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FINITE VOLUME METHODS ON SPHERES AND SPHERICAL CENTROIDAL VORONOI MESHES*

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Abstract. We study in this paper a finite volume approximation of linear convection-diffusion equations defined on a sphere using the spherical Voronoi meshes, in particular the spherical centroidal Voronoi meshes. The high quality of spherical centroidal Voronoi meshes is illustrated through both theoretical analysis and computational experiments. In particular, we show that the L^2 error of the approximate solution is of quadratic order when the underlying mesh is given by a spherical centroidal Voronoi mesh. We also demonstrate numerically the high accuracy and the superconvergence of the approximate solutions.

Key words. finite volume method, spherical Voronoi tessellations, spherical centroidal Voronoi tessellations, error estimates, convection-diffusion equations

AMS subject classifications. 65N15, 65N50, 65D17

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1. Introduction.

The numerical solution of partial differential equations defined on spheres is an active research subject in the scientific community. The subject is related to a number of important applications such as weather forecasting and climate modeling. For example, the numerical solution of linear convection-diffusion equations and nonlinear shallow water equations in spherical geometry can be used to test numerical algorithms for more complex atmospheric circulation models. Though these models were often solved with spectral methods or traditional finite difference methods in spherical coordinates, methods that use quasi-uniform tessellations of the sphere are gradually gaining popularity as the grid-based methods offer great potential when combined with massive parallelism and local adaptivity.

To get efficient and accurate numerical solutions of PDEs, it is well known that grid quality plays an important role and high quality grid generation is often a significant part of the overall solution process. In this regard, there were many recent studies on the approximations of PDEs defined on spheres using various spherical grids, such as grids based on Bucky-balls [19], icosahedral grids [2, 32, 33], skipped grids [22], grids from a gnomonic (cubed sphere) mapping [24], etc. In standard Euclidean geometry, the so-called Voronoi–Delaunay grids have always been very popular grids used in both finite element and finite volume methods [28]. Other spherical grids have also been studied; see, for example, [16].

In [8, 9], we proposed a high quality spherical grid based on the spherical centroidal Voronoi tessellation (SCVT), which can be used for both data assimilation purposes and for the numerical solution of PDEs on spheres. A very recent study made in [31] on both the global and the local uniformity of spherical grids indicated

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that the SCVT grid with a uniform density measures better than many other variations, and, when used to discretize a model Poisson equation on the sphere, the SCVT-based grid tends to produces the smallest local truncation errors among all the grids under consideration. In [9], a finite volume approximation to a second order linear elliptic equation using the spherical Voronoi meshes was studied, and a first order error estimate for the discrete H^1 norm was obtained under some grid regularity assumptions. Preliminary numerical experiments demonstrated the good performance of the finite volume scheme when implemented with the spherical centroidal Voronoi meshes (SCVMs) that include both the SCVT and its dual (Delaunay) triangular grid. The SCVM enjoys some optimization properties [8] and they can also be defined with a nonuniform density function. They offer excellent local grid regularity and global mesh conformity as well as flexible mesh adaptivity. Thus, the SCVM naturally becomes an optimal grid in some sense, or at least a practically *safe* choice for discretizing PDEs on the sphere.

In this paper, we make further attempts to substantiate the optimality of SCVMs both theoretically and computationally. Our main results include a carefully designed finite volume scheme for a general second order convection-diffusion equation defined on a sphere. When implemented with the SCVM, we present a rigorous quadratic order L^2 error estimate for such a discrete scheme whose proof relies critically on the geometric properties of the SCVT. We further demonstrate through experiments the superconvergent properties of the numerical solutions and their gradients solved using our modified finite volume scheme and the SCVT-based grid. All these findings provide compelling reasons for regarding the SCVTs with the uniform density as arguably the best alternative for near uniform partitions of the sphere and the SCVT-based grids the optimal triangular grids to use for the numerical solution of many PDEs defined on spheres.

We point out that the conclusions given in this paper can be readily adapted to problems defined on the two-dimensional (2d) Euclidean plane. The analysis for the spherical case is somewhat more involved than the planar case since we must deal with the differences between spherical triangles and planar triangles.

The paper is organized as follows: we first introduce the model equation, along with some notation used in the paper. Then in section 2, we briefly recall the basic theory of the spherical centroidal Voronoi meshes. Some discrete function spaces and a finite volume scheme for linear convection-diffusion equations on the sphere given in [9] are discussed in section 3. With a suitable modification to the finite volume scheme, a rigorous L^2 error estimate is given in section 4 for SCVMs. In section 5, a superconvergent gradient recovery scheme is provided, and in section 6 we present some numerical experiments. Some concluding remarks are given in section 7.

We now introduce the model equation to be considered. First, let $\mathbb{S}^2$ denote the sphere (surface of the ball) having radius $r > 0$, i.e., $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = r\}$. Let ∇_s denote the tangential gradient operator [13, 17] on $\mathbb{S}^2$ defined by

$$\nabla_s u(\mathbf{x}) = (\nabla_{s,1}, \nabla_{s,2}, \nabla_{s,3}) u(\mathbf{x}) = \nabla u(\mathbf{x}) - (\nabla u(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}}) \vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}} ,$$

where $\nabla = (D_1, D_2, D_3)$ denotes the general gradient operator in $\mathbb{R}^3$ and $\vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}}$ is the unit outer normal vector to $\mathbb{S}^2$ at $\mathbf{x} = (x_1, x_2, x_3)$. We consider the second order elliptic equation on the sphere given by

$$\nabla_s \cdot \big(- a(\mathbf{x}) \nabla_s u(\mathbf{x}) + \vec{\mathbf{v}}(\mathbf{x}) u(\mathbf{x}) \big) + b(\mathbf{x}) u(\mathbf{x}) = f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbb{S}^2 . \tag{1.1}$$

Note that since $\mathbb{S}^2$ has no boundary, there is no boundary condition imposed.

We use the standard notation $L^p(\mathbb{S}^2)$, $W^{m,p}(\mathbb{S}^2)$ for Sobolev spaces on $\mathbb{S}^2$ (viewed as a compact, 2d Riemannian manifold) [17], equipped with norms $\|\cdot\|_{L^p(\mathbb{S}^2)}$ and $\|\cdot\|_{W^{m,p}(\mathbb{S}^2)}$. We set $H^m(\mathbb{S}^2) = W^{m,2}(\mathbb{S}^2)$ and use the standard inner product $(u,v) = \int_{\mathbb{S}^2} u(\mathbf{x})v(\mathbf{x})\, ds(\mathbf{x})$ for $u, v \in L^2(\mathbb{S}^2)$.

Let the data in (1.1) satisfy the following assumptions.

Assumption 1. $f \in L^2(\mathbb{S}^2)$, $a \in C^1(\mathbb{S}^2)$, $b \in L^\infty(\mathbb{S}^2)$, and $\vec{\mathbf{v}} \in C^1(\mathbb{S}^2, \mathbb{R}^3)$ such that $a(\mathbf{x}) \geq \alpha_1 > 0$, $b(\mathbf{x}) \geq 0$, and $\nabla_s \cdot \vec{\mathbf{v}}(\mathbf{x}) + b(\mathbf{x}) \geq \alpha_2 > 0$, a.e.

For any $u, v \in H^1(\mathbb{S}^2)$, define the bilinear functional $\mathcal{A}$ such that

$$\begin{aligned}\mathcal{A}(u,v) = &\int_{\mathbb{S}^2} a(\mathbf{x})\big(\nabla_s u(\mathbf{x}) \cdot \nabla_s v(\mathbf{x})\big) + u(\mathbf{x})\big(\vec{\mathbf{v}}(\mathbf{x}) \cdot \nabla_s v(\mathbf{x})\big)\, ds(\mathbf{x}) \\ &+ \int_{\mathbb{S}^2} b(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\, ds(\mathbf{x}).\end{aligned} \tag{1.2}$$

We easily see, for some constant $C > 0$, that

$$\mathcal{A}(u,v) \leq C\|u\|_{H^1(S^2)}\|v\|_{H^1(S^2)}.$$

The problem (1.1) has a unique weak solution $u \in H^2(\mathbb{S}^2)$ such that

$$\mathcal{A}(u,v) = (f,v), \quad \forall\, v \in H^1(\mathbb{S}^2) \tag{1.3}$$

and u satisfies the H^2 regularity estimate $\|u\|_{H^2(\mathbb{S}^2)} \leq C\|f\|_{L^2(\mathbb{S}^2)}$ for some constant $C > 0$. Though the same conclusion holds under weaker conditions on $\vec{\mathbf{v}}$ and b (and $\nabla_s \cdot \vec{\mathbf{v}}(\mathbf{x}) + b(\mathbf{x})$), for simplicity Assumption 1 is made throughout the paper.

2. Spherical centroidal Voronoi meshes. Let $d(\mathbf{x}, \mathbf{y})$ denote the geodesic distance between $\mathbf{x}$ and $\mathbf{y}$ on $\mathbb{S}^2$, i.e., $d(\mathbf{x}, \mathbf{y}) = r \arccos[(\mathbf{x} \cdot \mathbf{y})/r^2]$, where arccos denotes the inverse cosine. We also use $m(\cdot)$ to denote the standard measure (surface area or curve length) of the argument. Given a set of distinct points $\{\mathbf{x}_i\}_{i=1}^n \subset \mathbb{S}^2$, the corresponding spherical Voronoi regions $\{V_i\}_{i=1}^n$ are defined by

$$V_i = \big\{\mathbf{x} \in \mathbb{S}^2 \mid d(\mathbf{x}_i, \mathbf{x}) < d(\mathbf{x}_j, \mathbf{x}) \text{ for } j = 1, \dots, n \text{ and } j \neq i\big\}, \quad 1 \leq i \leq n\,.$$

$\{V_i\}_{i=1}^n$ forms a *Voronoi tessellation* or *Voronoi diagram* of $\mathbb{S}^2$ associated with the generators $\{\mathbf{x}_i\}_1^n$. Each Voronoi cell V_i is an open convex spherical polygon on $\mathbb{S}^2$ with geodesic arcs making up its boundary. It is also well known that the dual tessellation (in a graph-theoretical sense) to a Voronoi tessellation of $\mathbb{S}^2$ consists of spherical triangles which form the *Delaunay triangulation*.

Given a density function ρ defined on $\mathbb{S}^2$, for any spherical region $V \subset \mathbb{S}^2$, the *constrained mass centroid* $\mathbf{x}^c$ of V on $\mathbb{S}^2$ is given by the solution of

$$\min_{\mathbf{x} \in V} F(\mathbf{x})\,, \qquad \text{where} \qquad F(\mathbf{x}) = \int_V \rho(\mathbf{y})|\mathbf{y} - \mathbf{x}|^2\, ds(\mathbf{y})\,. \tag{2.1}$$

As in [7, 8, 9], a Voronoi tessellation of $\mathbb{S}^2$ is called a *constrained centroidal Voronoi tessellation* (CCVT) of $\mathbb{S}^2$ or, specifically, SCVT if and only if the points $\{\mathbf{x}_i\}_{i=1}^m$ which serve as the generators of the associated spherical Voronoi tessellation $\{V_i\}_{i=1}^k$ are also the constrained mass centroids of those Voronoi regions. For any set of points $\{\widetilde{\mathbf{x}}_i\}_{i=1}^n$ on $\mathbb{S}^2$ and any spherical tessellation $\{\widetilde{V}_i\}_{i=1}^n$ of $\mathbb{S}^2$, the corresponding *energy*

$$\mathcal{K}\big(\{\widetilde{\mathbf{x}}_i, \widetilde{V}_i\}_{i=1}^n\big) = \sum_{i=1}^n \int_{\widetilde{V}_i} \rho(\mathbf{x})\|\mathbf{x} - \widetilde{\mathbf{x}}_i\|^2\, ds(\mathbf{x})$$

is minimized only if $\{\widetilde{\mathbf{x}}_i, \widetilde{V}_i\}_{i=1}^n$ are a SCVT [8]. Consequently, SCVMs have many good geometric properties [8, 9]. A constant density function ρ leads to *uniformly* distributed SCVTs, and a nonconstant density function provides systematically a nonuniform distribution of points while the accumulation of SCVT generators still remains locally regular.

Constructing a constrained mass centroid from (2.1) may be cumbersome. In [8], it has been shown that one can compute first the standard centroid $\mathbf{x}_i^*$ of V_i in $\mathbb{R}^3$, then compute $\mathbf{x}_i^c$ using the fact that it is the projection of $\mathbf{x}_i^*$ onto $\mathbb{S}^2$ along the normal direction at $\mathbf{x}_i^c$. We refer to [8, 9, 21] for both deterministic and probabilistic algorithms for the construction of SCVTs. Figure 2.1 shows some examples of SCVTs associated with a constant density. More examples, including SCVTs with nonuniform densities, can be found in [9].

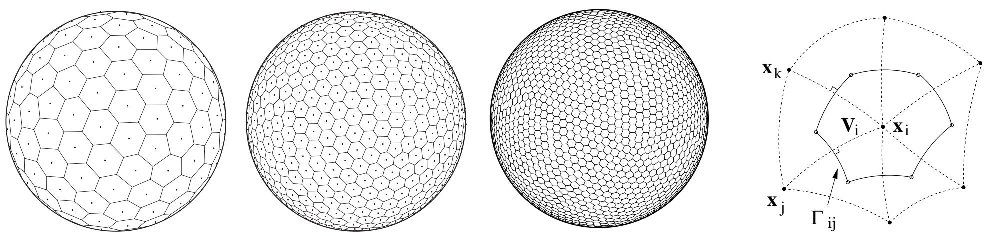

FIG. 2.1. *SCVTs for a constant density function with* 162, 642, *and* 2562 *generators, and an illustration of a spherical Voronoi region and its dual triangles.*

Given a spherical Voronoi mesh $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$, following [9], we refer to a pair of generators $\mathbf{x}_i$ and $\mathbf{x}_j$ as *neighbors* if and only if $\Gamma_{i,j} = \overline{V}_i \cap \overline{V}_j \neq \emptyset$. For Voronoi meshes, $\Gamma_{i,j}$ can only be a point or a geodesic arc on the sphere. For each $\mathbf{x}_i$, let χ_i be the set of the indices of its neighbors $\mathbf{x}_j$'s such that $m(\Gamma_{i,j}) > 0$. Let $\overline{\mathbf{x}_i\mathbf{x}_j}$ be the vector from $\mathbf{x}_i$ to $\mathbf{x}_j$, and let $\widetilde{\mathbf{x}_i\mathbf{x}_j}$ be the geodesic arc joining $\mathbf{x}_i$ and $\mathbf{x}_j$. From the construction of spherical Voronoi tessellations, it is known that $\widetilde{\mathbf{x}_i\mathbf{x}_j}$ is perpendicular to Γ_{ij} and the plane determined by $\Gamma_{i,j}$ and the origin bisects $\widetilde{\mathbf{x}_i\mathbf{x}_j}$ at its midpoint $\mathbf{x}_{ij}$ [9]; see Figure 2.1. Thus, $|\mathbf{x}_i - \mathbf{x}| = |\mathbf{x}_j - \mathbf{x}|$ for $\mathbf{x} \in \Gamma_{ij}$ and for $k = i, j$, $\vec{\mathbf{n}}_{\mathbf{x},V_k}$ is parallel to $\overline{\mathbf{x}_i\mathbf{x}_j}$ where $\vec{\mathbf{n}}_{\mathbf{x},V_k}$ is the outer unit normal vector to the boundary of V_k, taken to lie in the tangent plane of $\mathbb{S}^2$ at $\mathbf{x}$.

Let $h_i = \max_{\mathbf{y}\in V_i} d(\mathbf{x}_i, \mathbf{y})$, and we define the *mesh quality norm* by $h = \max_i h_i$. h gives the maximum geodesic distance between any particular generator $\mathbf{x}_i$ and the points in its associated cell V_i, and it has been used in [8] for the polynomial interpolation on the sphere.

Given a Voronoi mesh $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^m$, we define the *mesh regularity norm* σ by

$$\sigma = \min_{1\leq i\leq n} \sigma_i, \quad \text{where} \quad \sigma_i = \min_{j\in\chi_i} \sigma_{ij} \quad \text{and} \quad \sigma_{ij} = d(\mathbf{x}_i, \mathbf{x}_j)/(2h_i)\,. \tag{2.2}$$

If $\mathbf{x}_i, \mathbf{x}_j$, and $\mathbf{x}_k$ are neighbors for each other in $\mathcal{W}$, we denote by $\tilde{T}_{ijk}$ the spherical triangle determined by $\mathbf{x}_i, \mathbf{x}_j$, and $\mathbf{x}_k$, and by T_{ijk} the corresponding planar triangle (see Figure 2.1). In addition, let $\tilde{\mathcal{T}} = \{\tilde{T}_{ijk} \,|\, ijk \in \Sigma\}$, $\mathcal{T} = \{T_{ijk} \,|\, ijk \in \Sigma\}$ where $\Sigma = \{\, ijk \,|\, i, j, k \text{ are neighbors in } \mathcal{W}\}$. $\tilde{\mathcal{T}}$ gives the spherical Delaunay triangulation of $\mathbb{S}^2$ associated with the generators $\{\mathbf{x}_i\}_{i=1}^n$.

Meshes of the type $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$ are used as control (or finite) volumes for the discretization method discussed below. Our use of these meshes is particularly motivated by the covolume mesh approaches in the numerical solution of PDEs [28, 29] and for applications to nonlinear problems [6, 12, 30].

3. A finite volume method on spherical Voronoi tessellations. Without loss of generality, we consider the case of the unit sphere S^2 in what follows.

3.1. Some definitions and geometric properties. For any $\mathbf{x}$, let $\mathcal{P}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ be the projection onto the unit sphere S^2. $\mathcal{P}$ is also a one-to-one smooth function that maps $\mathbf{S}^* = \cup_{T_{ijk}\in\mathcal{T}} T_{ijk}$ to $\mathbb{S}^2 = \cup_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \tilde{T}_{ijk}$.

Clearly, $\mathbf{S}^* \cap \mathbb{S}^2 = \{\mathbf{x}_i\}_{i=1}^n$. Moreover, for any $\mathbf{x} \in \mathbb{S}^2$ and $\mathbf{x}', \mathbf{x}'' \in \tilde{T}_{ijk}$, we have

$$\begin{cases} |\mathbf{x} - \mathcal{P}^{-1}(\mathbf{x})| \le ch^2, \\ (1 - ch^2) d(\mathbf{x}', \mathbf{x}'') \le |\mathcal{P}^{-1}(\mathbf{x}') - \mathcal{P}^{-1}(\mathbf{x}'')| \le (1 + ch^2) d(\mathbf{x}', \mathbf{x}''), \\ m(T_{ijk}) \le m(\tilde{T}_{ijk}) \le (1 + ch^2) m(T_{ijk}), \end{cases} \tag{3.1}$$

where c is a generic constant for h small.

Let $\Omega = \{\mathbf{x} \,|\, 1 - ch^2 < |\mathbf{x}| < 1 + ch^2\}$; then $\cup T_{ijk} \subset \Omega$ and $\cup \tilde{T}_{ijk} \subset \Omega$. For any $u \in H^2(\mathbb{S}^2)$, define the function Eu, the extension of u in Ω, by $Eu(\mathbf{y}) = u(\mathbf{y}/|\mathbf{y}|)$ for any $\mathbf{y} \in \Omega$. The following results have been shown in [9].

PROPOSITION 1. *For any* $\mathbf{y} \in \Omega$ *and* $\mathbf{x} = \mathbf{y}/|\mathbf{y}| \in \mathbb{S}^2$, *and* $i, j = 1, 2, 3$,

$$\begin{cases} \nabla_s u(\mathbf{x}) = \nabla Eu(\mathbf{x}), \quad \nabla(D_i Eu)(\mathbf{x}) = \nabla_s(\nabla_{s,i} u)(\mathbf{x}) - (\nabla_{s,i} u(\mathbf{x}))\vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}}, \\ |\mathbf{y}|\ \nabla Eu(\mathbf{y}) = \nabla Eu(\mathbf{x}), \quad |\mathbf{y}|^2 D_i D_j Eu(\mathbf{y}) = D_i D_j Eu(\mathbf{x}). \end{cases} \tag{3.2}$$

By (3.1) and Proposition 1, the following result can be obtained using a proof similar to that used in Lemma 1 in [13].

PROPOSITION 2. *There exists a generic constant* $c > 0$ *such that for any* $ijk \in \Sigma$,

$$\begin{cases} C_1 \|u\|_{L^2(\tilde{T}_{ijk})} \le \|Eu|_{\mathbf{S}^*}\|_{L^2(T_{ijk})} \le C_2 \|u\|_{L^2(\tilde{T}_{ijk})}, \\ C_3 \|u\|_{H^1(\tilde{T}_{ijk})} \le \|Eu|_{\mathbf{S}^*}\|_{H^1(T_{ijk})} \le C_4 \|u\|_{H^1(\tilde{T}_{ijk})}, \\ \|Eu|_{\mathbf{S}^*}\|_{H^2(T_{ijk})} \le C_5 \|u\|_{H^2(\tilde{T}_{ijk})}. \end{cases} \tag{3.3}$$

We call u^L a piecewise linear function on $\mathbf{S}^*$ if and only if

$$u^L(\mathbf{x}^*) = \lambda_i u^L(\mathbf{x}_i) + \lambda_j u^L(\mathbf{x}_j) + \lambda_k u^L(\mathbf{x}_k), \quad \forall\ \mathbf{x}^* \in T_{ijk},$$

where $\lambda_i, \lambda_j, \lambda_k$ are the barycentric coordinates of $\mathbf{x}^*$ in the planar triangle T_{ijk}.

Let $\mathcal{V}_{\mathcal{W}}$ be the space of piecewise constant functions associated with a spherical Voronoi mesh $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$,

$$\mathcal{V}_{\mathcal{W}} = \{u \mid u(\mathbf{x}) \ \text{ is constant on each cell } \ V_i\}, \tag{3.4}$$

and denote by $\mathcal{U}_{\mathcal{W}}$ the space of all functions u_h on $\mathbb{S}^2$ such that $u_h(\mathbf{x}) = u^L(\mathcal{P}^{-1}(\mathbf{x}))$ for $\mathbf{x} \in \mathbb{S}^2$, where u^L is a piecewise linear function on $\mathbf{S}^*$ with $\{u^L(\mathbf{x}_i) = u_h(\mathbf{x}_i)\}_{i=1}^n$, i.e., $Eu_h(\mathbf{x}^*) = u^L(\mathbf{x}^*)$ for any $\mathbf{x}^* \in \mathbf{S}^*$.

If we interpret the Sobolev space on $\mathbf{S}^*$ in the piecewise sense, then it is easy to get $u_h \in H^1(\mathbb{S}^2)$ for any $u_h \in \mathcal{U}_{\mathcal{W}}$ using (3.2) and the fact that $Eu_h = u^L \in H^1(\mathbf{S}^*)$.

We now state some standard estimates on $\mathcal{P}_{\mathcal{U}}(u)$ and $\mathcal{P}_{\mathcal{V}}(u)$ which are the interpolants on $\mathcal{U}_{\mathcal{W}}$ and $\mathcal{V}_{\mathcal{W}}$, respectively, of a function u defined on $\mathbb{S}^2$.

PROPOSITION 3. *For any* $u \in H^2(\mathbb{S}^2)$, *there exists a generic constant* $C > 0$ *such that*

$$\begin{cases} \|u - \mathcal{P}_{\mathcal{U}}(u)\|_{L^2(\mathbb{S}^2)} + h\|u - \mathcal{P}_{\mathcal{U}}(u)\|_{H^1(\mathbb{S}^2)} \le Ch^2 \|u\|_{H^2(\mathbb{S}^2)}, \\ \|u - \mathcal{P}_{\mathcal{V}}(u)\|_{L^2(\mathbb{S}^2)} \le Ch\|u\|_{H^2(\mathbb{S}^2)}. \end{cases} \tag{3.5}$$

Proof. Note that $\mathcal{P}_{\mathcal{U}}(u)(\mathbf{x}) = u^L(\mathcal{P}^{-1}(\mathbf{x}))$ with

$$u^L(\mathbf{x}^*) = \lambda_i u(\mathbf{x}_i) + \lambda_j u(\mathbf{x}_j) + \lambda_k u(\mathbf{x}_k) \quad \forall\, \mathbf{x}^* \in T_{ijk}.$$

Using the estimate for the linear interpolation on planar triangles and the relation

$$u(\mathbf{x}) - \mathcal{P}_{\mathcal{U}}(u)(\mathbf{x}) = \tilde{u}(\mathcal{P}^{-1}(\mathbf{x})) - u^L(\mathcal{P}^{-1}(\mathbf{x})),$$

where $\tilde{u} = Eu|_{\mathbf{S}^*}$, we obtain by (3.1) and Proposition 2 that

$$\begin{aligned}
\|u - \mathcal{P}_{\mathcal{U}}(u)\|_{L^2(\mathbb{S}^2)} &= \left(\sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |u(\mathbf{x}) - \mathcal{P}_{\mathcal{U}}(u)(\mathbf{x})|^2 \, ds(\mathbf{x}) \right)^{1/2} \\
&= \left(\sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |\tilde{u}(\mathcal{P}^{-1}(\mathbf{x})) - u^L(\mathcal{P}^{-1}(\mathbf{x}))|^2 \, ds(\mathbf{x}) \right)^{1/2} \\
&\le C \left(\sum_{T_{ijk} \in \mathcal{T}} \int_{T_{ijk}} |\tilde{u}(\mathbf{x}^*) - u^L(\mathbf{x}^*)|^2 \, ds(\mathbf{x}^*) \right)^{1/2} \\
&\le Ch^2 \left(\sum_{T_{ijk} \in \mathcal{T}} \|\tilde{u}\|^2_{H^2(T_{ijk})} \right)^{1/2} \\
&\le Ch^2 \left(\sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}} \|u\|^2_{H^2(\tilde{T}_{ijk})} \right)^{1/2} = Ch^2 \|u\|_{H^2(\mathbb{S}^2)} .
\end{aligned} \tag{3.6}$$

The other estimates can be proved in similar manners. We omit the details. ∎

For given functions $u, v \in \mathcal{V}_{\mathcal{W}}$, or $\mathcal{U}_{\mathcal{W}}$, we define, similar to [27], the discrete inner products and norms associated with a spherical Voronoi mesh $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$ by the following:

$$\begin{cases}
(u, v)_{\mathcal{W}} = \displaystyle\sum_{i=1}^n m(V_i) u(\mathbf{x}_i) v(\mathbf{x}_i)\,, \quad \|u\|^2_{0,\mathcal{W}} = (u, u)_{\mathcal{W}}, \\
|u|^2_{1,\mathcal{W}} = \dfrac{1}{2} \displaystyle\sum_{i=1}^n \sum_{j \in \chi_i} m(\Gamma_{ij}) d(\mathbf{x}_i, \mathbf{x}_j) \left(\frac{u(\mathbf{x}_i) - u(\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|} \right)^2, \\
\|u\|^2_{1,\mathcal{W}} = \|u\|^2_{0,\mathcal{W}} + |u|^2_{1,\mathcal{W}}.
\end{cases}$$

Norms for general function spaces can also be defined.

We conclude with some norm equivalence results under mesh regularity assumptions. For convenience, we assume that all three angles of T_{ijk} are less than 90°. This is generally valid for the triangles in the SCVMs with sufficiently large (no smaller than 42, for example, for the constant density) number of vertices (generators) or, equivalently, sufficient small h. Using (3.1), Propositions 1 and 2, and similar arguments as those in Proposition 1 of section 2.1 in [26], we have the following.

PROPOSITION 4. *For any $u_h \in \mathcal{U}_{\mathcal{W}}$, there exist some constants $\{C_i > 0\}_{i=1}^4$,*

$$\begin{cases}
C_1 \|u_h\|_{0,\mathcal{W}} \le \|u_h\|_{L^2(\mathbb{S}^2)} \le C_2 \|u_h\|_{0,\mathcal{W}}, \\
C_3 \|u_h\|_{1,\mathcal{W}} \le \|u_h\|_{H^1(\mathbb{S}^2)} \le C_4 \|u_h\|_{1,\mathcal{W}}.
\end{cases} \tag{3.7}$$

The results of Proposition 4 are in fact valid for more general Voronoi–Delaunay meshes that satisfy the local mesh regular properties. Let $l(\tilde{T}_{ijk})$ be the maximum number of spherical Voronoi regions V_m having nonempty intersection $V_m \cap \tilde{T}_{ijk}$ for any spherical triangle $\tilde{T}_{ijk}$, and let $l(V_m)$ be the maximum number of spherical triangles $\tilde{T}_{ijk}$ needed to cover any spherical Voronoi region V_m; we need all the $\{l(\tilde{T}_{ijk})\}$ and $\{l(V_m)\}$ to be bounded above by a constant integer independent of h. Under those conditions and the mesh regularity conditions, the above equivalence of norms still holds.

3.2. A finite volume discretization scheme. Based on Green's formula, a finite volume method for (1.1) was proposed in [9]. Set $\{u_i^h = u^h(\mathbf{x}_i)\}_{i=1}^n$ and let the approximate flux $\mathcal{F}_{ij}$ be defined by

$$\mathcal{F}_{ij} = -m(\Gamma_{ij})a_{ij}\frac{u_j^h - u_i^h}{|\mathbf{x}_j - \mathbf{x}_i|} \approx \int_{\Gamma_{ij}} (-a(\mathbf{x})\nabla_s u(\mathbf{x})) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, d\gamma(\mathbf{x})\,, \tag{3.8}$$

where $a_{ij}m(\Gamma_{ij}) = \int_{\Gamma_{ij}} a(\mathbf{x})\, d\gamma(\mathbf{x})$. An up-wind approximate convection flux $\mathcal{V}_{i,j}$ was defined in [18] by

$$\mathcal{V}_{ij} = \beta_{ij}^+ u_i^h + \beta_{ij}^- u_j^h \approx \int_{\Gamma_{ij}} (\vec{\mathbf{v}}(\mathbf{x})u(\mathbf{x})) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, d\gamma(\mathbf{x})\,, \tag{3.9}$$

where $\beta_{ij}^+ = (\beta_{ij} + |\beta_{ij}|)/2$, $\beta_{ij}^- = (\beta_{ij} - |\beta_{ij}|)/2$, and $\beta_{ij} = \int_{\Gamma_{ij}} \vec{\mathbf{v}}(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, d\gamma(\mathbf{x})$.

For all V_i, let f_i and b_i denote, respectively, the mean value of f and b on V_i; i.e.,

$$f_i = \frac{1}{m(V_i)}\int_{V_i} f(\mathbf{x})\, ds(\mathbf{x}) \qquad \text{and} \qquad b_i = \frac{1}{m(V_i)}\int_{V_i} b(\mathbf{x})\, ds(\mathbf{x})\,. \tag{3.10}$$

The finite volume scheme given in [9] is defined as follows: find $u^h \in \mathcal{V}_{\mathcal{W}}$ such that

$$(\mathcal{L}^h u^h)_i = \frac{1}{m(V_i)}\sum_{j\in\chi_i}\big(\mathcal{F}_{ij} + \mathcal{V}_{ij}\big) + b_i u_i^h = f_i \qquad \text{for} \quad i = 1, \dots, n\,. \tag{3.11}$$

Since $\mathcal{F}_{ij} = -\mathcal{F}_{ji}$ and $\mathcal{V}_{ij} = -\mathcal{V}_{ji}$ for neighboring $\mathbf{x}_i$ and $\mathbf{x}_j$ with $m(\Gamma_{ij}) > 0$, the above scheme satisfies the discrete conservation law

$$\sum_{i=1}^n \sum_{j\in\chi_i} (\mathcal{F}_{ij} + \mathcal{V}_{ij}) = 0\,.$$

Note that an approximate convection flux of the form $\mathcal{V}_{i,j} = (u_i^h + u_j^h)\beta_{ij}/2$ leads to a central difference scheme. A stability condition such as

$$P_i = \max_{j\in\chi_i}\frac{|\beta_{ij}|\cdot|\mathbf{x}_i - \mathbf{x}_j|}{2m(\Gamma_{ij})a_{ij}} \le 1 \quad \text{for} \quad i = 1, \dots, n$$

is needed in such a case. P_i is called the local Peclet number [18, 27].

3.3. Previous results and a modified scheme. Assuming that $\mathcal{W}$ is *regular* in the sense that σ is not *too small*, i.e., it remains bounded from below as $h \to 0$, then the following result has been proved in [9].

THEOREM 1. *Let Assumption* 1 *be satisfied and the mesh be regular, and let* $\mathcal{F}_{ij}$, $\mathcal{V}_{ij}$, f_i, *and* b_i *be defined by* (3.8)–(3.10). *Then the discrete system* (3.11) *has a unique* *solution* $u^h \in \mathcal{V}_{\mathcal{W}}$*. Furthermore, assume that the unique solution* u *of* (1.1) *belongs to* $H^2(\mathbb{S}^2)$*; then there exists a constant* $C > 0$ *only depending on* a, $\vec{\mathbf{v}}$, b, *and* σ *such that*

$$\|e^h\|_{1,\mathcal{W}} \le Ch\|u\|_{H^2(\mathbb{S}^2)} , \tag{3.12}$$

where $e^h = \{e_i^h = u(\mathbf{x}_i) - u_i^h\}$.

Note that Theorem 1 holds for general regular spherical Voronoi meshes. For more existing studies on the finite volume methods, especially when applied to solve second order elliptic on the 2d plane, we refer to [1, 3, 4, 5, 12, 14, 15, 20, 26, 25, 28, 29, 34, 35].

To get second order accuracy for the L^2 estimates, the order of approximation for the convection term used in the original scheme needs to be improved with better integration rules. For this purpose, let us define the bilinear functionals $\mathcal{A}^*$ and $\mathcal{A}_{\mathcal{W}}$ such that

$$\mathcal{A}^*(u, v^h) = \sum_{i=1}^n v^h(\mathbf{x}_i)\mathcal{A}^*(u, \psi_i) , \quad \mathcal{A}_{\mathcal{W}}(u, v^h) = \sum_{i=1}^n v^h(\mathbf{x}_i)\mathcal{A}_{\mathcal{W}}(u, \psi_i) \tag{3.13}$$

for any $u \in H^2(\mathbb{S}^2) \cup \mathcal{U}_{\mathcal{W}}$ and $v^h \in \mathcal{V}_{\mathcal{W}}$, where

$$\mathcal{A}^*(u, \psi_i) = \int_{\partial V_i} (-\nabla_s u(\mathbf{x}) + \vec{\mathbf{v}}(\mathbf{x})u(\mathbf{x})) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i} \, d\gamma(\mathbf{x}) + \int_{V_i} b(\mathbf{x})P_{\mathcal{V}}(u)(\mathbf{x}) \, ds(\mathbf{x}) ,$$

$$\mathcal{A}_{\mathcal{W}}(u, \psi_i) = \sum_{j \in \chi_i} \mathcal{F}_{ij}(u) + \int_{\partial V_i} \mathcal{P}_{\mathcal{U}}(u)(\mathbf{x})(\vec{\mathbf{v}}(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}) \, d\gamma(\mathbf{x}) + m(V_i)b_i u(\mathbf{x}_i) .$$

Comparing $\mathcal{A}_{\mathcal{W}}$ with the finite volume scheme (3.11), we have in fact replaced only the convection term $\mathcal{V}_{ij}$ by $\int_{\partial V_i} \mathcal{P}_{\mathcal{U}}(u)(\mathbf{x})(\vec{\mathbf{v}}(\mathbf{x})\cdot\vec{\mathbf{n}}_{\mathbf{x},V_i}) \, d\gamma(\mathbf{x})$. Note that no change is made for a pure diffusion problem containing only the second order terms $\nabla_s \cdot (a(\mathbf{x})\nabla_s u(\mathbf{x}))$. Our discrete problem here is then as follows: find $u_h \in \mathcal{U}_{\mathcal{W}}$ such that

$$\mathcal{A}_{\mathcal{W}}(u_h, v^h) = (f, v^h) \quad \forall\, v^h \in \mathcal{V}_{\mathcal{W}} , \tag{3.14}$$

i.e.,

$$\mathcal{A}_{\mathcal{W}}(u_h, \psi_i) = f_i \qquad \text{for} \quad i = 1, 2, \dots , n.$$

Formulations like the above for finite volume methods have been used, for instance, in [26]. Combining Proposition 3 and Proposition 4, it can be shown that the error estimate of Theorem 1 still holds for the above u_h using analysis similar to that used in [9].

THEOREM 2. *Suppose that Assumption* 1 *is satisfied. Let* $\mathcal{F}_{ij}$ *be defined by* (3.8)*. Then the discrete system* (3.14) *has a unique solution* $u_h \in \mathcal{U}_{\mathcal{W}}$*. Furthermore, assume that the unique solution* u *of* (1.1) *belongs to* $H^2(\mathbb{S}^2)$*; then there exists a constant* $C > 0$ *only depending on* a, $\vec{\mathbf{v}}$, b, *and* σ *such that for* $e_h = u - u_h$*, we have*

$$\|e_h\|_{H^1(\mathbb{S}^2)} \le Ch\|u\|_{H^2(\mathbb{S}^2)} . \tag{3.15}$$

Proof. Notice that $\|\mathcal{P}_{\mathcal{U}}(u) - u_h\|_{1,\mathcal{W}} = \|e_h\|_{1,\mathcal{W}}$; then we have

$$\begin{aligned} \|e_h\|_{H^1(\mathbb{S}^2)} &= \|u - u_h\|_{H^1(\mathbb{S}^2)} \\ &\le \|u - \mathcal{P}_{\mathcal{U}}(u)\|_{H^1(\mathbb{S}^2)} + \|\mathcal{P}_{\mathcal{U}}(u) - u_h\|_{H^1(\mathbb{S}^2)} \\ &\le C_1 h\|u\|_{H^2(\mathbb{S}^2)} + C_2\|\mathcal{P}_{\mathcal{U}}(u) - u_h\|_{1,\mathcal{W}} \\ &= C_1 h\|u\|_{H^2(\mathbb{S}^2)} + C_2\|e_h\|_{1,\mathcal{W}} \le Ch\|u\|_{H^2(\mathbb{S}^2)}, \end{aligned}$$

where the conclusion of Theorem 1 has been used. ∎

4. L^2 error estimate on SCVMs.

An improved error estimate in the L^2 norm is generally expected for our finite volume approximations of second order elliptic equations. However, it is shown here that the quadratic order error estimate can only be proved when the grid satisfies certain geometric constraints. In fact, a part of the estimate depends critically on the property that if $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$ is an SCVT of $\mathbb{S}^2$ corresponding to a density function ρ, then

$$\int_{V_i} \rho(\mathbf{x})(\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) = 0, \quad \forall\, i = 1, 2, \dots, n,$$

where $\mathbf{x}_i^*$ is the standard mass centroid of V_i, whose projection $\mathbf{x}_i^c$ (through the standard map $\mathcal{P}$) onto the sphere coincides with $\mathbf{x}_i$. We note that so far we have not been able to extend the elegant analysis of the covolume schemes for planar Poisson equations in [28, 29] to our context, nor we have found any improvement of the results there for the SCVT-based meshes. Thus, we resort to a more traditional approach of obtaining estimates through appropriate weak forms.

For the rest of the section, only those schemes based on SCVMs are analyzed.

4.1. A technical lemma.

For the interpolation operator $\mathcal{P}_\mathcal{V}$, we present a better approximation result that requires the properties of the SCVMs.

LEMMA 1. *Suppose that $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$ is an SCVT of $\mathbb{S}^2$ with the density function ρ satisfying $\rho \in C^1(\mathbb{S}^2)$ and $\rho(\mathbf{x}) > 0$ for any $\mathbf{x} \in \mathbb{S}^2$. Then, for any $w \in H^2(\mathbb{S}^2)$, there exits a constant $C > 0$ such that*

$$\left| \int_{V_i} (w - \mathcal{P}_\mathcal{V}(w))\; ds(\mathbf{x}) \right| \le Ch^2 m(V_i)^{1/2} \|w\|_{H^2(V_i)}, \quad i = 1, \dots, n. \tag{4.1}$$

Proof. Let us assume that $w \in C^2(\mathbb{S}^2)$; then it is easy to see that $Ew \in C^2(\Omega)$. Consider the spherical Voronoi region V_i associated with $\mathbf{x}_i$, for any $\mathbf{x} \in V_i$. We have

$$\begin{aligned} w(\mathbf{x}_i) - w(\mathbf{x}) &= Ew(\mathbf{x}_i) - Ew(\mathbf{x}) \\ &= \nabla Ew(\mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x}) + \int_0^1 H(Ew)(t\mathbf{x} + (1-t)\mathbf{x}_i)(\mathbf{x}_i - \mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x}) t dt, \end{aligned}$$

where $H(Ew)(\mathbf{x})$ denotes the Hessian matrix of Ew at $\mathbf{x}$. Thus

$$\left| \int_{V_i} w - \mathcal{P}_\mathcal{V}(w)\, ds(\mathbf{x}) \right| \le E_1 + E_2 + E_3,$$

where, with $\mathbf{x}_i^*$ being the mass centroid of V_i in R^3 with the density ρ, we have

$$\begin{aligned} E_1 &= \left| \int_{V_i} \nabla Ew(\mathbf{x}) \cdot (\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) \right| = \left| \int_{V_i} \nabla_s w(\mathbf{x}) \cdot (\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) \right|, \\ E_2 &= \left| \int_{V_i} \nabla Ew(\mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x}_i^*)\, ds(\mathbf{x}) \right| = \left| \int_{V_i} \nabla_s w(\mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x}_i^*)\, ds(\mathbf{x}) \right|, \\ E_3 &= \int_{V_i} \int_0^1 |H(Ew)(t\mathbf{x} + (1-t)\mathbf{x}_i)(\mathbf{x}_i - \mathbf{x}) \cdot (\mathbf{x}_i - \mathbf{x})| t dt\, ds(\mathbf{x}). \end{aligned}$$

Using the property of the SCVT that $\int_{V_i} \rho(\mathbf{x})(\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) = 0$, we have

$$\begin{aligned} E_1 &= \left| \int_{V_i} \nabla_s w(\mathbf{x}) \cdot (\mathbf{x}_i^* - \mathbf{x}) - \frac{\rho(\mathbf{x})}{\rho(\mathbf{x}_i)} \Pi_{\mathcal{V}}(\nabla_s w) \cdot (\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) \right| \\ &\le \left| \int_{V_i} \frac{\rho(\mathbf{x}) - \rho(\mathbf{x}_i)}{\rho(\mathbf{x}_i)} \nabla_s w(\mathbf{x}) \cdot (\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) \right| \\ &\quad + \left| \int_{V_i} \frac{\rho(\mathbf{x})}{\rho(\mathbf{x}_i)} (\nabla_s w(\mathbf{x}) - \Pi_{\mathcal{V}}(\nabla_s w)) \cdot (\mathbf{x}_i^* - \mathbf{x})\, ds(\mathbf{x}) \right|, \end{aligned}$$

where $\Pi_{\mathcal{V}}$ denotes the L^2 projection on $\mathcal{V}_{\mathcal{W}}$. Denoting the two terms on the right-hand side of the last equation by E_4 and E_5, respectively, we have

$$\begin{aligned} E_4 &\le h \int_{V_i} \frac{\max_{\mathbf{x}\in\Omega} |\nabla E\rho(\mathbf{x})|}{|\rho(\mathbf{x}_i)|} |\nabla_s w(\mathbf{x})|\, \|\mathbf{x}_i^* - \mathbf{x}\|\, ds(\mathbf{x}) \\ &\le 2h^2 \int_{V_i} \frac{\max_{\mathbf{x}\in\Omega} |\nabla_s \rho(\mathbf{x})|}{|\rho(\mathbf{x}_i)|} |\nabla_s w(\mathbf{x})|\, ds(\mathbf{x}) \\ &\le Ch^2 \int_{V_i} |\nabla_s w(\mathbf{x})|\, ds(\mathbf{x}) \le Ch^2 m(V_i)^{1/2} \|w\|_{H^2(V_i)} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} E_5 &\le \int_{V_i} \frac{\max_{\mathbf{x}\in\Omega} |\rho(\mathbf{x})|}{|\rho(\mathbf{x}_i)|} |\nabla_s w(\mathbf{x}) - \Pi_{\mathcal{V}}(\nabla_s w)|\, \|\mathbf{x}_i^* - \mathbf{x}\|\, ds(\mathbf{x})| \\ &\le C \|\nabla_s w - \Pi_{\mathcal{V}}(\nabla_s w)\|_{L^2(V_i)} \left(\int_{V_i} \|\mathbf{x}_i^* - \mathbf{x}\|^2\, ds(\mathbf{x}) \right)^{1/2} \\ &\le Ch^2 m(V_i)^{1/2} \|w\|_{H^2(V_i)}. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$E_1 \le Ch^2 m(V_i)^{1/2} \|w\|_{H^2(V_i)}\ . \tag{4.4}$$

Consider E_2. Since $\mathbf{x}_i = \mathbf{x}_i^*/|\mathbf{x}_i^*|$, by (3.1) we know that $|\mathbf{x}_i - \mathbf{x}_i^*| < ch^2$. Thus

$$\begin{aligned} E_2 &\le Ch^2 \left(\int_{V_i} |\nabla_s w(\mathbf{x})|^2\, ds(\mathbf{x}) \right)^{1/2} \left(\int_{V_i} ds(\mathbf{x}) \right)^{1/2} \\ &\le Ch^2 m(V_i)^{1/2} \|w\|_{H^1(V_i)}. \end{aligned} \tag{4.5}$$

On the other hand, for E_3, let $V_i^t = \{\mathbf{x}^* = t\mathbf{x} + (1-t)\mathbf{x}_i \mid \mathbf{x} \in V_i\}$. By changing variable $\mathbf{x}^* = t\mathbf{x} + (1-t)\mathbf{x}_i$ and using $ds(\mathbf{x}) \le 2ds(\mathbf{x}^*)/t^2$, we get

$$E_3 \le 2h^2 \int_0^1 \int_{V_i^t} (|H(Ew)|/t)\, ds(\mathbf{x}^*) dt\ .$$

Obviously, $m(V_i^t) \le t^2 m(V_i)$. By a proof similar to that given in [9], we get

$$\begin{aligned} E_3 &\le 2h^2 \int_0^1 \left(\int_{V_i^t} |H(Ew)|^2\, ds(\mathbf{x}^*) \right)^{1/2} m(V_i)^{1/2} dt \\ &\le Ch^2 m(V_i)^{1/2} \|w\|_{H^2(V_i)}\ . \end{aligned} \tag{4.6}$$

Finally, we obtain (4.1) for $u \in H^2(\mathbb{S}^2)$ by combining (4.4) and (4.5) with (4.6) and invoking a density argument. ∎

Note that in the planar case we have $E_2 = 0$ instead of (4.5).

4.2. Estimates for the weak forms. For simplicity, we assume $a(\mathbf{x}) = 1$. We first compare the bilinear forms (1.2) and (3.13). Let $\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}}$ be the unit outer normal at $\mathbf{x} \in \partial\tilde{T}_{ijk}$ of the boundary of $\tilde{T}_{ijk}$ that is tangent to $\mathbb{S}^2$. By Green's formula, for $\mathcal{W} \in H^2(\mathbb{S}^2)$ we have

$$\begin{aligned}
\mathcal{A}(u-u_h, \mathcal{P}_{\mathcal{U}}(w)) &= \int_{\mathbb{S}^2} \nabla_s(u-u_h)\cdot\nabla_s\mathcal{P}_{\mathcal{U}}(w)\, ds(\mathbf{x})\\
&= \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \left(\int_{\tilde{T}_{ijk}} \nabla_s(u-u_h)\cdot\nabla_s\mathcal{P}_{\mathcal{U}}(w) + (u-u_h)(\vec{\mathbf{v}}\cdot\nabla_s\mathcal{P}_{\mathcal{U}}(w))\, ds(\mathbf{x})\right)\\
&\quad + \int_{\mathbb{S}^2} b(u-u_h)\mathcal{P}_{\mathcal{U}}(w)\, ds(\mathbf{x})\\
&= \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \left(\int_{\tilde{T}_{ijk}} -\triangle_s(u-u_h)\mathcal{P}_{\mathcal{U}}(w) + (\nabla_s\cdot(u-u_h)\vec{\mathbf{v}})\mathcal{P}_{\mathcal{U}}(w)\, ds(\mathbf{x})\right.\\
&\quad \left. + \int_{\partial\tilde{T}_{ijk}} (\nabla_s(u-u_h)\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})\mathcal{P}_{\mathcal{U}}(w) - (u-u_h)(\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})\mathcal{P}_{\mathcal{U}}(w)\, d\gamma(\mathbf{x})\right)\\
&\quad + \int_{\mathbb{S}^2} b(u-u_h)\mathcal{P}_{\mathcal{U}}(w)\, ds(\mathbf{x})\ ,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\mathcal{A}^*(u-u_h, \mathcal{P}_{\mathcal{V}}(w)) &= \sum_{i=1}^{n} \mathcal{P}_{\mathcal{V}}(w)(\mathbf{x}_i)\mathcal{A}^*(u-u_h,\psi_i)\\
&= \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \left(\int_{\tilde{T}_{ijk}} -\triangle_s(u-u_h)\mathcal{P}_{\mathcal{V}}(w) + (\nabla_s\cdot(u-u_h)\vec{\mathbf{v}})\mathcal{P}_{\mathcal{V}}(w)\, ds(\mathbf{x})\right.\\
&\quad \left. + \int_{\partial\tilde{T}_{ijk}} (\nabla_s(u-u_h)\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})\mathcal{P}_{\mathcal{V}}(w) - (u-u_h)(\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})\mathcal{P}_{\mathcal{V}}(w)\, d\gamma(\mathbf{x})\right)\\
&\quad + \int_{\mathbb{S}^2} bP_{\mathcal{V}}(u-u_h)\mathcal{P}_{\mathcal{V}}(w)\, ds(\mathbf{x}).
\end{aligned} \tag{4.8}$$

We now compare the first term of each functional.

LEMMA 2. *There is a constant $C > 0$ such that for $u \in H^3(\mathbb{S}^2)$ and $w \in H^2(\mathbb{S}^2)$,*

$$\left| \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} \triangle_s u(\mathcal{P}_{\mathcal{U}}(w) - \mathcal{P}_{\mathcal{V}}(w))\, ds(\mathbf{x}) \right| \le Ch^2 \|u\|_{H^3(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)}\ . \tag{4.9}$$

Proof. Letting E denote the left-hand side of (4.9), we have

(4.10)

$$\begin{aligned}
E &= \left| \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} \triangle_s u(\mathcal{P}_{\mathcal{U}}(w) - \mathcal{P}_{\mathcal{V}}(w))\, ds(\mathbf{x}) \right|\\
&\le \left| \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} \triangle_s u(\mathcal{P}_{\mathcal{U}}(w) - w)\, ds(\mathbf{x}) \right| + \left| \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} \triangle_s u(\mathcal{P}_{\mathcal{V}}(w) - w)\, ds(\mathbf{x}) \right|\\
&\le \left| \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} \triangle_s u(\mathcal{P}_{\mathcal{U}}(w) - w)\, ds(\mathbf{x}) \right| + \left| \sum_{i=1}^{n} \int_{V_i} \Pi_{\mathcal{V}}(\triangle_s u)(\mathcal{P}_{\mathcal{V}}(w) - w)\, ds(\mathbf{x}) \right|\\
&\quad + \left| \sum_{i=1}^{n} \int_{V_i} (\triangle_s u(\mathbf{x}) - \Pi_{\mathcal{V}}(\triangle_s u))(\mathcal{P}_{\mathcal{V}}(w) - w)\, ds(\mathbf{x}) \right|,
\end{aligned}$$

where $\Pi_{\mathcal{V}}$ denotes the L^2 projection on $\mathcal{V}_{\mathcal{W}}$. Using Proposition 3 and the Cauchy–Schwarz inequality, we get

$$\left|\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\tilde{T}_{ijk}}\triangle_s u(\mathcal{P}_{\mathcal{U}}(w)-w)\;ds(\mathbf{x})\right|\le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}.$$

Using Lemma 1, we have

$$\begin{aligned}\left|\sum_{i=1}^n\int_{V_i}\Pi_{\mathcal{V}}(\triangle_s u)(\mathcal{P}_{\mathcal{V}}(w)-w)\;ds(\mathbf{x})\right| &\le Ch^2\sum_{i=1}^n|\Pi_{\mathcal{V}}(\triangle_s u)|_{V_i}|\,m(V_i)^{1/2}\|w\|_{H^2(V_i)}\\ &= Ch^2\|\Pi_{\mathcal{V}}(\triangle_s u)\|_{L^2(\mathbb{S}^2)}\|w\|^2_{H^2(\mathbb{S}^2)}\\ &\le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}\end{aligned}$$

$$\left|\sum_{i=1}^n\int_{V_i}(\triangle_s u(\mathbf{x})-\Pi_{\mathcal{V}}(\triangle_s u))(\mathcal{P}_{\mathcal{V}}(w)-w)\;ds(\mathbf{x})\right|\le Ch^2\|u\|_{H^3(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}\;.$$

Thus we obtain the estimate (4.9) in the lemma. ▯

Now we are ready to show the following.

LEMMA 3. *Let $u_h\in\mathcal{U}_{\mathcal{W}}$ be the unique solution of the discrete system* (3.14) *and assume that the unique variational solution u of* (1.1) *belongs to $H^3(\mathbb{S}^2)$. Then, for any $w\in H^2(\mathbb{S}^2)$, there exists a constant $C>0$ such that*

$$|\mathcal{A}(u-u_h,\mathcal{P}_{\mathcal{U}}(w))-\mathcal{A}^*(u-u_h,\mathcal{P}_{\mathcal{V}}(w))|\le Ch^2\|u\|_{H^3(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}. \tag{4.11}$$

Proof. By equation (4.8) we obtain

$$\mathcal{A}(u-u_h,\mathcal{P}_{\mathcal{U}}(w))-\mathcal{A}^*(u-u_h,\mathcal{P}_{\mathcal{V}}(w))=E_1+E_2+E_3+E_4+E_5+E_6, \tag{4.12}$$

where

$$\begin{aligned}E_1&=-\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\tilde{T}_{ijk}}\triangle_s u(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))\;ds(\mathbf{x})\;,\\ E_2&=\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\tilde{T}_{ijk}}\triangle_s u_h(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))\;ds(\mathbf{x})\;,\\ E_3&=\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\partial\tilde{T}_{ijk}}(\nabla_s(u-u_h)\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))\;d\gamma(\mathbf{x})\;,\\ E_4&=\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\tilde{T}_{ijk}}(\nabla_s\cdot(u-u_h)\vec{\mathbf{v}})(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))\;ds(\mathbf{x})\;,\\ E_5&=\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}}\int_{\partial\tilde{T}_{ijk}}(u-u_h)(\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))\;d\gamma(\mathbf{x})\;,\\ E_6&=\int_{\mathbb{S}^2}b((u-u_h)\mathcal{P}_{\mathcal{U}}(w)-P_{\mathcal{V}}(u-u_h)\mathcal{P}_{\mathcal{V}}(w))\;ds(\mathbf{x})\;.\end{aligned}$$

Consider E_1. By Lemma 2 we have

$$|E_1|\le Ch^2\|u\|_{H^3(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}. \tag{4.13}$$

As for E_2, using Proposition 3 and Theorem 2 we have

$$
\begin{aligned}
|E_2| &\le \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |\triangle_s u_h(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))|\, ds(\mathbf{x}) \\
&\le Ch \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |\nabla_s u_h|\, |(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))|\, ds(\mathbf{x}) \qquad (4.14)\\
&\le Ch^2\|u_h\|_{H^1(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)} \le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}.
\end{aligned}
$$

According to the continuity of $\nabla_s u$ on each $\partial\tilde{T}_{ijk}$, we have

$$
\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\partial\tilde{T}_{ijk}} (\nabla_s u\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))\, d\gamma(\mathbf{x}) = 0,
$$

and thus we get

$$
E_3 = \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\partial\tilde{T}_{ijk}} (\nabla_s u_h\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))\, d\gamma(\mathbf{x})\,. \qquad (4.15)
$$

Additionally, on each edge $\tilde{L}$ of $\tilde{T}_{ijk}$, by symmetry with respect to the midpoint of $\tilde{L}$, we have that $\nabla_s u_h(z_1)\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}}$ is an even function for $\mathbf{x}\in\tilde{L}$ while $\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w)$ is odd. Thus,

$$
\int_{\tilde{L}} (\nabla_s u_h\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))\, d\gamma(\mathbf{x}) = 0.
$$

Thus we have

$$
E_3 = 0. \qquad (4.16)
$$

About E_4, we have by Theorem 2 that

$$
\begin{aligned}
|E_4| &\le \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |(\nabla_s\cdot(u-u_h)\vec{\mathbf{v}})(\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w))|\, ds(\mathbf{x}) \\
&\le \sup_{\mathbf{x}\in\mathbb{S}^2}(|\vec{\mathbf{v}}|+|\nabla_s\vec{\mathbf{v}}|) \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\tilde{T}_{ijk}} |\nabla_s(u-u_h)|\, |\mathcal{P}_{\mathcal{U}}(w)-\mathcal{P}_{\mathcal{V}}(w)|\, ds(\mathbf{x}) \qquad (4.17)\\
&\le Ch\|u-u_h\|_{H^1(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)} \le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}.
\end{aligned}
$$

About E_5, using Trace theorem [17] and Theorem 2 we have

$$
\begin{aligned}
|E_5| &\le \sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\partial\tilde{T}_{ijk}} |(u-u_h)(\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}_{\mathbf{x},\tilde{T}_{ijk}})(\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w))|\, d\gamma(\mathbf{x}) \\
&\le \sup_{\mathbf{x}\in\mathbb{S}^2}(|\vec{\mathbf{v}}|)\left(\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\partial\tilde{T}_{ijk}} |u-u_h|^2\, d\gamma(\mathbf{x})\right)^{1/2} \\
&\quad\cdot\left(\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \int_{\partial\tilde{T}_{ijk}} |\mathcal{P}_{\mathcal{V}}(w)-\mathcal{P}_{\mathcal{U}}(w)|^2\, d\gamma(\mathbf{x})\right)^{1/2} \qquad (4.18)\\
&\le C\left(\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} \|u-u_h\|^2_{H^1(\tilde{T}_{ijk})}\right)^{1/2}\left(\sum_{\tilde{T}_{ijk}\in\tilde{\mathcal{T}}} h^2\|w\|^2_{H^2(\tilde{T}_{ijk})}\right)^{1/2}. \\
&\le Ch\|u-u_h\|_{H^1(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)} \le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}.
\end{aligned}
$$

About E_6, by Proposition 3 and Theorem 2 we have

$$
\begin{aligned}
|E_6| &\le \left| \int_{\mathbb{S}^2} b(u-u_h)(\mathcal{P}_{\mathcal{U}}(w) - \mathcal{P}_{\mathcal{V}}(w))\, ds(\mathbf{x}) \right| \\
&\quad + \left| \int_{\mathbb{S}^2} b((u-u_h) - \mathcal{P}_{\mathcal{V}}(u-u_h))(\mathcal{P}_{\mathcal{V}}(w))\, ds(\mathbf{x}) \right| \\
&\le C\|u-u_h\|_{L^2(\mathbb{S}^2)}\, \|\mathcal{P}_{\mathcal{U}}(w) - \mathcal{P}_{\mathcal{V}}(w)\|_{L^2(\mathbb{S}^2)} \\
&\quad + Ch\|u-u_h\|_{H^1(\mathbb{S}^2)}\, \|\mathcal{P}_{\mathcal{V}}(w)\|_{L^2(\mathbb{S}^2)} \\
&\le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}.
\end{aligned}
\tag{4.19}
$$

Combining (4.12)–(4.14) and (4.16)–(4.19), we get (4.11). ∎

We see from the above proof that the extra regularity of $u \in H^3(\mathbb{S}^2)$ is only required for estimating the term E_1 (which is given in Lemma 2); the other terms merely require $u \in H^2(\mathbb{S}^2)$.

LEMMA 4. *Let $u_h \in \mathcal{U}_{\mathcal{W}}$ be the unique solution of the discrete system* (3.14) *and assume that the unique variational solution u of* (1.1) *belongs to $H^2(\mathbb{S}^2)$. Then, for any $w \in H^2(\mathbb{S}^2)$, there exists a constant $C > 0$ such that*

$$
|\mathcal{A}_{\mathcal{W}}(u_h, \mathcal{P}_{\mathcal{V}}(w)) - \mathcal{A}^*(u_h, \mathcal{P}_{\mathcal{V}}(w))| \le Ch^2\|u\|_{H^2(\mathbb{S}^2)}\|w\|_{H^2(\mathbb{S}^2)}\ . \tag{4.20}
$$

Proof. Since $\mathcal{P}_{\mathcal{V}}(u_h)|_{V_i} = u_h(\mathbf{x}_i)$ and $\mathcal{P}_{\mathcal{V}}(w)|_{V_i} = w(\mathbf{x}_i)$, we have

$$
\sum_{i=1}^n \int_{\partial V_i} (u_h - \mathcal{P}_{\mathcal{U}}(u_h))(\vec{\mathbf{v}} \cdot n_{\mathbf{x},V_i})\mathcal{P}_{\mathcal{V}}(w)\, d\gamma(\mathbf{x}) = 0,
$$

$$
\sum_{i=1}^n \left(\int_{V_i} b\mathcal{P}_{\mathcal{V}}(u_h)\mathcal{P}_{\mathcal{V}}(w)\, ds(\mathbf{x}) - m(V_i)b_i u_h(\mathbf{x}_i)w(\mathbf{x}_i) \right) = 0.
$$

Thus

$$
\begin{aligned}
&\mathcal{A}^*(u_h, \mathcal{P}_{\mathcal{V}}(w)) - \mathcal{A}_{\mathcal{W}}(u_h, \mathcal{P}_{\mathcal{V}}(w)) \\
&\quad = \sum_{i=1}^n \left(\int_{\partial V_i} (-\nabla_s u_h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i})\mathcal{P}_{\mathcal{V}}(w)\, d\gamma(\mathbf{x}) - \sum_{j\in\chi_i} \mathcal{F}_{ij}(u_h)\mathcal{P}_{\mathcal{V}}(w) \right) \\
&\quad = \sum_{i=1}^n \sum_{j\in\chi_i} m(\Gamma_{ij})\xi_{ij} w(x_i)\, ,
\end{aligned}
\tag{4.21}
$$

with

$$
\begin{aligned}
\xi_{ij} &= -\frac{1}{m(\Gamma_{ij})} \int_{\partial\Gamma_{ij}} \nabla_s u_h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, d\gamma(\mathbf{x}) + \frac{u_h(\mathbf{x}_i) - u_h(\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|} \\
&= -\frac{1}{m(\Gamma_{ij})} \int_{\partial\Gamma_{ij}} \nabla_s u_h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, d\gamma(\mathbf{x}) + \nabla E u_h(\mathbf{x}^*) \cdot \vec{\mathbf{n}}_{\mathbf{x},V_i}\, ,
\end{aligned}
$$

for any $\mathbf{x} \in \Gamma_{ij}$, $\mathbf{x}^* = \mathcal{P}^{-1}(\mathbf{x})$. Since $|\mathbf{x} - \mathbf{x}^*| \le Ch^2$, by Proposition 1

$$
|\nabla E u_h(\mathbf{x}^*) - \nabla_s u_h(\mathbf{x})| \le Ch^2 |\nabla E u_h(\mathbf{x}^*)|.
$$

Then we get

$$
\xi_{ij} \le \frac{Ch^2|u_h(\mathbf{x}_i) - u_h(\mathbf{x}_j)|}{|\mathbf{x}_i - \mathbf{x}_j|}\ . \tag{4.22}
$$

It is also easy to find that

$$E = \sum_{i=1}^{n} \sum_{j\in\chi_i} m(\Gamma_{ij})\xi_{ij} w(\mathbf{x}_i) = -\frac{1}{2}\sum_{i=1}^{n} \sum_{j\in\chi_i} m(\Gamma_{ij})\xi_{ij}|\mathbf{x}_i - \mathbf{x}_j| \frac{w(\mathbf{x}_i) - w(\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|} \ .$$

By Proposition 4 and Theorem 2 we have that

$$\begin{aligned} |E| &\le \sum_{i=1}^{n} \sum_{j\in\chi_i} m(\Gamma_{ij})\xi_{ij} d(\mathbf{x}_i,\mathbf{x}_j) \frac{|w(\mathbf{x}_i) - w(\mathbf{x}_j)|}{|\mathbf{x}_i - \mathbf{x}_j|} \\ &\le 2\left(\frac{1}{2}\sum_{i=1}^{n} \sum_{j\in\chi_i} m(\Gamma_{ij}) d(\mathbf{x}_i,\mathbf{x}_j)\xi_{ij}^2\right)^{1/2} \\ &\quad \cdot \left(\frac{1}{2}\sum_{i=1}^{n} \sum_{j\in\chi_i} m(\Gamma_{ij}) d(\mathbf{x}_i,\mathbf{x}_j)\left(\frac{w(\mathbf{x}_i) - w(\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|}\right)^2\right)^{1/2} \\ &\le Ch^2 |u_h|_{1,\mathcal{W}} |w|_{1,\mathcal{W}} \le Ch^2 \|u_h\|_{H^1(\mathbb{S}^2)} \|\mathcal{P}_{\mathcal{U}}(w)\|_{H^1(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)}. \end{aligned} \tag{4.23}$$

Combining (4.21) and (4.23), we thus get (4.20). ∎

We note that the results of the above lemmas hold for more general $a = a(\mathbf{x})$ as well, but some slight modifications of the proofs are needed.

4.3. Main result. We now present our main result on the L^2 error estimate.

THEOREM 3. *Let Assumption* 1 *be satisfied and additionally we assume that* $b \in H^1(\mathbb{S}^2)$. *Suppose that* $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$ *is an SCVM of* $\mathbb{S}^2$ *with the density function* ρ *satisfying* $\rho \in C^1(\mathbb{S}^2)$ *and* $\rho(\mathbf{x}) > 0$ *for any* $\mathbf{x} \in \mathbb{S}^2$. *Let* $\mathcal{F}_{ij}$ *be defined by* (3.8). *Then the discrete system* (3.14) *has a unique solution* $u_h \in \mathcal{U}_{\mathcal{W}}$. *Furthermore, assume that the unique solution* u *of* (1.1) *belongs to* $H^3(\mathbb{S}^2)$. *Then there exists a constant* $C > 0$ *only depending on* ρ, a, $\vec{\mathbf{v}}$, b, *and* σ *such that*

$$\|e_h\|_{L^2(\mathbb{S}^2)} \le Ch^2 \|u\|_{H^3(\mathbb{S}^2)}, \tag{4.24}$$

where $e_h(\mathbf{x}) = u(\mathbf{x}) - u_h(\mathbf{x})$.

Proof. Since $u - u_h \in H^1(\mathbb{S}^2)$, according to (1.3) we know that there exists a weak solution $w \in H^2(\mathbb{S}^2)$ satisfying

$$\mathcal{A}(w, v) = (u - u_h, v) \quad \forall\, v \in H^1(\mathbb{S}^2).$$

Putting $v = u - u_h$ in the above equality, we get

$$||u - u_h||^2_{L^2(\mathbb{S}^2)} = (u - u_h, u - u_h) = \mathcal{A}(w, u - u_h). \tag{4.25}$$

Furthermore, from the H^2 regularity estimate, we have

$$||w||_{H^2(\mathbb{S}^2)} \le C||u - u_h||_{L^2(\mathbb{S}^2)} \tag{4.26}$$

for some constant $C > 0$.

For the interpolants $\mathcal{P}_{\mathcal{U}}(w)$ and $\mathcal{P}_{\mathcal{V}}(w)$, we have

$$\mathcal{A}^*(u, \mathcal{P}_{\mathcal{V}}(w)) + \int_{\mathbb{S}^2} b(u - P_{\mathcal{V}}(u))\mathcal{P}_{\mathcal{V}}(w)\, ds(\mathbf{x}) = (f, \mathcal{P}_{\mathcal{V}}(w))$$

and

$$\mathcal{A}_{\mathcal{W}}(u_h, \mathcal{P}_{\mathcal{V}}(w)) = (f, \mathcal{P}_{\mathcal{V}}(w)) \, .$$

Consequently, we get

$$\begin{aligned} \|u-u_h\|^2_{L^2(\mathbb{S}^2)} \le\; & |\mathcal{A}(u-u_h, w-\mathcal{P}_{\mathcal{U}}(w))| \\ & + |\mathcal{A}(u-u_h, \mathcal{P}_{\mathcal{U}}(w)) - \mathcal{A}^*(u-u_h, \mathcal{P}_{\mathcal{V}}(w))| \\ & + |\mathcal{A}^*(u_h, \mathcal{P}_{\mathcal{V}}(w)) - \mathcal{A}_{\mathcal{W}}(u_h, \mathcal{P}_{\mathcal{V}}(w))| \\ & + \left| \int_{\mathbb{S}^2} b(u - P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| . \end{aligned} \tag{4.27}$$

According to Theorem 2, Proposition 3, and (4.26), we get

$$\begin{aligned} |\mathcal{A}(u-u_h, w-\mathcal{P}_{\mathcal{U}}(w))| &\le C\|u-u_h\|_{H^1(\mathbb{S}^2)} \, \|w-\mathcal{P}_{\mathcal{U}}(w)\|_{H^1(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \, \|w\|_{H^2(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \, \|u-u_h\|_{L^2(\mathbb{S}^2)} . \end{aligned} \tag{4.28}$$

By Lemma 3 and (4.26), we get

$$\begin{aligned} |\mathcal{A}(u-u_h, \mathcal{P}_{\mathcal{U}}(w)) - \mathcal{A}^*(u-u_h, \mathcal{P}_{\mathcal{V}}(w))| &\le Ch^2 \|u\|_{H^3(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^3(\mathbb{S}^2)} \|u-u_h\|_{L^2(\mathbb{S}^2)} \, . \end{aligned} \tag{4.29}$$

Again, by Lemma 4 and (4.26), we have

$$\begin{aligned} |\mathcal{A}_{\mathcal{W}}(u_h, \mathcal{P}_{\mathcal{V}}(w)) - \mathcal{A}^*(u_h, \mathcal{P}_{\mathcal{V}}(w))| &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \|u-u_h\|_{L^2(\mathbb{S}^2)} \, . \end{aligned} \tag{4.30}$$

It is easy to see that

$$\begin{aligned} \left| \int_{\mathbb{S}^2} b(u-P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| &= \left| \sum_{i=1}^n \int_{V_i} b(u-P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| \\ &\le \left| \sum_{i=1}^n \int_{V_i} \Pi_{\mathcal{V}}(b)(u-P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| \\ &\quad + \left| \sum_{i=1}^n \int_{V_i} (b-\Pi_{\mathcal{V}}(b))(u-P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| . \end{aligned}$$

Using a proof similar to Lemmas 1 and 2, we can get

$$\begin{aligned} \left| \int_{\mathbb{S}^2} b(u-P_{\mathcal{V}}(u)) \mathcal{P}_{\mathcal{V}}(w) \, ds(\mathbf{x}) \right| &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)} \\ &\le Ch^2 \|u\|_{H^2(\mathbb{S}^2)} \|u-u^h\|_{L^2(\mathbb{S}^2)} . \end{aligned} \tag{4.31}$$

Combining (4.28), (4.29), and (4.30), we get

$$\|u-u_h\|^2_{L^2(\mathbb{S}^2)} \le Ch^2 \|u\|_{H^3(\mathbb{S}^2)} \|u-u_h\|_{L^2(\mathbb{S}^2)},$$

which means

$$\|e_h\|_{L^2(\mathbb{S}^2)} = \|u-u_h\|_{L^2(\mathbb{S}^2)} \le Ch^2 \|u\|_{H^3(\mathbb{S}^2)} .$$

We end the proof by noting that the case of general $a(\mathbf{x})$ can be verified similarly. ∎

Remark 1. Extra regularity on the exact solution is required to get the quadratic order error estimates, though in the standard finite element literature such a requirement is not needed in general. This is seen as a consequence of the quadrature approximations to the standard weak forms of the equations. From the proof of Lemma 3, it is easy to see that the regularity in $H^3(\mathbb{S}^2)$ can in fact be further weakened to, for instance, $W^{3,p}(\mathbb{S}^2)$ for $p > 1$.

Remark 2. The quadratic order error estimates depend critically on the properties of the SCVMs. The proof is not valid for a general spherical Voronoi mesh. Such an L^2 error estimate has not been given in the literature even for the planar finite volume methods based on the general Voronoi–Delaunay meshes. With other choices of the covolumes which are not of the Voronoi–Delaunay type, a quadratic order estimate has been proved in [26] for 2d diffusion equations. A first order L^2 error estimate has been given in [20].

5. Superconvergent gradient recovery. In this section, we discuss how to postprocess the finite volume solutions to obtain their tangential gradients in the longitude and latitude directions. Let us rewrite (1.1) in the spherical coordinate system (ϕ, θ) defined by $\mathbf{x} = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$ for $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$. Ignoring the radial component, we may denote $\vec{\mathbf{v}}(\phi, \theta) = (v_1(\phi, \theta), v_2(\phi, \theta))$, where v_1 and v_2 are the (orthogonal) components of $\vec{\mathbf{v}}$ in the ϕ and θ directions, respectively, on the tangential surface of $\mathbb{S}^2$ at $\mathbf{x}$. We also have

$$\nabla_s u(\phi, \theta) = \left(\frac{1}{r} \frac{\partial u}{\partial \phi}, \; \frac{1}{r \sin\phi} \frac{\partial u}{\partial \theta} \right), \quad \nabla_s \cdot \vec{\mathbf{v}}(\phi, \theta) = \frac{1}{r \sin\phi} \left(\frac{\partial}{\partial \phi}(v_1 \sin\phi) + \frac{\partial v_2}{\partial \theta} \right).$$

For any generator $\mathbf{x}_i$ in $\mathcal{W}$, let $V_{\mathbf{x}_i} = \cup_{\mathbf{x}_i \in T_{ijk}} T_{ijk}$ and $\vec{\mathbf{n}}_{N,\mathbf{x}_i}$ be the unit vector on $S_{\mathbf{x}_i}$ along the ϕ direction. A map $H_{\mathbf{x}_i} : V_{\mathbf{x}_i} \to \mathbb{R}^2$ is defined by first projecting $V_{\mathbf{x}_i}$ onto the tangential plane $S_{\mathbf{x}_i}$ of $\mathbb{S}^2$ at $\mathbf{x}_i$ ($S_{\mathbf{x}_i} \perp \vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}_i}$ at $\mathbf{x}_i$) and then moving $S_{\mathbf{x}_i}$ to the (x, y)-plane via an affine map such that $\vec{\mathbf{n}}_{\mathbb{S}^2,\mathbf{x}_i}$ is mapped to the z-axis and $\vec{\mathbf{n}}_{N,\mathbf{x}_i}$ to the x-axis; see Figure 5.1. Let $V_i' = H_{\mathbf{x}_i}(V_{\mathbf{x}_i})$.

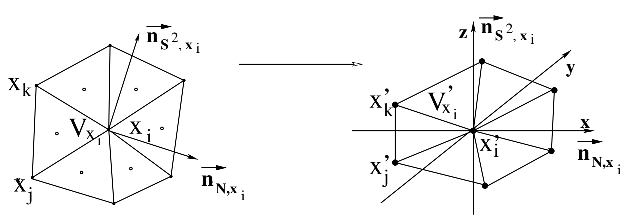

FIG. 5.1. *The mapping* $H_{\mathbf{x}_i}$.

On each planar triangle $\triangle \mathbf{x}_i' \mathbf{x}_j' \mathbf{x}_k' = H_{\mathbf{x}_i}(T_{ijk})$, we uniquely determine a linear function $\overline{u}_{T_{ijk}}$ by setting $\overline{u}_{T_{ijk}}(\mathbf{x}_i') = u_h(\mathbf{x}_i)$, $\overline{u}_{T_{ijk}}(\mathbf{x}_j') = u_h(\mathbf{x}_j)$, and $\overline{u}_{T_{ijk}}(\mathbf{x}_k') = u_h(\mathbf{x}_k)$. Now we define

$$\overline{\nabla}_s u_h(\mathbf{x}_i) = \frac{1}{q} \sum_{T_{ijk} \subset V_{\mathbf{x}_i}} \left(\frac{\partial \overline{u}_{T_{ijk}}}{\partial x}, \; \frac{\partial \overline{u}_{T_{ijk}}}{\partial y} \right), \tag{5.1}$$

where $q = \text{Card}(\{T_{ijk} \mid T_{ijk} \subset V_{\mathbf{x}_i}\})$. We also let

$$D(u, u_h) = \left(\sum_{i \in I} |\nabla_s u(\mathbf{x}_i) - \overline{\nabla}_s u_h(\mathbf{x}_i)|^2 m(V_i) \right)^{1/2}. \tag{5.2}$$

The index set I may be taken to be the set of all Voronoi generators or a large portion of the generator set. In light of the recent studies on the finite element gradient recovery [36] at mesh symmetric points, the close relationship between finite element and finite volume schemes [3, 34], and the nice properties of SCVMs, we expect that for the finite volume solution with SCVMs, there exists the estimate $D(u, u_h) = O(h^2)$. Such results are to be numerically investigated in the next section.

6. Numerical experiments. Let $\mathbb{S}^2$ be the unit sphere. We now present numerical results that are summarized in the following two examples, with each example containing two separate experiments (corresponding to two different exact solutions) but with one identical exact solution. In our experiments, the finite volume meshes are taken to be the SCVMs corresponding to a constant density function with various different numbers of generators.

For our first example, we choose the exact solution to be

$$u_1(\phi, \theta) = \sin^2 \phi \cos^2 \theta \tag{6.1}$$

and study two different model problems whose data are given in Table 6.1.

TABLE 6.1

Data for model problems		$a(\phi,\theta)$	$v_1(\phi,\theta)$	$v_2(\phi,\theta)$	$b(\phi,\theta)$
I	no convection	1	0	0	1
II	convection dominated	0.05	$1+\sin\phi$	$1+\sin\theta$	$3.0+\sin^2\phi$

Approximate solutions were obtained using the finite volume scheme (3.11) with the central difference scheme and the uniformly distributed SCVM based on the constant density function $\rho = 1$ (as in Figure 2.1). In Table 6.2, errors in the approximate solution are listed against the number of generators.

TABLE 6.2

n		$\|u_1 - u_{1,h}\|_{L^2(\mathbb{S}^2)}$	$D(u_1, u_{1,h})$	$\|u_2 - u_{2,h}\|_{L^2(\mathbb{S}^2)}$	$D(u_2, u_{2,h})$
162	I	4.038E-02	1.331E-01	4.032E-01	4.313E-00
	II	6.406E-02	1.655E-01	5.962E-01	4.314E-00
642	I	1.021E-02	3.444E-02	1.370E-01	1.178E-00
	II	1.612E-02	4.214E-02	1.241E-01	1.115E-00
2562	I	2.556E-03	8.788E-03	2.687E-02	3.115E-01
	II	3.994E-03	1.161E-02	3.033E-02	2.908E-01
10242	I	6.362E-04	2.221E-03	7.445E-03	7.902E-02
	II	1.004E-03	3.072E-03	7.577E-03	7.346E-02
40962	I	1.631E-04	5.269E-04	2.080E-04	1.745E-02
	II	2.375E-04	8.132E-04	2.072E-04	1.797E-02

For the second example, the exact solution of (1.1) is chosen to be

$$u_2(\phi, \theta)) = \sin^2(2\phi) \cos(4\theta). \tag{6.2}$$

Errors in the approximate solution are again given in Table 6.2. As the exact solution (6.2) is more complex than (6.1), the largest 2% of the pointwise gradient errors $|\nabla_s u_2(\mathbf{x}_i) - \overline{\nabla}_s u_{2,h}(\mathbf{x}_i)|$ was removed from the estimate when computing the $D(u_2, u_{2,h})$. These relatively larger errors concentrate near the 12 defect points of the SCVM (i.e., those Voronoi cells with only 5 neighbors) where the mesh lacks perfect symmetry.

From the numerical values given in the tables we see that, for both the L^2 errors and the gradient recovery errors $D(u, u_h)$, the trend of quadratic order convergence is very evident as we refine the mesh.

7. Conclusion. High quality spherical grids have many applications. Many strategies have already been studied in atmospheric and geophysical simulations for producing good spherical grids [31]. Though many of these choices produce good quality grids, in general the recently proposed concept of SCVT [8, 9] yields grids superior to most of existing ones. Our study here on a finite volume approximation of linear convection diffusion equations based on the SCVM demonstrated further their optimality from both theoretical and computational standpoints.

Further studies can be carried out to explore the local energy equipartition property and hierarchical SCVMs for multiresolution analysis, to validate superconvergent gradient recovery through analytical means. The application of the SCVM to Ginzburg–Landau models has been studied recently [10, 11] and we expect to find many more applications to other complex physical problems in the future.

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