMERICAL METHOD FOR A MEAN FIELD MODEL OF SUPERCONDUCTING VORTICES*

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Abstract. In the mean field models of superconductivity, the individual vortex-like structures occurring in practical type-II superconductors are averaged and a vortex density is solved for them. The numerical solution of the mean field models makes large-scale simulations of vortex phenomena possible. In this paper, we present a simple convergence analysis for a numerical method based on hybrid finite element/finite volume/finite difference approximations in the two-dimensional case, by providing various different interpretations to the discretization scheme.

Key words. mean field model, superconductivity, vortices, finite element, conforming and nonconforming, finite volume, covolume, finite difference, staggered grid, convergence analysis

AMS subject classifications. 65N99, 82D55

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1. Introduction.

Let the superconducting sample occupy a connected polygonal domain $\Omega \in \mathbb{R}^2$ with boundary Γ. Let ω and u represent the vortex density and the average magnetic field, respectively. Then, on $\Omega_T = \Omega \times (0, T)$, the model equations we study, after proper scaling, is given by the system

$$\omega_t - \nabla \cdot (w \nabla u) = 0 \quad \text{in} \quad \Omega_T, \tag{1.1}$$

$$-\Delta u + u = \omega \quad \text{in} \quad \Omega_T. \tag{1.2}$$

The system (1.1)–(1.2) is one of the more sophisticated mean field models presented in [4, 5, 6] for the motion of vortices in a type-II superconductor. Simpler models of this type such as the Bean model have existed for some time [2]. The models in [4, 5, 6] were derived by taking appropriate limits within the Ginzburg–Landau formalism. The various mechanisms for the *pinning* of vortices can also be introduced in the models [6].

In the last few years, the mathematical analysis and the numerical studies of the mean field–type models have received much attention [12, 14, 15, 18, 20, 21, 22]. Both theoretical analysis and numerical studies can be found in [20, 21]. In [12], finite element approximation is studied based on a penalty regularization of a model with the pinning effects in the variational inequality formulation. For the particular model considered here, the existence and uniqueness of solutions and the regularity estimates are obtained in [18, 22]. A formal derivation of (1.1)–(1.2) based on the Ginzburg–Landau models is also given in [14]. In [15], a finite element/finite volume approximation was developed, estimates on the discrete solutions were provided, and convergence analysis was given for the steady state model in the two space dimensional case and the time-dependent model in one space dimension. Nevertheless, a complete convergence theory for the approximation of time-dependent models in two

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or higher space dimensions has not been provided. Here, we fill this gap by presenting a simple convergence analysis of the numerical solutions to the weak solution of the two-dimensional mean field model (1.1)–(1.2), based on a hybrid finite element/finite volume (or simply finite difference) approximation. The analysis is presented in a few standard steps. While sectarian bias turns to let a particular discretization method be viewed only from one's favorite angle, we demonstrate that by using the connections (or in some cases, the equivalence) between the finite volume (or covolume), the finite element (conforming and nonconforming), and the finite difference (on standard grid or staggered grid) approximations, the proofs of some key estimates become much simpler.

With (1.1)–(1.2), a very important feature of the mean field models in [4, 5, 6] is ignored with regard to the pinning effects in the superconductors. We hope that the techniques presented here will be useful in the analysis of other types of approximations, such as those in [12], for more general mean field–type models that contain more physically relevant features.

The paper is organized as follows: In section 2, the weak formulation of the model and a numerical method are presented. The main convergence theorem is stated in section 3 along with some useful identities. A number of technical estimates on the discrete solutions are given in section 4. The proof of the main theorem is given in section 5. Further remarks and comments are given in section 6.

2. Weak formulation and the numerical method. Let $(\cdot,\cdot)$ denote the standard L^2 inner product in Ω and let $\langle\cdot,\cdot\rangle$ denote the L^2 inner product in Ω_T. Let $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ denote the standard Sobolev space and the dual space of functions of the variable $\mathbf{x} \in \Omega$.

Let the initial condition of (1.1)–(1.2) be given by

$$\omega|_{t=0} = \omega_0 \geq 0 \quad \text{in} \quad \Omega \tag{2.1}$$

and the boundary condition be given by

$$u = H_1 \quad \text{on} \quad \partial\Omega_T, \tag{2.2}$$

where $\partial\Omega_T = \{(\mathbf{x}, t) : \mathbf{x} \in \partial\Omega, t \in (0, T)\}$ and H_1 is a constant external applied magnetic field. For simplicity, we assume that $\text{supp}(\omega_0)$ is a proper subset of Ω. We assume further that in the time interval $[0, T]$ of interests, Ω is sufficiently large such that the weak solution ω satisfies $\text{supp}(\omega(\cdot, t)) \subset \Omega$ for all $t \in [0, T]$. By the assumption on the support of ω, we bypass some technical difficulty related to the pass of limit in a nonlinear boundary condition considered in [6].

The notation we use here is similar to that in [13] (see also [15, 19, 26]). Let Σ denote a regular triangulation of Ω with vertices $\{\mathbf{x}_j\}$, edges $\{s_{ij}\}$, and triangles $\{\tau_{ijk}\}$. A dual tessellation is formed by joining the circumcenters of adjacent triangles. Simple geometrical property implies that the line joining the circumcenters is normal to the common edge and bisects it. The set of dual figures, denoted by Σ', is a set of polygons which are named *covolumes*. The interior of the polygon associated with all the edges connecting the vertex $\mathbf{x}_j$ is denoted by Ω_j (see Figure 2.1). Under the *locally equiangular* assumption [13, 19], we have the following:

1. each Ω_j is convex,
2. $\Omega_j \cap \Omega_k = \emptyset$ if $j \neq k$.

Let $h_{ij} = |s_{ij}|$ denote the length of the edge s_{ij} and $h'_{ij} = |s'_{ij}|$ denote the length of the corresponding edge s'_{ij} of the covolume which is normal to s_{ij} and bisects it. If

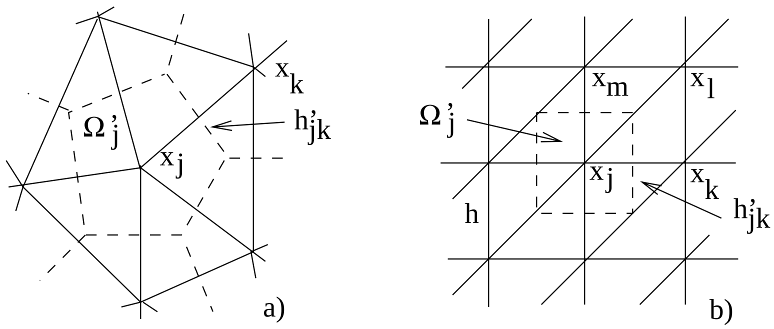

FIG. 2.1. *Triangular cells and its dual:* (a) *unstructured grid,* (b) *uniform grid.*

a vertex $\mathbf{x}_i$ is on the boundary of Ω, the region Ω_i is modified to include only the portion that is inside Ω. Similarly, $h'_{ij} = |s'_{ij}|$ includes only the segment contained inside Ω for two adjacent vertices on Γ. The area of Ω_j is denoted by $|\Omega_j|$.

For a quasi-uniform grid, we use h as a typical mesh parameter. Let $\mathcal{U}^h$ be the space of continuous piecewise linear functions defined on the triangular grid Σ and $\mathcal{W}^h$ be the space of piecewise constant functions defined on the dual grid Σ'. Let I_i denote the set of indices of the interior vertices and I_b be the set of indices of the vertices belonging to the boundary Γ. Define

$$\mathcal{U}_0^h = \{v^h \in \mathcal{U}^h \mid v^h(\mathbf{x}_j) = 0, \ \forall\, j \in I_b\},$$
$$\mathcal{W}_0^h = \{w^h \in \mathcal{W}^h \mid w^h(\mathbf{x}_j) = 0, \ \forall\, j \in I_b\}.$$

Similar to that in [15], the numerical method we consider in this paper is based on a hybrid upwinding finite difference scheme and a finite element approximation with numerical integration or, equivalently, a finite volume (or *covolume*) approximation [13, 19, 26]. For $\{u^h, \omega^h\}$ in $\mathcal{U}^h \times \mathcal{W}^h$, let their values at the vertex $\mathbf{x}_j$ be denoted by $u_j = u^h(\mathbf{x}_j)$ and $\omega_j = \omega^h(\mathbf{x}_j)$. Define the discrete inner product $(\cdot,\cdot)_h$ by

$$(v, w)_h = \sum_j v(\mathbf{x}_j) w(\mathbf{x}_j) |\Omega_j|$$

for any pair $\{v, w\}$ having proper definitions of their values at the vertices.

The approximate schemes for the mean field model (1.1)–(2.2) are given by the following: solve for $u^h - H_1 \in \mathcal{U}_0^h$, $\omega^h \in \mathcal{W}_0^h$ such that

$$\frac{d}{dt}\omega_j|\Omega_j| - \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left(\omega_k[u_k - u_j]_+ + \omega_j[u_k - u_j]_-\right) = 0 \qquad \forall\, j \in I_i, \tag{2.3}$$

$$(\nabla u^h,\, \nabla \varphi^h) + (u^h - \omega^h,\, \varphi^h)_h = 0 \qquad \forall\, \varphi^h \in \mathcal{U}^h, \tag{2.4}$$

where $[f]_+ = \max(f, 0)$ and $[f]_- = \min(f, 0)$ and

$$\sum_{k\to j} := \text{ sum over all } k\text{s with } \Omega_k \text{ sharing a common edge } s'_{jk} \text{ with } \Omega_j.$$

The initial conditions are given by

$$w_0^h(\mathbf{x}_j) = w_j^0 = \frac{1}{|\Omega_j|}\int_{\Omega_j} w_0(\mathbf{x}) d\mathbf{x} \qquad \forall\, j \in I_i. \tag{2.5}$$

We can rewrite (2.4) as

$$\frac{1}{|\Omega_j|}\sum_{k\to j}\frac{h'_{jk}}{h_{jk}}\,[u_j-u_k]+u_j=\omega_j \qquad \forall\, j\in I_i. \tag{2.6}$$

In the finite volume terminology, it is also possible to integrate (1.2) in Ω_j and use the divergence theorem to derive (2.6) by replacing the normal derivatives on the edges s'_{jk} by differences along the edges s_{jk} [13, 19].

In the case of a uniform triangulation obtained by dividing along diagonals in a given direction of the cells in a uniform rectangular mesh, the dual cells can be defined by identical squares shifted by a mesh size in each coordinate direction (which form a staggered grid in the finite difference terminology; see Figure 2.1). Equation (2.6) becomes a standard finite difference approximation with a five-point stencil for the Laplace operator.

For the convergence analysis, we limit our attention to the case where Ω is a square domain with a uniform triangulation obtained by dividing along $x=y$ diagonals of the cells in a uniform rectangular mesh on Ω. We retain the notation for the general mesh in the discussion as many estimates remain valid for quasi-uniform, locally equiangular meshes. For brevity, we focus on the semidiscrete approximation only.

3. The convergence theorem. For $p\geq 1$, let $\mathcal{X}_p=L^p(0,T;H_0^1(\Omega))$, $\mathcal{Y}=H^1(0,T;H_0^1(\Omega))$, $\mathcal{Z}=H^1(0,T;H^{-1}(\Omega))$, and $\mathcal{W}_{p,q}=L^p(0,T;L^q(\Omega))$ denote the Sobolev spaces of functions of the variables $(\mathbf{x},t)\in\Omega_T$ [24]. For any Banach space B, its norm is denoted by $\|\cdot\|_B$, except for $\|\cdot\|_p$, which denotes the norm of $L^p(\Omega)$.

The weak formulation of (1.1)–(2.2) is formulated as [15, 22]:

$$(\omega_0,\,\phi(\mathbf{x},0))+\langle\omega,\,\phi_t-\nabla u\nabla\phi\rangle=0 \qquad \forall\,\phi\in C_0^\infty(\Omega\times[0,T)), \tag{3.1}$$

$$\langle\nabla u,\,\nabla\varphi\rangle+\langle u-\omega,\,\varphi\rangle=0 \qquad \forall\,\varphi\in C(0,T;H_0^1(\Omega)). \tag{3.2}$$

We now state the main theorem.

THEOREM 3.1. *Let T, H_1, and $\omega_0\in L^\infty(\Omega)$ be given. Then, any subsequence of $\{u^h,\omega^h\}$ has a subsequence $\{u^{h_k},\omega^{h_k}\}$ such that as $h_k\to 0$,*

$$u^{h_k}\to u \quad \text{weakly (or weakly *) in} \quad \mathcal{X}_\infty\cap\mathcal{W}_{\infty,\infty}\cap\mathcal{Y}, \tag{3.3}$$

$$u^{h_k}\to u \quad \text{strongly in} \quad \mathcal{W}_{6,6}\cap\mathcal{X}_2, \tag{3.4}$$

$$\omega^{h_k}\to \omega \quad \text{weakly (or weakly *) in} \quad \mathcal{Z}\cap\mathcal{W}_{\infty,\infty}, \tag{3.5}$$

where (u,ω) is a weak solution of (3.1)–(3.2). If the weak solution of (3.1)–(3.2) is unique, then the whole sequence (u^h,ω^h) converges as $h\to 0$.

It is obvious that the second conclusion of the theorem follows from the first with the assumption on the uniqueness of the weak solution (and thus, the uniqueness of the subsequence limit). The question on the existence and uniqueness of the weak solution has been considered in [18] for a two-dimensional free boundary problem and in [22] for boundary value problems in a fixed domain.

Here, we do not intend to involve ourselves into further discussion on the uniqueness of the weak solution. For this reason, our proof is focused on the verification of the convergence of subsequences and the fact that the limit of such subsequence is indeed a weak solution of (3.1)–(3.2).

We give a few useful identities to be used in the proof of the main theorem.

First of all, for any $w^h \in \mathcal{U}^h$, we define the discrete Laplace operator Δ_h as $f^h = \Delta_h w^h \in \mathcal{W}_0^h$ if

$$f^h(\mathbf{x}_j) = \frac{1}{|\Omega_j|}\left(\sum_{k\to j}\frac{h'_{jk}}{h_{jk}}[w_k - w_j]\right) \quad \forall\, j \in I_i.$$

With the operator Δ_h, (2.6) is conveniently written as

$$-(\Delta_h u^h)_j + u_j = \omega_j \qquad \forall\, j \in I_i.$$

Notice that for the uniform mesh we consider here, $h'_{jk} = h_{jk} = h$ for all interior edges (and dual-edges with $h'_{jk} \neq 0$). Though the expressions in the discrete equations can be simplified by using a standard approximation of the Laplace operator by the finite difference operator with five-point stencil on a uniform rectangular mesh, we prefer to keep the present notation for general unstructured grid. This is not only for the sake of avoiding double indices but also for providing possible extensions of our analysis to more general cases.

We now have a discrete analog of Green's identity.

LEMMA 3.2. *For any $v^h \in \mathcal{U}_0^h$ and $w^h \in \mathcal{U}^h$,*

$$(\nabla v^h, \nabla w^h) = -(v^h, \Delta_h w^h)_h. \tag{3.6}$$

Proof. The lemma follows from straightforward calculation of the inner product of the basis functions, similar to that performed in [13, 15] and [19]. ∎

Similarly, we can easily derive the summation by the following part formula.

LEMMA 3.3. *For $w^h \in \mathcal{W}_0^h$, $\varphi^h \in \mathcal{W}_0^h, u^h \in \mathcal{W}^h$,*

$$\begin{aligned}&\sum_{j\in I_i}\sum_{k\to j}\varphi_j\frac{h'_{jk}}{h_{jk}}\left(\omega_k[u_k-u_j]_+ + \omega_j[u_k-u_j]_-\right)\\ &\quad = \sum_{jk}\frac{h'_{jk}}{h_{jk}}(\varphi_j-\varphi_k)\left(\omega_k[u_k-u_j]_+ + \omega_j[u_k-u_j]_-\right),\end{aligned} \tag{3.7}$$

where

$$\sum_{jk} := \text{ sum over all distinct edges } s_{jk}. \qquad ∎$$

4. Estimates for the numerical solution. First of all, let us derive some maximum norm estimates. The key is to notice that for locally equiangular mesh, the matrix presenting the operator $-\Delta_h$ is an M-matrix; that is, its inverse has nonnegative entries.

LEMMA 4.1. *For given H_1 and $\omega_0 \in L^\infty(\Omega)$, let $M(t) = \max(\|\omega^h(t)\|_\infty, H_1)$, $L(t) = \max(\|u^h(t)\|_\infty, H_1)$. Then*

$$\omega^h(t) \geq 0, \tag{4.1}$$

$$L(t) \leq M(t), \tag{4.2}$$

$$M(t) \text{ is nonincreasing in time.} \tag{4.3}$$

Proof. Note first that (2.3) can be rewritten in the vector form as

$$\frac{d}{dt}\vec{\omega}(t) + D(t)\vec{\omega}(t) - B(t)\vec{\omega}(t) = \vec{0},$$

where $D(t)$ has a diagonal matrix form containing nonnegative entries, $B(t)$ has zeros on the diagonal and nonnegative off-diagonal entries. We may take a large enough constant $\lambda > 0$ such that the entries of $D(t)$ are bounded by λ, at least for small time. Then, taking integration factor, we get

$$\frac{d}{dt}(e^{\lambda t}\vec{\omega}) = \left(\lambda I - D(t) + \hat{B}\right) e^{\lambda t}\vec{\omega},$$

where $\hat{B}$ remains to have zero diagonals and nonnegative off-diagonal entries. Thus, we get, $e^{\lambda t}\vec{\omega}(t) \geq \vec{0}$, which implies

$$\omega^h(t) \geq 0.$$

By continuation in time, we have the same conclusion for any $t > 0$.

Now, let $|u_{j_0}| = \|u^h\|_\infty$, if $j_0 \in I_b$; then

$$\max(\|u^h\|_\infty, H_1) = H_1 \leq \max(\|\omega^h\|_\infty, H_1).$$

If $j_0 \in I_i$, then for $u_{j_0} \geq 0$,

$$\sum_{k\to j_0} \frac{h'_{jk}}{h_{jk}} [u_{j_0} - u_k] \geq 0,$$

and by (2.6), we have $|u_{j_0}| = u_{j_0} \leq \omega_{j_0} \leq \|\omega^h\|_\infty$. Similarly, for $u_{j_0} \leq 0$,

$$\sum_{k\to j_0} \frac{h'_{jk}}{h_{jk}} [u_{j_0} - u_k] \leq 0;$$

therefore, $|u_{j_0}| = -u_{j_0} \leq -\omega_{j_0} \leq \|\omega^h\|_\infty$. Thus, in either case, we have $L(t) \leq M(t)$.

It now remains to show that $M(t)$ is nonincreasing. We rewrite (2.3) as

$$\frac{d}{dt}\omega_j|\Omega_j| - \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left((\omega_k - \omega_j)[u_k - u_j]_+ + \omega_j(u_k - u_j)\right) = 0.$$

Using (2.6), we get

$$\frac{d}{dt}\omega_j|\Omega_j| - \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left((\omega_k - \omega_j)[u_k - u_j]_+\right) - |\Omega_j|(u_j - \omega_j)\omega_j = 0.$$

Let $\omega_{j1} = \|\omega^h\|_\infty$; if $\omega_{j1} \leq H_1$, we are done; otherwise, we get from the previous result that

$$u_{j1} - \omega_{j1} \leq L(t) - M(t) \leq 0.$$

Since $(\omega_k - \omega_{j1})[u_k - u_{j1}]_+ \leq 0$, for all k, we get

$$\frac{d}{dt}\omega_{j1} \leq 0,$$

which implies that $M(t)$ is nonincreasing. □

Note that if the standard finite element approximation is used to discretize (1.2) without the use of numerical integration in (2.4), it would be much more involved to analyze the positivity preserving property for ω^h.

Using standard ODE theory, the uniform-in-time estimates in Lemma 4.1 also imply the global existence and uniqueness of the ODE system (2.3)–(2.4).

COROLLARY 4.2. *For any given $T > 0$ and given initial condition ω_0^h in* (2.5), *there exists a unique solution to* (2.3)–(2.4) *in* $(0, T)$. ◻

We now derive some uniform energy estimates on the discrete solution. We begin with estimates on the spatial derivatives. If we let $\phi^h = u^h - H_1$ in (2.4) and use the pointwise estimates given in the above, we immediately get the uniform-in-time estimate on $\|u^h\|_{H^1(\Omega)}$. In fact, we let

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 d\Omega + \frac{1}{2}|v - H_1|_h^2,$$

where

$$|v|_h^2 = \sum_j \frac{1}{|\Omega_j|}\left(\int_{\Omega_j} v(\mathbf{x})d\mathbf{x}\right)^2.$$

Then, we have an even sharper result.

LEMMA 4.3. *For any $t > 0$,*

$$\frac{d}{dt}\mathcal{J}(u^h(t)) + \sum_{jk}\frac{h'_{jk}}{h_{jk}}\left(\omega_k[u_k - u_j]_+^2 + \omega_j[u_k - u_j]_-^2\right) = 0.$$

Proof. We perform direct calculation:

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}(u^h(t)) &= (\nabla u^h, \nabla u_t^h) + (u^h - H_1, u_t^h)_h \\
&= -(u^h - H_1, \Delta_h u_t^h)_h + (u^h - H_1, u_t^h)_h \\
&= (u^h - H_1, \omega_t^h)_h \\
&= \sum_{j\in I_i}\sum_{k\to j}(u_j - H_1)\frac{h'_{jk}}{h_{jk}}\left(\omega_k[u_k - u_j]_+ + \omega_j[u_k - u_j]_-\right) \\
&= \sum_{jk}\frac{h'_{jk}}{h_{jk}}\left((u_j - u_k)\omega_k[u_k - u_j]_+ + (u_j - u_k)\omega_j[u_k - u_j]_-\right) \\
&= -\sum_{jk}\frac{h'_{jk}}{h_{jk}}\left(\omega_k[u_k - u_j]_+^2 + \omega_j[u_k - u_j]_-^2\right).
\end{aligned}$$

Here, we have used (2.3), (2.6), and the identities in Lemmas 3.2 and 3.3. ◻

Following from the above lemmas, we get some uniform bounds on the discrete solutions.

COROLLARY 4.4. *For given T, H_1 and given $\omega_0 \in L^\infty(\Omega)$, there exists some constant $c > 0$ such that*

$$\|u^h(t)\|_{\mathcal{X}_\infty} \le c, \tag{4.4}$$

$$\|u^h(t)\|_{\mathcal{W}_{\infty,\infty}} \le c, \tag{4.5}$$

$$\|\omega^h(t)\|_{\mathcal{W}_{\infty,\infty}} \le c \tag{4.6}$$

uniformly with respect to t and h. ◻

Now we consider the time derivative estimate.

LEMMA 4.5. *For given T, H_1 and given $\omega_0 \in L^\infty(\Omega)$, there exists a constant $c > 0$ such that*

$$\|\omega_t^h\|_{L^\infty(0,T;H^{-1}(\Omega))} \le c$$

uniformly with respect to T and h.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and φ^h be the L^2 projection of φ in $\mathcal{W}^h$. Consider

$$\begin{aligned}
|(\omega_t^h,\, \varphi)| &= |(\omega_t^h,\, \varphi^h)_h| \\
&= \left| \sum_j \sum_{k \to j} \varphi_j \left(\omega_k [u_k - u_j]_+ + \omega_j [u_k - u_j]_-\right) \right| \\
&= \left| \sum_{jk} \frac{h'_{jk}}{h_{jk}} (\varphi_j - \varphi_k) \left(\omega_k [u_k - u_j]_+ + \omega_j [u_k - u_j]_-\right) \right| \\
&\le 2\|\omega^h\|_\infty \left\{ \sum_{jk} (\varphi_j - \varphi_k)^2 \right\}^{1/2} \left\{ \sum_{jk} (u_k - u_j)^2 \right\}^{1/2}.
\end{aligned}$$

Now, using the fact that we have a uniform mesh, we get

$$\sum_{jk} (u_k - u_j)^2 \le \|\nabla u^h\|_2^2.$$

Let $\hat{\varphi}^h$ be the linear interpolant of φ^h in $\mathcal{U}^h$; then, using the standard theory on finite element approximations (see, e.g., [3, 8, 17]), one may verify that

$$\sum_{jk} (\varphi_j - \varphi_k)^2 \le \|\nabla \hat{\varphi}^h\|_2^2 \le c\|\varphi\|_{H^1(\Omega)}^2$$

for some constant c, independent of h and t. Thus,

$$|(\omega_t^h, \varphi)| \le c\|\varphi\|_{H^1(\Omega)} \|u^h\|_{H^1(\Omega)} \|\omega^h\|_\infty.$$

By Corollary 4.4, we get

$$\|\omega_t^h\|_{L^\infty(0,T;H^{-1}(\Omega))} = \sup_{t \in (0,T)} \sup_{\varphi \ne 0} \frac{|(\omega_t^h, \varphi)|}{\|\varphi\|_{H^1(\Omega)}} \le c$$

for some constant c, independent of h and t. ∎

Notice that though we used the properties of the uniform mesh, the above lemma remains valid for unstructured regular, quasi-uniform triangulation that satisfies the following assumption: for some constant $\alpha > 0$,

$$\frac{h'_{jk}}{h_{jk}} \le \alpha \quad \forall\, h'_{jk} \ne 0. \tag{4.7}$$

This assumption is slightly stronger than the locally equiangular assumption [13, 19].

Now, (2.4) implies that

$$(\nabla u_t^h,\, \nabla \varphi^h) + (u_t^h - \omega_t^h,\, \varphi^h)_h = 0 \qquad \forall\, \varphi^h \in \mathcal{U}^h.$$

Taking $\varphi^h = u_t^h$ and applying the above lemma, we get the following corollary.

COROLLARY 4.6. *For given T, H_1 and given $\omega_0 \in L^\infty(\Omega)$, there exists a generic constant $c > 0$ such that*

$$\|u_t^h(t)\|_{\mathcal{X}_\infty} \le c$$

uniformly with respect to T and h. □

The uniform estimates are sufficient for us to apply the standard compactness argument [24] as $h \to 0$; however, in order to be able to pass to the limit in the nonlinear terms, we need one more estimate.

LEMMA 4.7. *For given T, H_1 and $\omega_0 \in L^\infty(\Omega)$, there exists a constant $c > 0$ such that*

$$\sum_{jk} \frac{h'_{jk}}{h_{jk}} \int_0^T (\omega_j - \omega_k)^2 |u_j - u_k| dt \le c \tag{4.8}$$

uniformly with respect to h.

Proof. Let us multiply ω_j to (2.3) and sum over $j \in I_i$ to get

$$\begin{aligned}
\frac{d}{dt} \sum_{j\in I_i} \frac{\omega_j^2}{2} |\Omega_j| &= \sum_{j\in I_i} \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left(\omega_k\omega_j [u_k - u_j]_+ + \omega_j^2 [u_k - u_j]_-\right) \\
&= \sum_{jk} \frac{h'_{jk}}{h_{jk}} \left[\omega_j\omega_k([u_k - u_j]_+ + [u_j - u_k]_+)\right] + \sum_{j\in I_i}\sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \omega_j^2 [u_k - u_j]_- \\
&= \sum_{jk} \frac{h'_{jk}}{h_{jk}} \omega_j\omega_k |u_k - u_j| + \frac{1}{2} \sum_{j\in I_i}\sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \omega_j^2 [(u_k - u_j) - |u_k - u_j|] \\
&= \frac{1}{2} \sum_{jk} \frac{h'_{jk}}{h_{jk}} (\omega_j - \omega_k)^2 |u_k - u_j| + \frac{1}{2} \sum_{j\in I_i}\sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \omega_j^2 (u_k - u_j) \\
&= -\frac{1}{2} \sum_{jk} \frac{h'_{jk}}{h_{jk}} (\omega_j - \omega_k)^2 |u_k - u_j| + \frac{1}{2} \sum_{j\in I_i} |\Omega_j| \omega_j^2 (u_j - \omega_j),
\end{aligned}$$

where we have used (2.6) in the last step.

Now, integrating in time and applying the estimates (4.4)–(4.6), we get

$$\sum_{jk} \frac{h'_{jk}}{h_{jk}} \int_0^T (\omega_j - \omega_k)^2 |u_j - u_k| dt \le |\omega_0^h|_h^2 + T \max(\|\omega_0\|_\infty, H_1)^3.$$

This implies the inequality (4.8) given in the lemma. □

The above estimate controls the spatial variation of ω^h to some extent, and it is a useful fact in dealing with the nonlinearity due to the upwinding schemes used in the approximation.

Before we conclude this section, we note that much of the results we presented so far remains valid for unstructured triangulation that satisfies the assumption (4.7). It is also possible to derive similar estimates for fully discrete approximations as in [15]. With the uniform estimates, we can discuss the long-time asymptotics of the discrete solution as $t \to \infty$, again, similar to that given in [15] for a fully discrete approximation. We omit the details.

5. Proof of the main convergence theorem. We first apply the compactness argument based on the estimates obtained earlier to get the convergence of the discrete solutions to a weak limit.

PROPOSITION 5.1. *Let T, H_1, and $\omega_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ be given. Then, any subsequence of $\{u^h, \omega^h\}$ has a a subsequence $\{u^{h_k}, \omega^{h_k}\}$ such that as $h_k \to 0$,*

$$u^{h_k} \to u \quad \text{weakly (or weakly *) in} \quad \mathcal{X}_\infty \cap \mathcal{W}_{\infty,\infty} \cap \mathcal{Y}, \tag{5.1}$$

$$u^{h_k} \to u \quad \text{strongly in} \quad \mathcal{W}_{6,6}, \tag{5.2}$$

$$\omega^{h_k} \to \omega \quad \text{weakly (or weakly *) in} \quad \mathcal{Z} \cap \mathcal{W}_{\infty,\infty} \tag{5.3}$$

for some limit $\{u, \omega\}$ in the appropriate spaces.

Proof. The existence of the weak (or weak *) limit $\{u, \omega\}$ follows from the uniform estimates and the weak (or weak *) compactness of the corresponding Sobolev spaces. In addition, the weak convergence in $\mathcal{Y}$ implies the strong convergence of a subsequence of $\{u^{h_k}\}$ in $\mathcal{W}_{6,6}$, which in particular implies the almost everywhere convergence in Ω_T. □

We now try to verify that the limit is a weak solution of (3.1)–(3.2).

PROPOSITION 5.2. *The limit $\{u, \omega\}$ satisfies* (3.2).

Proof. For simplicity, we drop the subscript k in the sequence $\{u^{h_k}, \omega^{h_k}\}$. First of all, for $\varphi \in C([0,T]; H_0^1(\Omega))$, by the weak convergence of u^h and its uniform bound, we have

$$\lim_{h\to 0} \langle \nabla u^h,\, \nabla \hat{\varphi}^h \rangle = \lim_{h \to 0} \langle \nabla u^h,\, \nabla \varphi \rangle = \langle \nabla u,\, \nabla \varphi \rangle.$$

Conversely, if we let $\bar{u}^h$ be the piecewise constant interpolation of u^h with respect to Σ', let φ^h be the piecewise constant projection of φ in $\mathcal{W}_0^h$, and let $\hat{\varphi}^h \in \mathcal{U}_0^h$ be the piecewise linear interpolation of φ^h with respect to Σ, we have, as $h \to 0$,

$$\begin{aligned}
\langle \nabla u^h,\, \nabla \hat{\varphi}^h \rangle &= \int_0^T (\omega^h - u^h,\, \hat{\varphi}^h)_h dt \\
&= \int_0^T (\omega^h - u^h,\, \varphi^h)_h dt \\
&= \int_0^T (\omega^h,\, \varphi) dt - \int_0^T (\bar{u}^h,\, \varphi) dt \\
&= \int_0^T (\omega^h - u^h,\, \varphi) dt + \int_0^T (u^h - \hat{u}^h,\, \varphi) dt \\
&\to \langle \omega,\, \varphi \rangle - \langle u,\, \varphi \rangle
\end{aligned}$$

since, based on the estimates on u^h, $\bar{u}^h - u^h \to 0$ in $L^2(\Omega_T)$. Therefore, we get (3.2). □

Using the above proposition and the convergence results in Proposition 5.1, we can derive the strong convergence $\{u^{h_k}\}$ in a stronger topology.

COROLLARY 5.3. *For the subsequence $\{u^{h_k}\}$ considered above,*

$$u^{h_k} \to u \quad \text{strongly in} \quad \mathcal{X}_2 \qquad \text{as} \quad k \to \infty.$$

Proof. Using (2.4), and taking $\varphi^h = u^h - H_1$, we get

$$(\nabla u^{h_k},\, \nabla u^{h_k}) + (u^{h_k} - \omega^{h_k},\, u^{h_k} - H_1)_h = 0.$$

Clearly,

$$\lim_{k\to\infty}\int_0^T (u^{h_k}-\omega^{h_k},H_1)_h dt = \langle u-\omega, H_1\rangle.$$

Notice that $|\cdot|_h$ is an equivalent norm to $\|\cdot\|_2$ in $\mathcal{U}^h$, in the sense that there exist two positive constants c_1, c_2, independent of h, such that

$$c_1\|v^h\|_2 \le |v^h|_h \le c_2\|v^h\|_2 \qquad \forall\, v^h\in\mathcal{U}^h.$$

So, for any $v^h\in\mathcal{W}_0^h$, we have

$$\begin{aligned}
&|(u^h\,,\,u^h)_h-(u,u)| = |(u^h\,,\,u^h)_h-(v^h\,,\,v^h)_h| + |(u,u)-(v^h\,,\,v^h)|\\
&\quad\le |u^h-v^h|_h|u^h+v^h|_h + \|u-v^h\|_2\|u+v^h\|_2\\
&\quad\le c\|u^h-v^h\|_2\|u^h+v^h\|_2 + \|u-v^h\|_2\|u+v^h\|_2\\
&\quad\le c\|u^h-u\|_2\|u^h+v^h\|_2 + \|u-v^h\|_2(\|u^h+v^h\|_2+\|u+v^h\|_2)\,.
\end{aligned}$$

By the strong convergence of $\{u^{h_k}\}$, it is easy to see that by choosing v^{h_k} converging strongly to u in $L^2(\Omega_T)$, we have

$$\lim_{k\to\infty}\int_0^T (u^{h_k},\,u^{h_k})_h dt = \langle u,u\rangle.$$

Coupled with the weak convergence of $\{\omega^{h_k}\}$, we get

$$\lim_{k\to\infty}\int_0^T (u^{h_k},\,\omega^{h_k})_h dt = \langle u,\omega\rangle.$$

Thus

$$\lim_{k\to\infty}\int_0^T (\nabla u^{h_k},\,\nabla u^{h_k}) dt = \langle \omega-u, u-H_1\rangle = \langle\nabla u,\nabla u\rangle.$$

Thus, we get the strong convergence of $\{u^{h_k}\}$ in $\mathcal{X}_2$. ◻

We now turn to check on (3.1).

PROPOSITION 5.4. *The limit* $\{u,\omega\}$ *satisfies* (3.1).

Proof. For simplicity, we again avoid the use of the subscripts for the sequence $\{h_k\}$. We consider for $\phi\in C_0^\infty(\Omega\times[0,T))$, as $h\to 0$,

$$\langle\omega_t^h,\,\phi\rangle = -\langle\omega^h,\,\phi_t\rangle + (\omega_0^h,\,\phi(\mathbf{x},0)) \to -\langle\omega,\,\phi_t\rangle + (\omega_0,\,\phi(\mathbf{x},0)). \tag{5.4}$$

Meanwhile, let ϕ^h be the piecewise constant projection of ϕ in $\mathcal{W}_0^h$,

$$\begin{aligned}
(\omega_t^h,\,\phi) &= (\omega_t^h,\,\phi^h)_h\\
&= \sum_{j\in I_i}\sum_{k\to j}\phi_j\frac{h'_{jk}}{h_{jk}}\left(\omega_k[u_k-u_j]_+ + \omega_j[u_k-u_j]_-\right)\\
&= \sum_{jk}\frac{h'_{jk}}{h_{jk}}(\phi_j-\phi_k)\left(\omega_k[u_k-u_j]_+ + \omega_j[u_k-u_j]_-\right)\\
&= \sum_{jk}\frac{h'_{jk}}{h_{jk}}(\phi_j-\phi_k)\left(\frac{\omega_k+\omega_j}{2}(u_k-u_j) + \frac{\omega_j-\omega_k}{2}|u_k-u_j|\right)
\end{aligned}$$

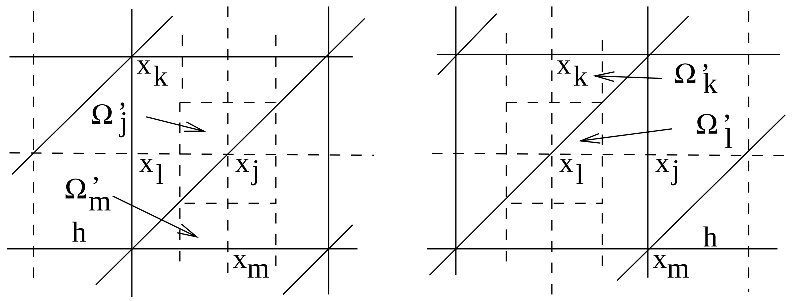

FIG. 5.1. *Two kinds of coarse triangulation with a horizontal shift of distance h.*

$$= \sum_j \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left[\frac{\omega_j}{2}(\phi_j-\phi_k)(u_k-u_j)\right] \\ +\frac{1}{2}\sum_{jk}\frac{h'_{jk}}{h_{jk}}\left[(\phi_j-\phi_k)(\omega_j-\omega_k)|u_k-u_j|\right].$$

Unfortunately, in general, we have

$$\sum_j \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left[\frac{\omega_j}{2}(\phi_j-\phi_k)(u_k-u_j)\right] \neq (\omega^h \nabla u^h,\ \nabla \hat{\phi}^h),$$

where $\hat{\phi}^h$ is the continuous piecewise linear interpolation of ϕ^h in $\mathcal{U}_0^h$. So we need to find an alternative expression. We now make use of the following observation. Let us define two kinds of coarse triangulation: Σ_0^{2h} and Σ_1^{2h}. Let all even mesh lines of Σ in both x and y directions form the mesh lines for Σ_0^{2h}. We then divide all the square cells along the $x=y$ diagonals to get the triangulation Σ_0^{2h}. The triangulation Σ_1^{2h} is obtained by shifting the triangulation Σ_1^{2h} in the horizontal direction by a distance h. The precise definition of Σ_i^{2h} (i=1,2) near the boundary Γ can be ignored as we are only interested in the estimates for ϕ with compact support in Ω. We then divide all the indices of the grid points in Σ into two disjoint subsets I^0 and I^1. A grid point is to have index in I^0 (respectively, I^1) if it lies on an even (respectively, odd) mesh line in the x direction. We now define $\tilde{U}_i^{2h}$ $(i=1,2)$ to be the nonconforming piecewise linear finite element spaces on Σ_i^{2h} with the midpoint of each triangular edge as the nodal point. By our division, each grid point with the index in I^i is a nodal point defining the space $\tilde{U}_i^{2h}$ for $i=1,2$ (though not vice versa; see Figure 5.1). Then, by using the fact that ϕ has compact support in Ω, we have, for small enough h,

$$\sum_{j\in I^i} \sum_{k\to j} \frac{h'_{jk}}{h_{jk}} \left[\frac{\omega_j}{2}(\phi_j-\phi_k)(u_k-u_j)\right] = \sum_{j\in I^i}\int_{\Omega_j} \omega^h \nabla \Pi_i^{2h} u^h \nabla \Pi_i^{2h}\phi^h \, dxdy,$$

where $\Pi_i^{2h}u^h$ and $\Pi_i^{2h}\phi^h$ are the piecewise linear interpolation of u^h and ϕ^h at the nodal sets of the nonconforming finite element spaces $\tilde{U}_i^{2h}$ $(i=1,2)$.

Integrating in time, we get

$$\langle \omega_t^h,\, \phi\rangle = \sum_{i=1,2}\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\omega^h\nabla\Pi_i^{2h}u^h\nabla\Pi_i^{2h}\phi^h\, dxdydt$$
$$+\frac{1}{2}\sum_{jk}\left(\frac{h'_{jk}}{h_{jk}}\int_0^T(\phi_j-\phi_k)(\omega_j-\omega_k)|u_k-u_j|\, dt\right).$$

Now, by the standard finite element theory [3], we can get

$$\sum_{j\in I^i}\int_0^T\int_{\Omega_j}|\nabla(\Pi_i^{2h}\phi^h-\phi)|^2\, dxdydt \to 0 \quad \text{as } h\to 0.$$

In addition, for any smooth $v\in C_0^\infty(\Omega_T)$ and its continuous piecewise linear interpolation v^h with respect to the triangulation Σ, by applying the triangle inequality and the Holder's inequality, we get

$$\sum_{j\in I^i}\int_0^T\int_{\Omega_j}|\nabla(\Pi_i^{2h}u^h-u)|^2\, dxdydt$$
$$\le c\left\{\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\left[|\nabla(\Pi_i^{2h}u^h-\Pi_i^{2h}v)|^2+|\nabla(\Pi_i^{2h}v-v)|^2+|\nabla(v-u)|^2\right] dxdydt\right\}$$
$$\le c\|\nabla(u^h-v^h)\|^2_{L^2(\Omega_T)}+c\|\nabla(v-u)\|^2_{L^2(\Omega_T)}+c\sum_{j\in I^i}\int_0^T\int_{\Omega_j}|\nabla(\Pi_i^{2h}v-v)|^2 dxdydt$$
$$\le c\|\nabla(u^h-u)\|^2_{L^2(\Omega_T)}+c\|\nabla(v^h-v)\|^2_{L^2(\Omega_T)}+c\|\nabla(v-u)\|^2_{L^2(\Omega_T)}$$
$$+c\sum_{j\in I^i}\int_0^T\int_{\Omega_j}|\nabla(\Pi_i^{2h}v-v)|^2 dxdydt,$$

where a property such as

$$\|\nabla\Pi_i^{2h}v\|_2\le c\|v^h\|_2 \qquad \text{for} \quad i=0,1,$$

for some constant c, independent of h, has also been used. Thus, due to the strong convergence of u^h to u, we can choose v to be close to u and choose h small enough to have u^h close to u and v^h close to v; then, as $h\to 0$, we get

$$\sum_{j\in I^i}\int_0^T\int_{\Omega_j}|\nabla(\Pi_i^{2h}u^h-u)|^2\, dxdydt\to 0.$$

Thus, as $h\to 0$,

$$\sum_{i=1,2}\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\omega^h\nabla\Pi_i^{2h}u^h\nabla\Pi_i^{2h}\phi^h\, dxdydt=\sum_{i=1,2}\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\omega^h\nabla u\nabla\phi\, dxdydt$$
$$+\sum_{i=1,2}\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\omega^h\left[\nabla(\Pi_i^{2h}u^h-u)\nabla\phi+\nabla\Pi_i^{2h}u^h\nabla(\Pi_i^{2h}\phi^h-\phi)\right] dxdydt$$
$$=\langle\omega^h\nabla u,\,\nabla\phi\rangle+c\|\phi\|_{\mathcal{X}_2}\left\{\sum_{i=1,2}\sum_{j\in I^i}\int_0^T\int_{\Omega_j}\|\nabla(\Pi_i^{2h}u^h-u)\|^2 dxdydt\right\}^{1/2}$$

$$+ c\|u^h\|_{\mathcal{X}_2} \left\{ \sum_{i=1,2} \sum_{j\in I^i} \int_0^T \int_{\Omega_j} |\nabla(\Pi_i^{2h}\phi^h - \phi)|^2 \, dxdydt \right\}^{1/2}$$
$$\to \langle \omega\nabla u, \, \nabla\phi\rangle.$$

By Jensen's inequality and the estimates derived in the previous section, we have

$$\begin{aligned}
&\sum_{jk} \left(\frac{h'_{jk}}{h_{jk}} \int_0^T |\phi_j - \phi_k| \, |\omega_j - \omega_k| \, |u_k - u_j| \, dt \right) \\
&\quad \le \left\{ \sum_{jk} \left(\frac{h'_{jk}}{h_{jk}} \int_0^T |\phi_j - \phi_k|^2 \, |u_k - u_j| \, dt \right) \right\}^{1/2} \\
&\qquad\qquad \cdot \left\{ \sum_{jk} \left(\frac{h'_{jk}}{h_{jk}} \int_0^T |\omega_j - \omega_k|^2 \, |u_k - u_j| \, dt \right) \right\}^{1/2} \\
&\quad \le c \max\{|\phi_j - \phi_k|\}^{1/2} \|\hat{\phi}^h\|_{H^1(\Omega)} \|u^h\|_{H^1(\Omega)} \to 0
\end{aligned}$$

as $h \to 0$. Putting the estimates together, we get

$$\lim_{h\to 0} \langle \omega_t^h, \, \phi\rangle = \langle \omega\nabla u, \, \nabla\phi\rangle. \tag{5.5}$$

Combining (5.4)–(5.5), we get that $\{u, \omega\}$ satisfies (3.1). ☐

By Propositions 5.2 and 5.4, we have completed the proof of the main theorem. As a consequence, we have also obtained the existence of the weak solution to problems (3.1)–(3.2) as a by-product.

COROLLARY 5.5. *For any* $T > 0$, *given* H_1 *and* $\omega_0 \in L^\infty(\Omega)$, *there exists a weak solution to* (3.1)–(3.2) *in* $(\mathcal{X}_\infty \cap \mathcal{W}_{\infty,\infty} \cap \mathcal{Y}) \times (\mathcal{Z} \cap \mathcal{W}_{\infty,\infty})$. ☐

We note that the construction of the coarse finite element spaces relies on the fact that we have a structured triangulation obtained from a uniform rectangular mesh. With the construction, we do not need to use additional (such as BV-type) estimates for the discrete solution ω^h. It remains open as to whether such necessary estimates can be derived for more general discretization of the two-dimensional problems, in particular, on unstructured grids. Similar difficulties have limited the number of conclusions made in [15, 22] on the convergence of numerical methods and the uniqueness of weak solutions to be valid only in the one-dimensional setting. (In a recent work [7], similar convergence results were proved for a numerical scheme based on the nonconforming finite element methods on unstructured grids. Combining the techniques used there with the a priori estimates derived in earlier sections, a convergence proof of the method presented in this paper can be obtained for the unstructured grids as well.)

6. Conclusion. The mathematical models used in the study of superconductivity range from the microscopic BCS model of Bardeen, Cooper, and Schrieffer [1] to the mezoscale Ginzburg–Landau model [16, 25], to the macroscopic models such as the critical state model and the Bean model [2]. While the BCS theory has provided the fundamental understanding of the superconductivity mechanism (at least in the low-T_c regime), the Ginzburg–Landau models have also been very useful in the studies of the vortex phenomena in type-II superconductors with the resolution of individual vortices and vortex lattices [10, 11, 25]. It is worth noting that the separation of individual vortices occurs on a typical length scale of 100 or so Å. A superconducting sample of a dimension of, say, one millimeter, will contain a huge number of vortex-like structures. Even with modern computational methodology, it is still impractical to carry out efficient numerical simulation based on the Ginzburg–Landau model in such a situation. Thus, it remains a challenging problem in scientific computation to perform large-scale simulations of the vortex phenomena for devices of realistic size. A possible alternative, at the scale of devices, is to use continuum models, such as the mean field model considered in this paper that do not attempt to resolve the individual superconducting vortices, but rather that determine an averaged, homogenized quantity such as ω, the density of these vortices.

Once more practical features such as the pinning effect and inhomogeneities are introduced, numerical simulations for the mean field models can be performed for devices with realistic scales. They will be helpful to the design of superconducting devices in gaining information on the flux penetration and vortex dynamics. In this paper, a convergence analysis is given for a semidiscrete hybrid finite element/finite volume/finite difference method of a mean field model in a simple situation where no pinning effect is considered. In addition, no discussion is made here on the nonlinear boundary condition for ω. We plan to study robust numerical methods, their implementation, and their rigorous analysis in more complicated situations in the future.

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