DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **8**, Number **3**, October **2007**

Website: http://AIMsciences.org

pp. **539–556**

ANALYSIS OF A PHASE FIELD NAVIER-STOKES VESICLE-FLUID INTERACTION MODEL

Qiang Du, Manlin Li and Chun Liu

Department of Mathematics
Pennsylvania Sate University
University Park, PA 16802, USA

Abstract. This paper is concerned with the dynamics of vesicle membranes in incompressible viscous fluids. Some rigorous theory are presented for the phase field Navier-Stokes model proposed in [7], which is based on an energetic variation approach and incorporates the effect of bending elasticity energy for the vesicle membranes. The existence and uniqueness results of the global weak solutions are established.

1. **Introduction.** The study of hydrodynamical and rheological properties of fluids involving vesicle membranes and cells is of interest in many biological and physiological applications. Considerable research efforts have been devoted to both experimental studies [1, 11, 13] and the development of mathematical models and computational codes of various degrees of physical relevance and sophistication regarding the membrane properties/configurations and the fluid constitution in recent years [2, 3, 4, 7, 14, 15, 16, 20, 21, 22, 24, 25, 26, 27].

Vesicle membranes are formed by lipid bilayers which play an essential role in biological functions. Their equilibrium shapes are often characterized by minimizing the bending elastic energy of the membrane [12, 19, 23, 28]:

$$E = \int_\Gamma \frac{k}{2}(H - c_0)^2 \, dS \tag{1}$$

where Γ is the surface of vesicle membrane, H is the mean curvature of Γ, c_0 the spontaneous curvature and k the bending modulus. It is known that the behavior of these vesicles, in both static configurations and under external flow fields, dramatically differs from that of those droplets whose shape is governed by the surface tension (with surface energy depending only on the surface area of the membranes). In this paper, we continue our earlier studies [7] and consider the phase field Navier-Stokes model for the vesicle shape dynamics, which is governed by the coupling of the hydrodynamic fluid flow and the bending elastic properties of the vesicle membrane. The resulting membrane configuration and the flow field reflect the competition and the coupling of the kinetic energy and membrane elastic energies.

2000 *Mathematics Subject Classification.* Primary: 58F15.

Key words and phrases. Phase Field, Navier-Stokes, Well-posedness, Weak Solution, Existence, Uniqueness, Vesicle Membrane, Fluid Interaction, Bending Elastic Energy.

The research of Q. Du and M. Li is supported in part by the NSF DMS-0409297, NSF ITR-0205232 and NSF CCF-0430349. The research of C. Liu is supported in part by the NSF DMS-0405850 and DMS-0509094.

In the phase field Navier-Stokes model, the description of the membrane is given in terms of a phase field function ϕ (see [8] and the subsequent works [5, 6, 7, 9, 10, 32] for details). The phase field function ϕ, roughly speaking, is a labeling function defined on computational domain Ω. The function ϕ takes value nearly $+1$ inside the vesicle membrane and -1 outside, with a thin transition layer of width characterized by a small positive parameter ϵ. The zero level surface of ϕ represents the surface of vesicle membrane. The advantage of introducing such a labeling function is to formulate the original Lagrangian description of the membrane evolution in the Eulerian (observer's) coordinates.

As in [8], we will approximate the elastic bending energy (1) by

$$E_\epsilon(\phi) = \frac{k}{2\epsilon}\int_\Omega \left(\epsilon\triangle\phi + (\frac{1}{\epsilon}\phi + c_0\sqrt{2})(1-\phi^2)\right)^2 dx \ .$$

For illustration purposes, the fluids both inside and outside the vesicle are taken to be an incompressible viscous Newtonian fluid and the elastic energy associated with the vesicle deformation mainly comes from the bending energy. We want to point out that it is easy to incorporate other physical considerations of the fluids and the membrane into our energetic variational approach. The vesicle deformation and the fluid velocity field are then regarded as the result of the competition between vesicle membrane bending energy and fluid kinetic energy, subject to the constraints that the volume and surface area of the vesicle are preserved. The equations governing the dynamics of the phase field function ϕ and the fluid velocity field u can be obtained via the energetic variation approach [7]. To enforce the two constraints, one may either adopt the Lagrange multiplier approach or use a penalty formulation [7]. Here, we focus on the latter, that is, we add two penalty terms to the elastic bending energy $E_\epsilon(\phi)$ to enforce the volume and surface area constraints respectively. As in [9] and [7], the modified energy is given by

$$E(\phi) = E_\epsilon(\phi) + \frac{1}{2}M_1(A(\phi)-\alpha)^2 + \frac{1}{2}M_2(B(\phi)-\beta)^2 \ , \tag{2}$$

where

$$A(\phi) = \int_\Omega \phi dx, \quad B(\phi) = \int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2 dx.$$

We define the following corresponding action functional which illustrate the competition between different part of the energies.

$$A[x(t,X)] = \int_0^T \int_\Omega \frac{1}{2}|x_t(t,X)|^2 dx - E(\phi(t,x(t,X)))dt \tag{3}$$

where $x(t,X)$ can be thought as the incompressible fluid trajectory in the Lagrangian coordinate and u being the fluid velocity field. The Least Action principle yields the actual force balance (linear momentum) equation [7]. We are thus led to the following phase field Navier-Stokes equation for ϕ and u:

$$\begin{cases} u_t + u\cdot\nabla u = \nabla p + \mu\triangle u + \frac{\delta E(\phi)}{\delta\phi}\nabla\phi \quad \text{in } [0,T]\times\Omega \ , \\ \text{div } u = 0 \quad \text{in } [0,T]\times\Omega \ , \\ \phi_t + u\cdot\nabla\phi = -\gamma\frac{\delta E(\phi)}{\delta\phi} \quad \text{in } [0,T]\times\Omega \ , \\ u(0,x) = \tilde{u}(x) \quad \text{in } \Omega \ , \\ \phi(0,x) = \tilde{\phi}(x) \quad \text{in } \Omega \end{cases} \tag{4}$$

where $\frac{\delta E(\phi)}{\delta\phi}$, the so-called chemical potential, denotes the variational derivative of $E(\phi)$ in the variable ϕ (its precise form is described later). The above equation is complemented by boundary conditions (BC). The particular BC considered in this paper is of the Dirichlet type for the phase field function ϕ and the no-slip boundary condition for the velocity field u:

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{5}$$

$$\phi = -1,\ \triangle\phi = 0 \quad \text{on } \partial\Omega\,. \tag{6}$$

The main objective of this paper is to provide a rigorous mathematical foundation to the above coupled phase field Navier-Stokes (PFNS) equation. In particular, we present the proof of existence and uniqueness of weak solution to (4). Our results indicate that we can essentially control the coupling between the velocity field and the phase field so that the natural (energy) solution spaces for the PFNS equations remain the same as that for the decoupled conventional incompressible Navier-Stokes equation and the simple phase field gradient flow for the bending elastic energy. We elect to only focus on the case $c_0 = 0$ in this paper, though the proof can be readily extended to the non-zero spontaneous curvature case.

We note that some formal analysis has been given in [7] on the sharp interface limit of the coupled PFNS model as the interfacial width parameter $\epsilon \to 0$. In particular, it is seen that under a general ansatz assumption, the extra term $\frac{\delta E(\phi)}{\delta\phi}\nabla\phi$ in the momentum equation leads to the well-known Willmore force acting between the background fluid and the membrane surface (see also [30]), though the corresponding well-posedness results for the limiting system are still open and under investigation.

2. **Main results and formal estimates.**

In this section, we state our main results concerning the well-posedness of the coupled PFNS model and the properties of their weak solutions.

Throughout the discussion, we use the space $H_d(\Omega)$ to denote the space of divergence free vector fields in $H_0^1(\Omega)$, and $L_d^2(\Omega)$ for the closure of divergence free subset of $C_c^\infty(\Omega)$ in $L^2(\Omega)$. $H_d^{-1}(\Omega)$ denotes the dual space of $H_d(\Omega)$. For notational convenience, for any given time T, we also use spaces like $L^p(0,T;L^q(\Omega))$ for functions of both the time and space variables as defined in [29]. In addition, we use $< \cdot,\cdot >$ to denote the inner product in (and duality pairing with respect to) $L^2(\Omega)$, and we also use the following trilinear form:

$$B(u,v,w) = \int_\Omega u\cdot\nabla v\cdot w\,dx\ . \tag{7}$$

The main results of this paper are the following existence and uniqueness theorems.

Theorem 1. ***Existence of Weak Solution.*** *Let Ω be an open, bounded subset of $\mathbb{R}^3$ either having a smooth boundary or being a convex polyhedra. There exists a pair of functions ϕ and u with*

1. $u \in L^2(0,T;H_d(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;H_d^{-1}(\Omega))$
2. $\phi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$

which is a weak solution to equation (4) with boundary condition (5-6), that is,

1. *for any* $\delta(x) \in H_d(\Omega)$, $\xi(x) \in L^2(\Omega)$, *and a.e.* $t \in [0,T]$, *we have*

$$\left\{ \begin{array}{lcl} < u_t, \delta > + B(u,u,\delta) & = & -\mu < \nabla u, \nabla \delta > + \int\limits_{\Omega} \frac{\delta E(\phi)}{\delta \phi} \nabla \phi \cdot \delta \, dx \, , \\ < \phi_t, \xi > + B(u,\phi,\xi) & = & -\gamma < \frac{\delta E(\phi)}{\delta \phi}, \xi > \ . \end{array} \right. \tag{8}$$

2. $u(0,x) = \tilde{u}(x)$, $\phi(0,x) = \tilde{\phi}(x)$ *where* $\tilde{u} \in L^2_d(\Omega)$ *and* $\tilde{\phi} + 1 \in H^2_0(\Omega)$.

Theorem 2. ***Uniqueness of Weak Solution.*** *For the weak solutions to equation (4) discussed in the previous existence theorem, if in addition we have the solution satisfying* $u \in L^8(0,T;L^4(\Omega))$, *then the weak solution is unique.*

A few remarks are first in order. First, we adopt suitable assumptions on the domain so that we can obtain the H^2 regularity for the Laplace operator with homogeneous boundary condition. Second, due to the standard theory for the conventional Navier-Stokes equations without the membrane stress [29] and the simple L^2 gradient flow of the elastic bending energy without the fluid transport [31], it is easy to see that the main task at hand is to analyze the coupling terms in the PFNS equation, which has similar spirits as that in the study of coupled systems for fluid and liquid crystal director [17]. Therefore, we need to consider (control) the contribution to the momentum equation of the additional stress tensor due to the membrane deformation and the contribution of the convection term to the phase field evolution. With the energy law established below, it turns out that the solution space $L^2(0,T;H_d(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;H_d^{-1}(\Omega))$ for the velocity field u remains the same as that for the conventional three dimensional incompressible Navier-Stokes equations. This reflects the fact that the membrane stress tensor does not pose any extra limitation on the regularity of the weak solution of the velocity field. Meanwhile, the solution space $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ for ϕ also coincides with the natural space for the solution of the simple L^2 gradient flow of the elastic bending energy, again showing that the effect of fluid transport on the phase field function can also be properly controlled. Third, due to the limited regularity in u, the issue of uniqueness of the weak solution remains open even for the conventional Navier-Stokes equations in three dimension without the membrane effect, thus, we do not have the proof of uniqueness of the global weak solutions for the coupled PFNS equation in general. However, with a better regularity assumption on the weak solutions, as in the case of the conventional Navier-Stokes equations [29], the uniqueness can be assured.

Before we turn to the proofs, let us also mention that we have chosen to work with the penalty formulation to incorporate the volume and surface area conservation of the vesicle membrane in time. The results presented here are for given penalty constants, so the constraints are satisfied only approximately. Careful examination of our proofs indicates that much of the estimates derived in the paper can be made to be independent of the penalty constants, thus allowing one to extract a suitable limit as the penalty constants approach infinity. Such a limiting process would lead the existence of solutions in the Lagrange multiplier formulation with the constraints exactly satisfied. However, a detailed account of the dependence and independence of the estimates on the penalty constants would require tedious book-keeping notation-wise. To avoid complication, we elect to ignore such dependence in the later presentation.

The detailed proofs of the above theorems are divided into several parts which are given in the later sections. The main steps include establishing solution estimates from the energy law and passing to the limit via a modified Galerkin procedure. For a detailed account of similar techniques, we refer to the discussion on the conventional incompressible Navier-Stokes equation given in [29]. We note that though we have used the boundary condition (5-6), much of our analysis can be carried out in other cases as well, such as the Neumann and periodic boundary conditions, and in particular, the case of an inhomogeneous velocity profile on the boundary which is often used in the study of cell deformation in shear and/or extensional flows.

2.1. **Formal derivation of the energy law.** The dissipation of the kinetic energy is one of the most basic property of the conventional incompressible Navier-Stokes equations, and it can be easily derived. It is thus interesting to note that a similar energy law holds for the coupled PFNS equation, with the membrane bending elastic energy being added to the kinetic energy to produce the total energy.

For convenience, let us denote

$$f(\phi) = -\epsilon\triangle\phi + \frac{1}{\epsilon}(\phi^2-1)\phi\ , \quad g(\phi) = -\triangle f(\phi) + \frac{1}{\epsilon^2}(3\phi^2-1)f(\phi)\ .$$

Then, we may rewrite the energy as

$$E(\phi) = \frac{k}{2\epsilon}\int_\Omega |f(\phi)|^2 dx + \frac{1}{2}M_1(A(\phi)-\alpha)^2 + \frac{1}{2}M_2(B(\phi)-\beta)^2\ .$$

The direct computation shows

$$\frac{\delta E(\phi)}{\delta\phi} = kg(\phi) + M_1(A(\phi)-\alpha) + M_2(B(\phi)-\beta)f(\phi)\ . \tag{9}$$

We now give a formal derivation of the energy law for smooth solutions u and ϕ of (4). Multiply u to the first equation in (4) and $\frac{\delta E(\phi)}{\delta\phi}$ to the second equation, then integrate over Ω, we get the following dissipative energy law

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|u|^2 dx + E(\phi)\right) = -\mu\int_\Omega |\nabla u|^2 dx - \gamma\int_\Omega \left|\frac{\delta E(\phi)}{\delta\phi}\right|^2 dx\ . \tag{10}$$

Immediately one can conclude that if u and ϕ are the solutions of the PFNS, we have the uniform bounds (with respect to any $T>0$) of the following type:

- $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_d(\Omega))$;
- $\phi \in L^\infty(0,T;H^2(\Omega))$;
- $f(\phi) \in L^\infty(0,T;L^2(\Omega))$;
- $\frac{\delta E(\phi)}{\delta\phi} \in L^2((0,T)\times(\Omega))$.

For weak solutions of (4), we can derive a weaker version of the energy law rigorously via a Galerkin procedure outlined below.

2.2. **Formal estimates of $\|u_t\|_{H_d^{-1}(\Omega)}$ and $\|\phi_t\|_{L^2(\Omega)}$.** Based on the bounds from the energy law and the PFNS equation, we may deduce better estimates on the time derivatives.

1. Let $v \in L^2(\Omega)$ and $\|v\|_{L^2(\Omega)} \leq 1$, from equation (4),

$$\left|\int_\Omega \phi_t v\, dx\right| \leq \left|\int_\Omega u\,\nabla\phi\, v\, dx\right| + \gamma\left|\int_\Omega \frac{\delta E(\phi)}{\delta\phi} v\, dx\right|\ .$$

And,

$$\left|\int_\Omega u\,\nabla\phi\, v\,dx\right| \le C\|v\|_{L^2(\Omega)}\|\nabla\phi\|_{L^3(\Omega)}\|u\|_{L^6(\Omega)} \le C'\|\phi\|_{H^2(\Omega)}\|u\|_{H_0^1(\Omega)}\ ,$$

$$\left|\int_\Omega \frac{\delta E(\phi)}{\delta\phi} v dx\right| \le \|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\ .$$

Therefore,

$$\|\phi_t\|_{L^2(\Omega)} \le C\left(\|\phi\|_{H^2(\Omega)}\|u\|_{H_0^1(\Omega)} + \|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\right).$$

2. We now take $v \in H_d(\Omega)$. Also from equation (4), we get

$$\begin{aligned}\left|\int_\Omega u_t\cdot v\,dx\right| &\le \left|\int_\Omega u\cdot\nabla u\cdot v\,dx\right| + \left|\int_\Omega \frac{\delta E(\phi)}{\delta\phi}\nabla\phi\cdot v\,dx\right| \\ &\quad +\mu\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\ , \\ \|u_t\|_{H_d^{-1}(\Omega)} &\le \|u\cdot\nabla u\|_{H^{-1}(\Omega)} + \mu\|\nabla u\|_{L^2(\Omega)} + \|\frac{\delta E(\phi)}{\delta\phi}\nabla\phi\|_{H^{-1}(\Omega)}\ .\end{aligned}$$

By a well known interpolation result [29], we have,

$$\|u\cdot\nabla u\|_{H^{-1}(\Omega)} \le C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u\|^{\frac{3}{2}}_{L^2(\Omega)}\ .$$

As for $\|\frac{\delta E(\phi)}{\delta\phi}\nabla\phi\|_{H^{-1}(\Omega)}$, we consider for any $v \in H_0^1(\Omega)$, with $\|v\|_{H_0^1(\Omega)} \le 1$. By the Sobolev inequality $\|v\|_{L^6(\Omega)} \le C\|v\|_{H_0^1(\Omega)}$, we get

$$\begin{aligned}\left|\int_\Omega \frac{\delta E(\phi)}{\delta\phi}\nabla\phi\cdot v\,dx\right| &\le \|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^3(\Omega)}\|v\|_{L^6(\Omega)} \\ &\le K_1\|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^3(\Omega)} \\ &\le K_2\|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\|\phi\|_{H^2(\Omega)}\ .\end{aligned}$$

Hence,

$$\|u_t\|_{H_d^{-1}(\Omega)} \le C\left(\|\nabla u\|^{\frac{3}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\frac{\delta E(\phi)}{\delta\phi}\|_{L^2(\Omega)}\|\phi\|_{H^2(\Omega)}\right)\ .$$

3. **Proof the existence of weak solutions.** This section is devoted to the proof of existence of weak solutions of the coupled PFNS equation (4) with boundary condition (5-6). We first outline a modified Galerkin approximation, then we consider its weak limit and verify it as a weak solution of (4).

3.1. **A modified Galerkin approximation.** Let us choose $\{\omega_n\} \subset L^2_d(\Omega)$ to be the eigenfunctions of Stokes operator, such that $\{\omega_n\}$ forms an orthonormal basis for $L^2_d(\Omega)$. Set $W_n = \text{Span}\{\omega_1, \omega_2, \ldots, \omega_n\}$. Apply the Galerkin approximation to the velocity field u, one can get the approximate equation for $u \in W_n$ and $\phi \in H^2(\Omega)$:

$$\begin{cases} u_t + \mathcal{P}_n(u \cdot \nabla u) & = & \mu \triangle u + \mathcal{P}_n(\frac{\delta E(\phi)}{\delta \phi} \nabla \phi) \ , \\ \phi_t + u \cdot \nabla \phi & = & -\gamma \frac{\delta E(\phi)}{\delta \phi} \ , \\ u(0) = \mathcal{P}_n \tilde{u}(x) \ , \\ \phi(0) = \tilde{\phi}(x) \end{cases} \tag{11}$$

where $\mathcal{P}_n$ is the L^2 projection operator to W_n. The following Lemma asserts the existence of a solution to the approximate equation (11). It also provides a uniform energy estimate on the solution (with respect to the dimension n).

Lemma 1. ***Existence of An Approximate Solution*** *There exists a pair of functions $u(t,x) \in W_n$, $\phi(t,x) \in H^2(\Omega)$ satisfying*

$$\begin{cases} < u_t, w > + B(u, u, w) & = & -\mu < \nabla u, \nabla w > + \int\limits_\Omega \frac{\delta E(\phi)}{\delta \phi} \nabla \phi \cdot w \, dx \ , \\ < \phi_t, v > + B(u, \phi, v) & = & -\gamma < \frac{\delta E(\phi)}{\delta \phi}, v > \ , \\ u(0) = \mathcal{P}_n \tilde{u}(x) \\ \phi(0) = \tilde{\phi}(x) \ , \end{cases} \tag{12}$$

$\forall w \in W_n$ and $v \in L^2(\Omega)$ for almost all $t \in [0,T]$. Furthermore, for a.e. $\hat{T} \in [0,T]$,

$$\int_0^{\hat{T}} \int_\Omega \mu |\nabla u|^2 + \gamma \left|\frac{\delta E(\phi)}{\delta \phi}\right|^2 dx dt + \int_\Omega \frac{1}{2} |u(\hat{T}, x)|^2 dx + E(\phi(\hat{T}, x)) \leq M \ ,$$

with a constant M independent of W_n.

Proof. 1. **Apply Galerkin approximation to ϕ**

Denote by $\{\upsilon_1, \upsilon_2, \ldots\}$ the eigenfunctions of operator $\triangle$ under the homogeneous Dirichlet boundary condition. They consist of an orthonormal basis of $L^2(\Omega)$. By the assumption on the domain Ω, we have that the eigenfunctions of the Laplace operator have H^2 regularity. Set $V_m = \text{Span}\{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$. Apply a modified Galerkin method to ϕ, we get an approximate equation to equation (12) as follows: find $u_m(x,t)$ and $\phi_m(x,t)$ of the form

$$u_m(x,t) = \sum_{i=1}^n d_i(t) \omega_i(x) \in W_n \quad \text{and} \quad \phi_m(x,t) + 1 = \sum_{j=1}^m h_j(t) \upsilon_j(x) \in V_m$$

such that for $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$,

$$\begin{cases} < u_m', \omega_k > + B(u_m, u_m, \omega_k) & = & -\mu < \nabla u_m, \nabla w_k > \\ & & + \int\limits_\Omega \pi_m(\frac{\delta E(\phi_m)}{\delta \phi_m}) \nabla \phi_m \cdot \omega_k \, dx \ , \\ < \phi_m', \upsilon_l > + B(u_m, \phi_m, \upsilon_l) & = & -\gamma \int\limits_\Omega \frac{\delta E(\phi_m)}{\delta \phi_m} \upsilon_l \, dx \ , \\ u_m(0,x) & = & \mathcal{P}_n \tilde{u}(x) \ , \\ \phi_m(0,x) & = & \pi_m(\tilde{\phi}(x) + 1) - 1 \ . \end{cases} \tag{13}$$

Here $'$ means differentiating in time, and π_m denotes a L^2 (also H^2 if applicable) projection to V_m. The solutions have natural dependence on the index n, but for convenience, we have suppressed this dependence in the notation of u_m and ϕ_m.

It is easy to see that the above finite dimensional ODE system has a solution local in time.

2. **Energy estimate**

In equation (13), replace w_k with u_m, υ_l with $\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})$, we have,

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int\limits_\Omega |u_m|^2dx\right) &= -\mu < \nabla u_m, \nabla u_m > + \int\limits_\Omega \pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\nabla\phi_m\cdot u_m\,dx \\ & \qquad < \phi_m', \pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m}) > + \int\limits_\Omega \pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\nabla\phi_m\cdot u_m\,dx \\ &= -\gamma\int\limits_\Omega \frac{\delta E(\phi_m)}{\delta\phi_m}\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})dx\ .\end{aligned}$$

Since

$$< \phi_m', \pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m}) > = < \pi_m(\phi_m'), \frac{\delta E(\phi_m)}{\delta\phi_m} > = < \phi_m', \frac{\delta E(\phi_m)}{\delta\phi_m} > = \frac{d}{dt}E(\phi_m),$$

the summation of the two expressions above gives the energy law,

$$\frac{d}{dt}\left(\frac{1}{2}\int\limits_\Omega |u_m|^2dx + E(\phi_m)\right) = -\mu\int\limits_\Omega |\nabla u_m|^2dx - \gamma\int\limits_\Omega \left|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\right|^2 dx\ . \tag{14}$$

It implies,

$$\begin{aligned}&\frac{1}{2}\int\limits_\Omega |u_m(\hat{T},x)|^2dx + E(\phi_m(\hat{T},x)) + \int\limits_0^{\hat{T}}\int\limits_\Omega \mu|\nabla u_m|^2 \\ &+\gamma\left|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\right|^2 dxdt \le \frac{1}{2}\int\limits_\Omega |u_m(0,x)|^2dx + E(\phi_m(0,x))\ .\end{aligned}$$

We know that $\|u_m(0,x)\|_{L^2(\Omega)} \le \|\tilde{u}(x)\|_{L^2(\Omega)}$. By the construction of $\phi_m(0,x)$, we also have that $\phi_m(0,x)$ converges to $\tilde{\phi}(x)$ in $H^2(\Omega)$ as $m\to\infty$. Hence there exists some constant M independent of W_n, such that,

$$\begin{aligned}&\frac{1}{2}\int\limits_\Omega |u_m(\hat{T},x)|^2dx + E(\phi_m(\hat{T},x)) + \int\limits_0^{\hat{T}}\int\limits_\Omega \mu|\nabla u_m|^2dxdt \\ &+\int\limits_0^{\hat{T}}\int\limits_\Omega \gamma\left|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\right|^2 dxdt \le M\ .\end{aligned}$$

Note that such an energy law essentially gives the existence of local solutions for all time.

3. **Compactness of** $\{u_m\}$ **and** $\{\phi_m\}$

The energy law ensures that $\|u_m(t)\|_{L^2(\Omega)}$ and $\|\phi_m(t)\|_{L^2(\Omega)}$ are uniformly bounded in m and $t\in[0,T]$. Thus the solution of ODE (13) actually exists global in time. Furthermore, the energy law also indicates

- u_m is uniformly bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_d(\Omega))$.
- ϕ_m is uniformly bounded in $L^\infty(0,T;H^2(\Omega))$ (thus in $L^2(0,T;H^2(\Omega))$)

- $\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})$ is uniformly bounded in $L^2((0,T)\times\Omega)$.

Similar to the previous formal derivation, we can rigorously obtaining estimates on $\|u_m'\|_{H_d^{-1}(\Omega)}$ and $\|\phi_m'\|_{L^2(\Omega)}$:

- u_m' is uniformly bounded $L^{\frac{4}{3}}(0,T;H_d^{-1}(\Omega))$.
- ϕ_m' is uniformly bounded $L^2((0,T)\times\Omega)$.

Therefore, using the Aubin-Lions type compact embedding results [29], there exist some ϕ, u, such that,

- u_m has a subsequence u_{m_k} converging to u weakly in $L^2(0,T;H_d(\Omega))$ and strongly in $L^2(0,T;L^2_d(\Omega))$.
- ϕ_m has a subsequence ϕ_{m_l} converging to ϕ weakly in $L^2(0,T;H^2(\Omega))$ and strongly in $L^2(0,T;W^{1,p}(\Omega))$ for $1\le p<6$.

For convenience, if there is no ambiguity, from now on we will identify u_m and ϕ_m with their subsequences.

4. **Passing weak limits of $\{u_m\}$ and $\{\phi_m\}$**

Choose $w(t,x)=\alpha(t)\delta(x)$, $v(t,x)=\alpha(t)\xi(x)$ where $\alpha\in C([0,T])$, $\delta\in W_n$, and $\xi\in V_m$, we have

$$\begin{aligned}\int_0^T <u_m',w>+B(u_m,u_m,w)\,dt &= \int_0^T <\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\nabla\phi_m,w>\ dt\\ &\quad -\int_0^T \mu<\nabla u_m,\nabla w>\ dt\\ \int_0^T <\phi_m',v>+B(u_m,\phi_m,v) &= -\gamma\int_0^T <\frac{\delta E(\phi_m)}{\delta\phi_m},v>\ dt\,.\end{aligned}$$

In which,

$$\begin{aligned}\frac{\delta E(\phi_m)}{\delta\phi_m} &= k\{-\triangle f(\phi_m)+\frac{1}{\epsilon^2}(3\phi_m^2-1)f(\phi_m)\}\\ &\quad +M_1(A(\phi_m)-\alpha)+M_2(B(\phi_m)-\beta)f(\phi_m)\\ &= k\{-\triangle[-\epsilon\triangle\phi_m+\frac{1}{\epsilon}(\phi_m^2-1)\phi_m]+\frac{1}{\epsilon^2}(3\phi_m^2-1)f(\phi_m)\}\\ &\quad +M_1(A(\phi_m)-\alpha)+M_2(B(\phi_m)-\beta)f(\phi_m) \qquad (15)\\ &= \epsilon k\triangle^2\phi_m+L(\phi_m)\end{aligned}$$

where $L(\phi_m)$ denotes the lower order term. Note that the $\|L(\phi_m)\|_{L^2((0,T)\times\Omega)}$ is uniformly bounded by the uniform bound on ϕ_m in $L^\infty(0,T;H^2(\Omega))$. Then,

$$\begin{aligned}\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m}) &= \pi_m(\epsilon k\triangle^2\phi_m)+\pi_m(L(\phi_m))\\ &= \epsilon k\triangle^2\phi_m+\pi_m(L(\phi_m))\,.\end{aligned}$$

Together with the energy estimate (14), we have

$$\begin{aligned}&\|\epsilon k\triangle^2\phi_m\|_{L^2((0,T)\times\Omega)}\\ &\le \|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\|_{L^2((0,T)\times\Omega)}+\|\pi_m(L(\phi_m))\|_{L^2((0,T)\times\Omega)}\\ &\le \|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\|_{L^2((0,T)\times\Omega)}+\|L(\phi_m)\|_{L^2((0,T)\times\Omega)}\,.\end{aligned}$$

Therefore,

$$\|\frac{\delta E(\phi_m)}{\delta\phi_m}\|_{L^2((0,T)\times\Omega)} \leq \|\pi_m(\frac{\delta E(\phi_m)}{\delta\phi_m})\|_{L^2((0,T)\times\Omega)} + 2\|L(\phi_m)\|_{L^2((0,T)\times\Omega)} \ .$$

(a) **Weak limit of $\{\frac{\delta E(\phi_m)}{\delta\phi_m}\}$.**

We now prove that (a subsequence of) $\{\frac{\delta E(\phi_m)}{\delta\phi_m}\}$ converges to $\frac{\delta E(\phi)}{\delta\phi}$ weakly in $L^2((0,T)\times\Omega)$. By the energy law (14), $f(\phi_m)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, hence uniformly bounded in $L^2((0,T)\times\Omega)$. It is enough to show for $\forall g\in C_0^\infty([0,T]\times\Omega)$,

$$\begin{aligned}\lim_{m\to\infty}\int_0^T\int_\Omega f(\phi_m)g\,dxdt &= \int_0^T\int_\Omega f(\phi)g\,dxdt\ .\\ \int_0^T\int_\Omega f(\phi_m)g\,dxdt &= \int_0^T\int_\Omega[-\epsilon\triangle\phi_m+\frac{1}{\epsilon}(\phi_m^2-1)\phi_m]g\,dxdt\ .\end{aligned}$$

It is sufficient to check on the nonlinear terms only. Since ϕ_m is uniformly bounded in $L^\infty(0,T;H^2(\Omega))$, we have

$$\|\phi_m(t)\|_{C^{0,\frac{1}{2}}(\Omega)} \leq C\|\phi_m(t)\|_{H^2(\Omega)} \leq M$$

for $t\in[0,T]$. Hence, $\phi_m(t,x)$ is uniformly bounded in $[0,T]\times\Omega$. Furthermore, ϕ_m has as subsequence converging to ϕ strongly in $L^2(0,T,W^{1,p})$ for $1\leq p<6$. Then a subsequence of ϕ_m converges to ϕ almost everywhere in $[0,T]\times\Omega$. By the Lebesgue-Dominated Theorem,

$$\lim_{n\to\infty}\int_0^T\int_\Omega \phi_m^3 g\,dxdt = \int_0^T\int_\Omega \phi^3 g\,dxdt\ .$$

Similarly, we need to verify that

$$\lim_{m\to\infty}\int_0^T\int_\Omega \frac{\delta E(\phi_m)}{\delta\phi_m}g\,dxdt = \int_0^T\int_\Omega \frac{\delta E(\phi)}{\delta\phi}g\,dxdt\ . \tag{16}$$

To give more details, we follow (15) term by term. First, we have

$$\begin{aligned}\int_0^T\int_\Omega \triangle^2\phi_m g\,dxdt &= \int_0^T\int_\Omega \triangle\phi_m\triangle g\,dxdt \to \int_0^T\int_\Omega \triangle\phi\triangle g\,dxdt\\ &= \int_0^T\int_\Omega \triangle^2\phi g\,dxdt\end{aligned}$$

as $m\to\infty$.

Next, we have

$$\begin{aligned}&\left|\int_0^T\int_\Omega \triangle(\phi_m^3-\phi^3)g\,dxdt\right| = 3\left|\int_0^T\int_\Omega(\phi_m^2\nabla\phi_m-\phi^2\nabla\phi)\cdot\nabla g\,dxdt\right|\\ &\leq C\left|\int_0^T\int_\Omega(\phi_m^2-\phi^2)\nabla\phi_m\nabla g\,dxdt\right| + C\left|\int_0^T\int_\Omega\phi^2(\nabla\phi-\nabla\phi_m)\nabla g\,dxdt\right|\\ &\leq C'\|\phi_m^2-\phi^2\|_{L^2((0,T)\times\Omega)}\|\nabla\phi_m\|_{L^2((0,T)\times\Omega)} + C'\|\nabla\phi-\nabla\phi_m\|_{L^1((0,T)\times\Omega)}\ .\end{aligned}$$

Hence, as $m \to \infty$,

$$\left| \int_0^T \int_\Omega \triangle(\phi_m^3 - \phi^3) g \, dxdt \right| \to 0 \, .$$

Now, consider

$$\left| \int_0^T \int_\Omega (\phi_m^2 f(\phi_m) - \phi^2 f(\phi)) g \, dxdt \right| \le \left| \int_0^T \int_\Omega (f(\phi_m) - f(\phi)) \phi^2 g \, dxdt \right| + \left| \int_0^T \int_\Omega (\phi^2 - \phi_m^2) g f(\phi_m) dxdt \right| = I_1 + I_2 \, .$$

We have $I_1 \to 0$ since $f(\phi_m) \rightharpoonup f(\phi)$ weakly in $L^2((0,T)\times\Omega)$. In addition,

$$I_2 \le \|g\|_{L^\infty((0,T)\times\Omega)} \|f(\phi_m)\|_{L^2((0,T)\times\Omega)} \|\phi^2 - \phi_m^2\|_{L^2((0,T)\times\Omega)} \to 0 \, .$$

It is also easy to show,

$$\lim_{m\to\infty} \int_0^T \int_\Omega B(\phi_m) f(\phi_m) g \, dxdt = \int_0^T \int_\Omega B(\phi) f(\phi) g \, dxdt \, .$$

(b) **Verifying the approximate equation**.
Choose $g \in L^2((0,T) \times \Omega)$. Since $\frac{\delta E(\phi_m)}{\delta \phi_m}$ weakly converges to $\frac{\delta E(\phi)}{\delta \phi}$ and $\pi_m(g)$ converges strongly to g in $L^2((0,T) \times \Omega)$, we have

$$\begin{aligned}
&\left| \int_0^T < \pi_m(\frac{\delta E(\phi_m)}{\delta \phi_m}) - \frac{\delta E(\phi)}{\delta \phi}, g > dt \right| \\
&= \left| \int_0^T < \frac{\delta E(\phi_m)}{\delta \phi_m}, \pi_m(g) > - < \frac{\delta E(\phi)}{\delta \phi}, g >) \, dt \right| \\
&\le \left| \int_0^T < \frac{\delta E(\phi_m)}{\delta \phi_m} - \frac{\delta E(\phi)}{\delta \phi}, g > \, dt \right| + \left| \int_0^T < \frac{\delta E(\phi_m)}{\delta \phi_m}, \pi_m(g) - g > \, dt \right| \\
&\to 0 \, ,
\end{aligned}$$

as $m \to \infty$, We can conclude that $\pi_m(\frac{\delta E(\phi_m)}{\delta \phi_m})$ converges to $\frac{\delta E(\phi)}{\delta \phi}$ weakly in $L^2((0,T) \times \Omega)$.
Now, one can let $m \to \infty$ to recover

$$\begin{aligned}
\int_0^T < u_t, w > + B(u,u,w) \, dt &= - \int_0^T \mu < \nabla u, \nabla w > \, dt \\
&\quad + \int_0^T \int_\Omega \frac{\delta E(\phi)}{\delta \phi} \nabla \phi \cdot w \, dx \, dt \, , \\
\int_0^T < \phi_t, v > + B(u,\phi,v) &= -\gamma \int_0^T < \frac{\delta E(\phi)}{\delta \phi}, v > \, dt \, .
\end{aligned}$$

And,

$$\frac{1}{2}\int_\Omega |u(\hat{T},x)|^2 dx + E(\phi(\hat{T},x)) + \int_0^{\hat{T}}\int_\Omega \left(\mu|\nabla u|^2 + \gamma\left|\frac{\delta E(\phi)}{\delta\phi}\right|^2\right) dxdt$$
$$\le M\ .$$

Since $\alpha = \alpha(t)$ is arbitrarily chosen in $C([0,T])$, one can conclude for any $\delta \in W_n$ and $\xi \in V_m$,

$$\begin{cases} < u_t, \delta > + B(u,u,\delta) &= -\mu < \nabla u, \nabla\delta > + \int_\Omega \frac{\delta E(\phi)}{\delta\phi}\nabla\phi\cdot\delta\, dx\ , \\ < \phi_t, \xi > + B(u,\phi,\xi) &= -\gamma\int_\Omega \frac{\delta E(\phi)}{\delta\phi}\xi\, dx\ . \end{cases} \tag{17}$$

By density argument, it is also true for any $\delta \in W_n$ and $\xi \in L^2(\Omega)$. Setting $\alpha(0) = 1$ and $\alpha(T) = 0$, one can show that $u(0,x) = \mathcal{P}_n\tilde{u}(x)$ and $\phi(0,x) = \tilde{\phi}(x)$.

Finally, since for $i > n$

$$\int_0^T \int_\Omega u_m(t,x)\zeta(t)\omega_i(x)\, dx\, dt = 0$$

for any $\zeta(t) \in C([0,T])$, we have by taking $m \to \infty$ that

$$\int_\Omega u(t,x)\omega_i(x)\, dx = 0$$

for almost all $t \in [0,T]$ when $i > n$. Therefore $u \in W_n$. This completes the proof of this lemma.

□

3.2. **Proof of the existence theorem.** We now wrap up the proof of the existence theorem. According to Lemma 1, for any positive time $\hat{T} \in (0,T)$, and for each W_n, the equation (11) has a solution u_n and ϕ_n, such that

$$\int_0^{\hat{T}}\int_\Omega \left(\mu|\nabla u_n|^2 + \gamma|\frac{\delta E(\phi_n)}{\delta\phi_n}|^2\right) dxdt + \int_\Omega \frac{1}{2}|u_n(\hat{T},x)|^2 dx + E(\phi_n(\hat{T},x)) \le M \tag{18}$$

where M is independent of W_n. Hence,

- u_n is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_d(\Omega))$;
- ϕ_n is uniformly bounded in $L^\infty(0,T;H^2(\Omega))$ (thus in $L^2(0,T;H^2(\Omega))$);
- $\frac{\delta E(\phi_n)}{\delta\phi_n}$ is uniformly bounded in $L^2((0,T)\times\Omega)$.

Also,

- u'_n is uniformly bounded $L^{\frac{4}{3}}(0,T;H_d^{-1}(\Omega))$;
- ϕ'_n is uniformly bounded $L^2(0,T;L^2(\Omega))$.

Therefore, there exist some ϕ and u, such that,

- u_n has a subsequence u_{n_k} converging to u weakly in $L^2(0,T;H_d(\Omega))$ and strongly in $L^2((0,T)\times\Omega)$;
- ϕ_n has a subsequence ϕ_{n_l} converging to ϕ weakly in $L^2(0,T;H^2(\Omega))$ and strongly in $L^2(0,T;W^{1,p}(\Omega))$ for $1 \le p < 6$.

Similar to the claim in Lemma 1, (a subsequence of) $\frac{\delta E(\phi_n)}{\delta \phi_n}$ converges weakly to $\frac{\delta E(\phi)}{\delta \phi}$ in $L^2((0,T)\times\Omega)$.

Choose $w(t,x)=\alpha(t)\delta(x)$, $v(t,x)=\alpha(t)\xi(x)$ where $\alpha\in C([0,T])$, $\delta\in W_n$, and $\xi\in C(\Omega)$, then consider

$$\int_0^T < u_n', w > + B(u_n,u_n,w)\,dt = \int_0^T < \frac{\delta E(\phi_n)}{\delta\phi_n}\phi_n, w > \,dt - \int_0^T \mu < \nabla u_n, \nabla w > \,dt$$

$$\int_0^T < \phi_n', v > + B(u_n,\phi_n,v) = -\gamma\int_0^T < \frac{\delta E(\phi)}{\delta\phi}, v > \,dt\ .$$

Let $n\to\infty$, we get,

$$\int_0^T < u_t, w > + B(u,u,w)\,dt = \int_0^T < \frac{\delta E(\phi)}{\delta\phi}\phi, w > \,dt - \int_0^T \mu < \nabla u, \nabla w > \,dt\ ,$$

$$\int_0^T < \phi_t, v > + B(u_n,\phi,v)\,dt = -\gamma\int_0^T < \frac{\delta E(\phi)}{\delta\phi}, v > \,dt\ .$$

Because $\alpha(t)$ is an arbitrarily chosen function in $C([0,T])$, one can conclude for any $\delta\in W_n$, $\xi\in C(\Omega)$,

$$\begin{cases} < u_t, \delta > + B(u,u,\delta) &= -\mu < \nabla u, \nabla\delta > + \int_\Omega \frac{\delta E(\phi)}{\delta\phi}\nabla\phi\cdot\delta\,dx\ , \\ < \phi_t, \xi > + B(u,\phi,\xi) &= -\gamma < \frac{\delta E(\phi)}{\delta\phi}, \xi > \end{cases}$$

By a density argument, it is also true for any $\delta\in H_0^1(\Omega)$ and $\xi\in L^2(\Omega)$. Set $\alpha(0)=1$ and $\alpha(T)=0$, one can show $u(0,x)=\tilde{u}(x)$, and $\phi(0,x)=\tilde{\phi}(x)$. This concludes the proof of the main existence theorem.

4. **Uniqueness of weak solution.**

We now provide the proof of the uniqueness of the weak solution, under the additional regularity assumption on the velocity field.

We first introduce a few notations to simplify the later discussion. Define

$$G(\phi)=\frac{1}{2}\int_\Omega \left(k\epsilon|\triangle\phi|^2+\frac{k}{\epsilon}|\nabla\phi|^2+|\phi|^2\right)dx\ ,$$

then it is easy to check that $\frac{1}{C}\|\phi\|^2_{H^2(\Omega)}\le G(\phi)\le C\|\phi\|^2_{H^2(\Omega)}$.

Define also

$$M(\phi)=\frac{\delta G(\phi)}{\delta\phi}=k\epsilon\triangle^2\phi-\frac{k}{\epsilon}\triangle\phi+\phi,\quad N(\phi)=\frac{\delta E(\phi)}{\delta\phi}-M(\phi)\ .$$

Assume u_i, ϕ_i and p_i $(i=1,2)$ are two weak solutions to equation (4) satisfying the assumptions given in the uniqueness theorem. Let $\hat{u}=u_1-u_2$, $\hat{\phi}=\phi_1-\phi_2$ and $\hat{p}=p_1-p_2$. We derive a Gronwall type inequality for $\hat{u}$ and $\hat{\phi}$ to prove the uniqueness.

4.1. **Derivation of a Gronwall inequality.**

First, we have,

$$\begin{cases} \hat{u}'+\hat{u}\nabla u_1+u_2\nabla\hat{u} &= \nabla\hat{p}+\mu\triangle\hat{u}+(M(\phi_1)+N(\phi_1))\nabla\phi_1 \\ & \quad -(M(\phi_2)+N(\phi_2))\nabla\phi_2\ , \\ \hat{\phi}'+u_1\nabla\phi_1-u_2\nabla\phi_2 &= -\gamma(M(\hat{\phi})+N(\phi_1)-N(\phi_2))\ . \end{cases} \tag{19}$$

Multiply $\hat{u}$ to the first equation in (19) and $M(\hat{\phi})$ to the second one, integrate in space, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\hat{u}|^2 dx + \int_\Omega \hat{u}\nabla u_1 \hat{u} dx = -\mu \int_\Omega |\nabla \hat{u}|^2 dx$$
$$+ \int_\Omega (M(\phi_1)\nabla\phi_1 - M(\phi_2)\nabla\phi_2)\hat{u} + \int_\Omega (N(\phi_1)\nabla\phi_1 - N(\phi_2)\nabla\phi_2)\hat{u} dx \ ,$$

$$\frac{d}{dt}G(\hat{\phi}) + \int_\Omega (u_1\nabla\phi_1 - u_2\nabla\phi_2)M(\hat{\phi})dx = -\gamma\int_\Omega |M(\hat{\phi})|^2 dx - \gamma \int_\Omega M(\hat{\phi})(N(\phi_1) - N(\phi_2))dx$$

Hence,

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|\hat{u}|^2 dx + G(\hat{\phi})\right) + \gamma\int_\Omega |M(\hat{\phi})|^2 dx + \mu \int_\Omega |\nabla\hat{u}|^2 dx + B(\hat{u}, u_1, \hat{u})$$
$$= \int_\Omega (M(\phi_1)\nabla\phi_1 - M(\phi_2)\nabla\phi_2)\hat{u} - (u_1\nabla\phi_1 - u_2\nabla\phi_2)M(\hat{\phi})dx$$
$$+ \int_\Omega (N(\phi_1)\nabla\phi_1 - N(\phi_2)\nabla\phi_2)\hat{u} dx - \gamma\int_\Omega M(\hat{\phi})(N(\phi_1) - N(\phi_2))dx$$

Recall $B(u, v, w) = \int_\Omega u \cdot \nabla v \cdot w dx$.

Since $B(\hat{u}, u_1, \hat{u}) = -B(\hat{u}, \hat{u}, u_1)$, we have

$$\begin{aligned} |B(\hat{u}, u_1, \hat{u})| &= |B(\hat{u}, \hat{u}, u_1)| \\ &\leq \|\hat{u}\|_{L^4(\Omega)} \|\nabla\hat{u}\|_{L^2(\Omega)} \|u_1\|_{L^4(\Omega)} \\ &\leq \|\hat{u}\|^{1/4}_{L^2(\Omega)} \|\nabla\hat{u}\|^{7/4}_{L^2(\Omega)} \|u_1\|_{L^4(\Omega)} \\ &\leq \varrho\|\nabla\hat{u}\|^2_{L^2(\Omega)} + C(\varrho)\|\hat{u}\|^2_{L^2(\Omega)} \|u_1\|^8_{L^4(\Omega)} \end{aligned}$$

where ϱ is an arbitrary small positive number.

Direct calculation shows,

$$\int_\Omega (M(\phi_1)\nabla\phi_1 - M(\phi_2)\nabla\phi_2)\hat{u} - (u_1\nabla\phi_1 - u_2\nabla\phi_2)M(\hat{\phi})dx$$
$$= \int_\Omega (M(\phi_1)\nabla\hat{\phi} + M(\hat{\phi})\nabla\phi_2)\hat{u} - (u_1\nabla\hat{\phi} + \hat{u}\nabla\phi_2)M(\hat{\phi})dx$$
$$= \int_\Omega M(\phi_1)\nabla\hat{\phi}\,\hat{u} dx - \int_\Omega u_1\nabla\hat{\phi}\,M(\hat{\phi})dx = J_1 - J_2 \ ,$$

and

$$\begin{aligned} |J_1| &\leq \|M(\phi_1)\|_{L^2(\Omega)} \|\nabla\hat{\phi}\|_{L^4(\Omega)} \|\hat{u}\|_{L^4(\Omega)} \\ &\leq \varrho\|\hat{u}\|^2_{H^1_0(\Omega)} + \frac{C}{\varrho}\|M(\phi_1)\|^2_{L^2(\Omega)} \|\hat{\phi}\|^2_{H^2(\Omega)} \ , \end{aligned}$$

$$\begin{aligned}|J_2| &\leq \|M(\hat{\phi})\|_{L^2(\Omega)}\|u_1\|_{L^4(\Omega)}\|\nabla\hat{\phi}\|_{L^4(\Omega)} \\ &\leq \varrho\|M(\hat{\phi})\|^2_{L^2(\Omega)} + \frac{C}{\varrho}\|u_1\|^2_{H^1_0(\Omega)}\|\hat{\phi}\|^2_{H^2(\Omega)} \ .\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_\Omega (N(\phi_1)\nabla\phi_1 - N(\phi_2)\nabla\phi_2)\hat{u}dx \\ &= \int_\Omega (N(\phi_1)\nabla\hat{\phi}\cdot\hat{u}dx + \int_\Omega (N(\phi_1)-N(\phi_2))\nabla\phi_2\cdot\hat{u}dx \\ &= K_1 + K_2 \ ,\end{aligned}$$

with

$$\begin{aligned}|K_1| &\leq \|N(\phi_1)\|_{L^2(\Omega)}\|\nabla\hat{\phi}\|_{L^4(\Omega)}\|\hat{u}\|_{L^4(\Omega)} \\ &\leq C\|N(\phi_1)\|_{L^2(\Omega)}\|\hat{\phi}\|_{H^2(\Omega)}\|\hat{u}\|_{H^1_0(\Omega)} \\ &\leq \varrho\|\hat{u}\|^2_{H^1_0(\Omega)} + \frac{C'}{\varrho}\|N(\phi_1)\|^2_{L^2(\Omega)}\|\hat{\phi}\|^2_{H^2(\Omega)} \ ,\end{aligned}$$

and

$$\begin{aligned}|K_2| &\leq \|N(\phi_1)-N(\phi_2)\|_{L^2(\Omega)}\|\nabla\phi_2\|_{L^4(\Omega)}\|\hat{u}\|_{L^4(\Omega)} \\ &\leq C\|N(\phi_1)-N(\phi_2)\|_{L^2(\Omega)}\|\phi_2\|_{H^2(\Omega)}\|\hat{u}\|_{H^1_0(\Omega)} \\ &\leq \varrho\|\hat{u}\|^2_{H^1_0(\Omega)} + \frac{C'}{\varrho}\|N(\phi_1)-N(\phi_2)\|^2_{L^2(\Omega)}\|\phi_2\|^2_{H^2(\Omega)} \\ &\leq \varrho\|\hat{u}\|^2_{H^1_0(\Omega)} + \frac{C'}{\varrho}\|N(\phi_1)-N(\phi_2)\|^2_{L^2(\Omega)} \ .\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int_\Omega M(\hat{\phi})(N(\phi_1)-N(\phi_2))dx\right| &\leq \|M(\hat{\phi})\|_{L^2(\Omega)}\|N(\phi_1)-N(\phi_2)\|_{L^2(\Omega)} \\ &\leq \varrho\|M(\hat{\phi})\|^2_{L^2(\Omega)} + \frac{1}{4\varrho}\|N(\phi_1)-N(\phi_2)\|^2_{L^2(\Omega)} \ .\end{aligned}$$

We now use a claim (to be verified later):

$$\|N(\phi_1)-N(\phi_2)\|_{L^2(\Omega)} \leq C\|\phi_1-\phi_2\|_{H^2(\Omega)} = C\|\hat{\phi}\|_{H^2(\Omega)} \ . \tag{20}$$

Recall that $\|\hat{\phi}\|^2_{H^2(\Omega)} \leq CG(\hat{\phi})$. Using (20) and putting everything together, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}|\hat{u}|^2dx + G(\hat{\phi}) &\leq C\left(\int_\Omega \frac{1}{2}|\hat{u}|^2dx + G(\hat{\phi})\right) \\ &\cdot\left(\|M(\phi_1)\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^1_0(\Omega)} + \|u_1\|^8_{L^4(\Omega)} + \|N(\phi_1)\|^2_{L^2(\Omega)} + 1\right) \ .\end{aligned}$$

Using the estimates already derived and the extra assumption on the velocity field, we have

$$\left(\|M(\phi_1)\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^1_0(\Omega)} + \|u_1\|^8_{L^4(\Omega)} + \|N(\phi_1)\|^2_{L^2(\Omega)}\right)$$

is integrable in time. This implies if $\hat{u}(t) = 0$ and $\hat{\phi}(t) = 0$ at $t = 0$, then $\hat{u} = 0$ and $\hat{\phi} = 0$ for all time, which proves the uniqueness theorem.

4.2. **Proof of claim (20).** We now verify the claim (20) used in the above proof. By (9) and the definitions of M and N, we get

$$\begin{aligned} N(\phi) &= -\frac{k}{\epsilon}\triangle\phi^3 + \frac{2k}{\epsilon}\triangle\phi + \frac{3k}{\epsilon^2}\phi^2 f(\phi) - \frac{k}{\epsilon^2}f(\phi) - \phi \\ &\quad +M_1(A(\phi)-\alpha) + M_2(B(\phi)-\beta)f(\phi) \,. \end{aligned}$$

This leads to

$$\begin{aligned} \|N(\phi_1) - N(\phi_2)\|_{L^2(\Omega)} \le C\{&\|\triangle\phi_1^3 - \triangle\phi_2^3\|_{L^2(\Omega)} + \|\triangle\phi_1 - \triangle\phi_2\|_{L^2(\Omega)} \\ &+\|\hat{\phi}\|_{L^2(\Omega)} + \|\phi_1^2 f(\phi_1) - \phi_2^2 f(\phi_2)\|_{L^2(\Omega)} + \|f(\phi_1) - f(\phi_2)\|_{L^2(\Omega)} \\ &+\|A(\hat{\phi})\|_{L^2(\Omega)} + \|B(\phi_1)f(\phi_1) - B(\phi_2)f(\phi_2)\|_{L^2(\Omega)}\} \end{aligned}$$

According to the energy estimate derived in the proof of existence theorem, we have $\phi_i \in L^\infty(0,T;H^2(\Omega))$, $i = 1,2$. Therefore, for $i,j = 1,2$,

$$\begin{aligned} &\|\phi_i(t,x)\|_{L^\infty((0,T)\times\Omega)} \le M \,, \\ &\|\nabla(\phi_i(t,x)\phi_j(t,x))\|_{L^\infty((0,T);L^6(\Omega))} \le M \,, \\ &\|\Delta(\phi_i(t,x)\phi_j(t,x))\|_{L^\infty((0,T);L^2(\Omega))} \le M \end{aligned}$$

for some constant M. Now we carefully estimate the individual terms respectively. A generic time-independent constant C is used. Let $\tilde{\phi} = \phi_1^2 + \phi_1\phi_2 + \phi_2^2$, then

$$\begin{aligned} \|\triangle\phi_1^3 - \triangle\phi_2^3\|_{L^2(\Omega)} &\le \|\triangle\hat{\phi}\tilde{\phi}\|_{L^2(\Omega)} + \|\hat{\phi}\triangle\tilde{\phi}\|_{L^2(\Omega)} \\ &\quad +2\|\nabla\hat{\phi}\cdot\nabla\tilde{\phi}\|_{L^2(\Omega)} \le C\|\hat{\phi}\|_{H^2(\Omega)} \,. \end{aligned}$$

$$\begin{aligned} \|f(\phi_1) - f(\phi_2)\|_{L^2(\Omega)} &\le C(\|\triangle\hat{\phi}\|_{L^2(\Omega)} + \|\hat{\phi}\|_{L^2(\Omega)} + \|\phi_1^3 - \phi_2^3\|_{L^2(\Omega)}) \\ &\le C\|\hat{\phi}\|_{H^2(\Omega)} \,. \end{aligned}$$

$$\begin{aligned} \|\phi_1^2 f(\phi_1) - \phi_2^2 f(\phi_2)\|_{L^2(\Omega)} &\le \|(\phi_1^2 - \phi_2^2)f(\phi_1)\|_{L^2(\Omega)} + \|\phi_2^2(f(\phi_1) - f(\phi_2))\|_{L^2(\Omega)} \\ &\le C\|\hat{\phi}\|_{H^2(\Omega)} \,. \end{aligned}$$

$$\begin{aligned} |B(\phi_1) - B(\phi_2)| \le C(&\|\nabla(\phi_1+\phi_2)\|_{L^2(\Omega)})\|\nabla\hat{\phi}\|_{L^2(\Omega)} \\ &+C\left|\int_\Omega (\phi_1^2 + \phi_2^2 - 2)(\phi_1+\phi_2)\hat{\phi}dx\right| \le C\|\hat{\phi}\|_{H^2(\Omega)} \,. \end{aligned}$$

$$\begin{aligned} \|B(\phi_1)f(\phi_1) - B(\phi_2)f(\phi_2)\|_{L^2(\Omega)} &\le \|B(\phi_1)(f(\phi_1) - f(\phi_2))\|_{L^2(\Omega)} \\ &+\|(B(\phi_1) - B(\phi_2))f(\phi_2)\|_{L^2(\Omega)} \le C\|\hat{\phi}\|_{H^2(\Omega)} \,. \end{aligned}$$

Summing together, the claim (20) is verified.

5. **Conclusion.** In this paper, a system of coupled phase field Navier-Stokes equations modeling the deformation and evolution of three dimensional vesicle membranes in a fluid field is analyzed. A major characteristic of the current model studied here, in comparison of many other similar models studied previously in the literature on membrane fluid interactions, even in the phase field context, is the inclusion of the bending elastic energy which is crucial for vesicle bilayers. The resulting interaction mechanism is thus based on the competition between the elastic bending energy of the membrane, with prescribed bulk volume and surface area, and the kinetic energy in the surrounding fluid velocity fields. The variation of the elastic bending energy leads to an extra stress in the Navier-Stokes equation, which involves a nonlinear combination of higher order spatial derivatives of the phase field function. Our main results illustrate that these additional contributions can be properly controlled due to the establishment of the energy law. Such an energy dissipation mechanism is intrinsic in our derivation of the coupled system. The results provide a rigorous mathematical foundation to the coupled phase-field Navier-Stokes equations and their numerical simulations [7]. The results can be extended to cases with the Neumann and periodic boundary conditions, as well as inhomogeneous Dirichlet condition for the velocity field.

We note that our analysis largely relies on the damping term in the evolution of the phase field function and can not be readily extended to the case of a pure transport. Such cases were treated in [18] for the systems for viscoelastic materials. However, the nonlinear coupling terms involve higher derivatives in this paper. In addition, the various estimates derived in the paper are not uniform with respect to the small interfacial width parameter ϵ and thus cannot be used in the study of the sharp interface limits. Such issues pose interesting challenges for future studies. Extensions to interactions of vesicles with other types of fluids (possibly with different types inside and outside the vesicle) may also be considered. Finally, it will be also interesting to study the existence of classical solutions of the system in this paper, in particular, the existence for the large viscosity situations.

REFERENCES

[1] M. Abkarian, C. Lartigue, and A. Viallat, *Tank Treading and Unbinding of Deformable Vesicles in Shear Flow: Determination of the Lift Force*, Phys. Rev. Lett., 88 (2002), 068103

[2] J. Beaucourt, F. Rioual, T. Sion, T. Biben, and C. Misbah, *Steady to unsteady dynamics of a vesicle in a flow*, Phys. Rev. E, 69 (2004), 011906

[3] T. Biben, K. Kassner and C. Misbah, *Phase-field approach to three-dimensional vesicle dynamics*, Phys. Rev. E, 72 (2005), 041921

[4] T. Biben and C. Misbah, *Tumbling of vesicles under shear flow within an advected-field approach*, Phys. Rev. E, 67 (2003), 031908

[5] Q. Du, C. Liu, R. Ryham and X. Wang, *A phase field formulation of the Willmore problem*, Nonlinearity, 18 (2005), 1249-1267,

[6] Q. Du, C. Liu, R. Ryham and X. Wang, *Modeling the Spontaneous Curvature Effects in Static Cell Membrane Deformations by a Phase Field Formulation*, Communications in Pure and Applied Analysis, 4 (2005), 537-548.

[7] Q. Du, C. Liu, R. Ryham and X. Wang, *Modeling Vesicle Deformations in Flow Fields via Energetic Variational Approaches*, preprint, 2006.

[8] Q. Du, C. Liu, and X. Wang, *A phase field approach in the numerical study of the elastic bending energy for vesicle membranes*, Journal of Computational Physics, 198 (2004), 450-468.

[9] Q. Du, C. Liu, and X. Wang, *Retrieving Topological Information For Phase Field Models*, SIAM Journal on Applied Mathematics, 65 (2005), 1913-1932.

[10] Q. Du, C. Liu, and X. Wang, *Simulating the Deformation of Vesicle Membranes under Elastic Bending Energy in Three Dimensions*, Journal of Computational Physics, 212 (2006), 757-777.
[11] K. de Haas, C. Bloom, D. van den Ende, M. Duits, and J. Mellema, *Deformation of giant lipid bilayer vesicles in shear flow*, Phys. Rev. E, 56 (1997), 7132.
[12] W. Helfrich, *Elastic properties of lipid bilayers: theory and possible experiments*. Z. Naturforsch. C, 28 (1973), 693-703.
[13] V. Kantsler and V. Steinberg, *Orientation and Dynamics of a Vesicle in Tank-Treading Motion in Shear Flow*, Phys. Rev. Lett., 95 (2005), 258101
[14] S. Keller and R. Skalak, *Motion of a tank-treading ellipsoidal particle in a shear flow*, J. Fluid Mech., 120 (1982), 27-47
[15] M. Kraus, W. Wintz, U. Seifert, and R. Lipowsky. *Fluid Vesicles in Shear Flow*, Phys. Rev. Lett., 77 (1996), 3685-3688.
[16] H. Li, H. Yi, X. Shan and H. Fang, *Shape changes and motion of a vesicle in a fluid using a lattice Boltzmann model*, Arxiv preprint physics/0607074
[17] F. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure. Appl. Math., XLVIII (1995), 1-36.
[18] F. Lin, C. Liu and P. Zhang, *On Hydrodynamics of Viscoelastic Fluids*, Comm. Pure Appl. Math., Vol LVIII (2005), 1–35.
[19] R. Lipowsky *The morphology of lipid membranes*, Current Opinion in Structural Biology, 5 (1995), 531–540.
[20] Y. Liu, L. Zhang, X. Wang and W.K. Liu, *Coupling of Navier-Stokes equations with protein molecular dynamics and its application to hemodynamics*, Inter. J. Numerical Methods in Fluids, 46 (2004), 1237-1252
[21] H. Noguchi and G. Gompper, *Fluid Vesicles with Viscous Membranes in Shear Flow*, Phys. Rev. Lett., 93 (2004), 258102
[22] H. Noguchi and G. Gompper, *Meshless membrane model based on the moving least-squares method*, Phys. Rev. E, 73 (2006), 021903
[23] Z. Ou-Yang, J. Liu, and Y. Xie, "Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases," World Scientific, Singapore, 1999.
[24] C. Pozrikidis, *Interfacial dynamics for Stokes flow*, Journal of Computational Physics, 169 (2001), 250-301
[25] C. Pozrikidis, *Numerical Simulation of the Flow-Induced Deformation of Red Blood Cells*, Annals of Biomedical Engineering, 31 (2003), 1194-1205
[26] U. Seifert, *Fluid membranes in hydrodynamic flow fields: Formalism and an application to fluctuating quasispherical vesicles in shear flow*, European Physical Journal B, 8 (1999), 405-415
[27] U. Seifert, *Hydrodynamic Lift on Bound Vesicles*, Phys. Rev. Lett., 83 (1999), 876-879
[28] U. Seifert, K. Berndl and R. Lipowsky, *Configurations of fluid membranes and Vesicles*, Physical Rev A , 44 (1991), 1182-1202,
[29] R. Temam. "Navier-Stokes Equations, Theory and Numerical Analysis," American Mathematical Society, 2001
[30] X. Wang, *Optimal Profile of the phase field models for Willmore problems*, preprint, 2006.
[31] X. Wang and Q. Du, *Convergence of numerical approximations to a phase field bending elasticity model of membrane deformations*, to appear in Inter. J. Numer. Anal and Modeling, 2006.
[32] X. Wang and Q. Du, *Modelling and Simulations of Multi-component Lipid Membranes and Open Membranes via a Diffuse Interface Approach*, to appear in J. Mathematical Biology, 2006.

Received August 2006; revised February 2007.

E-mail address: qdu@math.psu.edu
E-mail address: li_m@math.psu.edu
E-mail address: liu@math.psu.edu