Numer. Math. (1999) 81: 539–560

Numerische Mathematik

Analysis and computation of a mean-field model for superconductivity

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Received January 21, 1997

Summary. A mean-field model for superconductivity is studied from both the analytical and computational points of view. In this model, the individual vortex-like structures occuring in practical superconductors are not resolved. Rather, these structures are homogenized and a vortex density is solved for. The particular model studied includes effects due to the pinning of vortices. The existence and uniqueness of solutions of a regularized version of the model are demonstrated and the behavior of these solutions as the regularization parameter tends to zero is examined. Then, semi-discrete and fully discrete finite element based discretizations are formulated and analyzed and the results of some computational experiments are presented.

Mathematics Subject Classification (1991): 65N30

1. Introduction

Superconductivity can be modelled at different scales ranging from the atomistic to that visible with the naked eye. The BCS model of Bardeen, Cooper, and Schrieffer [1] is universally accepted as a discrete, atomistic model for standard superconductors whose material properties are homogeneous and isotropic. The Ginzburg-Landau model of superconductivity [7] is a

⋆ The research was supported in part by NSF under Grant DMS-9500718 and a DAG grant from HKUST

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mezoscale, continuum, phenomenological, steady-state model that resolves phenomena occuring in these types of materials on the scale of 100 or so Å. Although this model cannot resolve at the atomistic scale, it is refined enough that it can resolve the individual vortex-like structures that occur in the most useful superconductors in particular electromagnetic and thermodynamic configurations. In certain restricted circumstances, Gor'kov [9] has derived the Ginzburg-Landau model from the BCS model. For details concerning the BCS and Ginzburg-Landau models and their connection, one may consult [13] and the references cited therein. Time dependent versions of the Ginzburg-Landau model have also been formulated; see, e.g., [10].

The Ginzburg-Landau model, although being a continuum model, cannot be used to model any other than the tiniest of devices. The reason for this is that the individual vortex-like structures mentioned above are separated by distance of the order of 100 or less Å, so that certainly a superconducting sample of dimension, say, a millimeter, will contain a huge number of such structures. Thus, for example, in such a situtation, it is hopeless to use the Ginzburg-Landau model as the basis of a numerical simulation of superconducting phenomena. Thus, at the macroscale, i.e., the scale of devices, one should use continuum models that do not attempt to resolve the individual superconducting vortices, but rather that determine an averaged, homogenized quantity such as the density of these vortices. Simple models of this type, e.g., the Bean model, have existed for some time now, and are very much in use in that community dedicated to the incorporation of superconductors in the design of practical devices. However, in many situations, these very simple models are known to provide incorrect results so that better macroscale models would certainly be of use.

Recently, Chapman and co-workers have developed, in a series of papers [2], [5], and [6], mean-field models for the motion of vortices in the mixed state of a type-II superconductor. In these models, the individual vortices are homogenized to give a vortex density, or vorticity. The case of vortices in an inviscid fluid was used as a paradigm problem to derive the models for superconductors. The models were derived by taking appropriate limits within the Ginzburg-Landau formalism. In this paper, our goal is to study one these models from both an analytic and computational point of view.

Let the superconducting sample occupy the bounded domain $\Omega \in \mathbb{R}^2$ having a smooth boundary Γ. Let ω and H represent the vortex density and the average magnetic field, respectively. Let ω_0 be the initial vortex density, H_1 be the external applied magnetic field, and J_c be the value of the critical current. Then, the mean-field model of [6] we study is given by the system

$$\omega = H - \lambda^2 \Delta H \quad \text{in } \Omega \times (0,T)\,, \tag{1.1}$$

$$\omega_t - \nabla \cdot (m \nabla H) = 0 \quad \text{in } \Omega \times (0,T)\,, \tag{1.2}$$

(1.3) $$\omega|_{t=0} = \omega_0 \quad \text{in } \Omega\,,$$

(1.4) $$H|_\Gamma = H_1 \quad \text{on } (0,T)\,,$$

(1.5) $$m \ge 0\,, \quad \text{and} \quad |\nabla H| \le J_{\rm c}$$

(1.6) $$\text{with} \quad |\nabla H| < J_{\rm c} \quad \text{whenever} \quad m = 0\,.$$

In (1.1), λ denotes a length scale, known as the penetration depth, which gives and indication of how rapidly the magnetic field varies in the superconductor.

The motion of the vortex-like structures in supercondutors induces an electric field and therefore resistance. Thus, the ability to suppress such motion is important to the usefullness of superconducting devices. There are various mechanisms for “pinning’ these vortices; see [3] and [4] for a discussion of two pinning mechanisms in the context of the Ginzburg-Landau model. The specific mean-field model (1.1)-(1.6) accounts for the pinning of superconducting vortices. For values of the current below the critical value $J_{\rm c}$, the pinning forces are sufficiently strong so that the vortices are not moved by the passage of the current, and perfect conductivity is retained. However, at the value $J_{\rm c}$, the Lorentz force on the vortices due to the current are sufficiently strong so that the pinning force is overcome and vortices move. This phenomona is embodied within the model through (1.5) and (1.6).

The plan for the rest of the paper is as follows. In Sect. 2, we introduce a regularization of the problem (1.1)-(1.6) which facilitates the application, at least in an approximate manner, of the conditions (1.5) and (1.6). In Sect. 3, we study the regularized problem, including showing that it has a unique solution. In Sect. 4, we study the behavior of the solutions of the regularized problem as the value of the regularization parameter tends to zero. In Sects. 5 and 6, we respectively study semi-discrete and fully-discrete finite element based algortihms for the numerical approximation of solutions of the regularized problem and, in Sect. 7, we present the results of some computational experiments.

2. Regularization

Let $L^p(0,T;W_0^{1,q}(\Omega))$ and $H^1(0,T;W_0^{1,q}(\Omega))$ denote the standard Sobolev spaces of real valued functions of the variables $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^2$. Then, let

$$\mathcal{V} = L^\infty(0,T;W_0^{1,\infty}(\Omega))$$

and

$$\mathcal{K} = \{\, \phi \in \mathcal{V} \;:\; |\nabla\phi| \le 1 \text{ a.e., } \phi|_\Gamma = 0 \,\}\,.$$

Let $(\cdot,\cdot)$ denote the standard L^2 inner product in Ω and $<\cdot,\cdot>$ denote standard L^2 inner product in $(0,T)\times\Omega$, where $(0,T)$ is a given bounded time interval.

We suppose the the boundary data H_1 is a constant on Γ, but not necessarily in time. The initial magnetic field H_0 is determined from the initial vorticity ω_0 by solving $H_0 - \lambda^2 \Delta H_0 = \omega_0$ along with $H_0|_\Gamma = H_1|_{t=0}$. We may set, without loss of generality, the critical current $J_c = 1$ by normalizing H and ω by J_c.

Let $u = H - H_1$ and $u_0 = H_0 - H_1$ and let ϵ be an arbitrary positive number. Then, we consider the regularized problem

$$(2.1)\quad \frac{\partial u_\epsilon}{\partial t} - \lambda^2 \Delta \frac{\partial u_\epsilon}{\partial t} + \frac{1}{\epsilon}\beta_0(u_\epsilon) = -\frac{\partial H_1}{\partial t} = g \quad \text{in } \Omega\times(0,T)\,,$$

$$(2.2)\quad u_\epsilon = 0 \quad \text{on } \Gamma\times(0,T)\,,$$

and

$$(2.3)\quad u_\epsilon|_{t=0} = u_0 \quad \text{in } \Omega\,,$$

where $\beta_0 : L^4(0,T;W_0^{1,4}(\Omega)) \to (L^4(0,T;W_0^{1,4}(\Omega)))'$ is defined by

$$(\beta_0(u),\phi) = \int_\Omega (|\nabla u|^2 - 1)^+ \nabla u\cdot\nabla\phi \qquad \forall\,\phi\in L^4(0,T;W_0^{1,4}(\Omega))\,.$$

Here $(\psi)^+ = \psi$ whenever $\psi \geq 0$ and $(\psi)^+ = 0$ otherwise. One can easily verify that β_0 is a monotone operator. We assume that the initial data u_0 satisfies

$$(2.4)\quad |\nabla u_0| \leq 1\,.$$

Note that, in terms of our original variables, (2.4) requires that $|\nabla H_0 - \nabla H_1| \leq J_c$, a requirement that can be practically and easily met by, e.g., choosing $|\omega_0|$ sufficiently small and $H_1|_{t=0} = 0$.

A weak formulation of (2.1)-(2.3) is then given by

$$(2.5)\quad (\frac{\partial u_\epsilon}{\partial t},\phi) + \lambda^2(\nabla\frac{\partial u_\epsilon}{\partial t},\nabla\phi) + \frac{1}{\epsilon}(\beta_0(u_\epsilon),\phi) = (g,\phi) \qquad \forall\,\phi\in W_0^{1,4}(\Omega).$$

Remark. We could also regularize the initial condition, e.g., replace (2.3) by

$$u_\epsilon|_{t=0} = u_{0,\epsilon} \quad \text{in } \Omega\,.$$

Then, all the results derived in the sequel remain valid if we assume that $|\nabla H_0| \leq J_c$ and that the reqularized initial condition $u_{0,\epsilon}\in W_0^{1,4}(\Omega)$ satisfies the conditions

$$\|u_{0,\epsilon}\|^2_{H^1(\Omega)} + \frac{1}{\epsilon}\|(|\nabla u_{0,\epsilon}|^2 - 1)^+\|^2_{L^2(\Omega)} \leq C\|u_0\|^2_{H^1(\Omega)}$$

and $u_{0,\epsilon}\to u_0$ in $H^1(\Omega)$ as $\epsilon\to 0$.

3. Energy estimates

In this section, we derive some estimates that will be of use in demonstrating the existence and uniqueness of the solution of the weak formulation (2.5).

Lemma 3.1 *Suppose* $g \in L^1(0,T;H^{-1}(\Omega))$ *and* $u_0 \in H_0^1(\Omega)$*, where* $0 < T < \infty$ *is a given time. Then, for* $t \in (0,T)$ *and for any* $\epsilon > 0$*, there exists a constant C that is independent of* ϵ *such that*

$$\|u_\epsilon\|_{L^\infty(0,t;W_0^{1,2}(\Omega))} \le C\{\|g\|_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)}\} \tag{3.1}$$

and

$$\|u_\epsilon\|^2_{L^4(0,t;W_0^{1,4}(\Omega))} \le C\{\|g\|_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)}\}\,. \tag{3.2}$$

Proof. Setting $\phi = u_\epsilon$ in (2.5) yields that

$$\frac{1}{2}\frac{d}{dt}(u_\epsilon, u_\epsilon) + \frac{\lambda^2}{2}\frac{d}{dt}(\nabla u_\epsilon, \nabla u_\epsilon) + \frac{1}{\epsilon}((|\nabla u_\epsilon|^2 - 1)^+ \nabla u_\epsilon, \nabla u_\epsilon) = (g, u_\epsilon)\,.$$

Integrating in time, we have

$$\begin{aligned}\frac{1}{2}(u_\epsilon(t), u_\epsilon(t)) &+ \frac{\lambda^2}{2}(\nabla u_\epsilon(t), \nabla u_\epsilon(t))\\ &+\frac{1}{\epsilon}\int_0^t\int_\Omega (|\nabla u_\epsilon|^2 - 1)^+ |\nabla u_\epsilon|^2 \, d\Omega dt\\ &= \int_0^t (g, u_\epsilon)\, dt + \frac{1}{2}(u_0, u_0) + \frac{\lambda^2}{2}(\nabla u_0, \nabla u_0)\\ &\le \|g\|_{L^1(0,t;H^{-1}(\Omega))}\|u_\epsilon\|_{L^\infty(0,t;H^1(\Omega))}\\ &+\frac{1+\lambda^2}{2}\|u_0\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.3}$$

It follows that, $\forall\, t \in [0,T]$ and for any $\delta > 0$,

$$\begin{aligned}\|u_\epsilon(t)\|^2_{H^1(\Omega)} \le C_1(\frac{1}{4\delta}\|g\|^2_{L^1(0,t;H^{-1}(\Omega))} + \delta\|u_\epsilon\|^2_{L^\infty(0,t;H^1(\Omega))})\\ +C_3\|u_{0,\epsilon}\|^2_{H^1(\Omega)}\,,\end{aligned}$$

where the constants C_i, $i = 1, 2$, are independent ϵ as $\epsilon \to 0$. Hence, by choosing δ sufficiently small, we have that

$$\|u_\epsilon\|_{L^\infty(0,t;W_0^{1,2}(\Omega))} \le C(\|g\|_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)})\,,$$

where C is independnet of ϵ as $\epsilon \to 0$; this proves (3.1).

From (3.1) and (3.3), we have that

$$\int_0^t\int_\Omega (|\nabla u_\epsilon|^2 - 1)^+ |\nabla u_\epsilon|^2 \, d\Omega dt \le C\epsilon\,(\|g\|^2_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|^2_{H^1(\Omega)})$$

and

$$\int_0^t \int_\Omega |\nabla u_\epsilon|^2 \le C(\|g\|^2_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|^2_{H^1(\Omega)})$$

so that, since $z^4 \le (z^2-1)^+ z^2 + z^2$, we have that

$$\int_0^t \int_\Omega |\nabla u_\epsilon|^4 \le C(\|g\|^2_{L^1(0,t;H^{-1}(\Omega))} + \|u_0\|^2_{H^1(\Omega)})$$

from which (3.2) follows. Here, again, C is a constant that is independent of ϵ as $\epsilon \to 0$. □

Lemma 3.2 *Suppose* $g \in L^2(0,T;H^{-1}(\Omega))$ *and* $u_0 \in W_0^{1,4}(\Omega)$, *where* $0 < T < \infty$ *is a given time. If* u_0 *satisfies* (2.4), *then, for any* $\epsilon > 0$, *there exists a constant C that is independent of* ϵ *such that*

$$\|\frac{\partial u_\epsilon}{\partial t}\|_{L^2(0,t;W_0^{1,2}(\Omega))} \le C(\|g\|_{L^2(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)}) \tag{3.4}$$

and

$$\|u_\epsilon\|^2_{L^\infty(0,t;W_0^{1,4}(\Omega))} \le C(\|g\|_{L^2(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)})\,. \tag{3.5}$$

Proof. Note that

$$\frac{1}{4\epsilon}\frac{d}{dt}\left|(|\nabla u_\epsilon|^2-1)^+\right|^2 = \frac{1}{\epsilon}(|\nabla u_\epsilon|^2-1)^+\nabla u_\epsilon \cdot \nabla\frac{\partial u_\epsilon}{\partial t}$$

so that setting $\phi = \partial u_\epsilon/\partial t$ in (2.5) yields that

$$\begin{aligned}(\frac{\partial u_\epsilon}{\partial t},\frac{\partial u_\epsilon}{\partial t}) &+ \lambda^2(\nabla\frac{\partial u_\epsilon}{\partial t},\nabla\frac{\partial u_\epsilon}{\partial t})\\ &+\frac{1}{4\epsilon}\frac{d}{dt}\Big((|\nabla u_\epsilon|^2-1)^+,(|\nabla u_\epsilon|^2-1)^+\Big)\\ &= (g,\frac{\partial u_\epsilon}{\partial t})\end{aligned}$$

which implies that

$$\|\frac{\partial u_\epsilon}{\partial t}\|^2_{H^1(\Omega)} + \frac{1}{2\epsilon}\frac{d}{dt}\|(|\nabla u_\epsilon|^2-1)^+\|^2_{L^2(\Omega)} \le C\|g\|^2_{H^{-1}(\Omega)}\,.$$

Integrating in time then yields that, using (2.4),

$$\begin{aligned}\int_0^t \|\frac{\partial u_\epsilon}{\partial t}\|^2_{H^1(\Omega)}\,dt &+ \frac{1}{2\epsilon}\|(|\nabla u_\epsilon(t)|^2-1)^+\|^2_{L^2(\Omega)}\\ &\le C\|g\|^2_{L^2(0,t,H^{-1}(\Omega))} + \frac{1}{2\epsilon}\|(|\nabla u_\epsilon|^2-1)^+\|^2_{L^2(\Omega)}\\ &\le C\|g\|^2_{L^2(0,t,H^{-1}(\Omega))}\end{aligned} \tag{3.6}$$

Hence, we have (3.4).

It is also follows from (3.6) that

$$\|(|\nabla u_\epsilon|^2 - 1)^+\|^2_{L^\infty(0,t;L^2(\Omega))} \le \epsilon C \|g\|^2_{L^2(0,t;H^{-1}(\Omega))} .$$

Then, since $z^4 \le [(z^2-1)^+]^2 + z^2$, combining with (3.1) yields that

$$\begin{aligned}\|\nabla u_\epsilon\|^4_{L^\infty(0,t;L^4(\Omega))} &\le \|(|\nabla u_\epsilon|^2 - 1)^+\|^2_{L^\infty(0,t;L^2(\Omega))} \\ &\quad + \|\nabla u_\epsilon\|^2_{L^\infty(0,t;L^2(\Omega))} \\ &\le C(\|g\|_{L^2(0,t;H^{-1}(\Omega))} + \|u_0\|_{H^1(\Omega)})\end{aligned}$$

and then (3.5) follows. □

Combing the above estimates, we can obtain the existence and uniqueness of the weak solution to the problem (2.5).

Theorem 3.1 *For given $\epsilon > 0$ and $0 < T < \infty$, let $g \in L^2(0,T;H^{-1}(\Omega))$, $u_0 \in W_0^{1,4}$, and u_0 satisfy* (2.4). *Then, there exists a unique solution $u_\epsilon \in C(0,T;W_0^{1,4}(\Omega)) \cap W^{1,\infty}(0,T;W_0^{1,2}(\Omega))$ of the problem* (2.5). *Moreover, the solution u_ϵ is uniformly bounded in the space and independent of ϵ as $\epsilon \to 0$.*

Proof. The theorem can be proved by standard techniques using the results of previous lemmas. For example, see [12]. □

Corollary 3.1 *Under the assumptions of Theorem 3.1, there exists a constant $C > 0$ that is independent of ϵ as $\epsilon \to 0$, such that*

$$\|\beta_0(u_\epsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \le c\epsilon . \tag{3.7}$$

Proof. The result follows from (2.5) and the previous lemmas. □

4. Passing to the limit

We now consider the limit of the sequence $\{u_\epsilon\}$ as $\epsilon \to 0$. Lemmas 3.1 and 3.2 show that the sequence is bounded uniformly in $L^4(0,T;W_0^{1,4}(\Omega)) \cap H^1(0,T;W_0^{1,2}(\Omega))$ and in $L^\infty(0,T;W_0^{1,4}(\Omega)) \cap W^{1,\infty}(0,T;W_0^{1,2}(\Omega))$. Hence, there exists a subsequence of $\{u_\epsilon\}$ that converges to u weakly in these spaces. Without loss of generality, we still use $\{u_\epsilon\}$ to denote the subsequence.

First, by Corollary 3.1, we have

$$\beta_0(u_\epsilon) \to 0 \text{ in } \left(L^2(0,T;H_0^1(\Omega))\right)' \quad \text{as } \epsilon \to 0 .$$

Hence,

$$| < \beta_0(u_\epsilon), \phi > | \to 0 \quad \forall\, \phi \in L^4(0,T; W_0^{1,4}(\Omega))$$

as $\epsilon \to 0$. In particular,

$$| < \beta_0(u_\epsilon), u > | \to 0\,.$$

Since $[(z^2-1)^+]^2$ is convex, the operator β_0 is semi-continuous and, therefore,

$$< \beta_0(u), u >= 0\,.$$

Thus, we obtain that $u \in \mathcal{K}$.

Next, we show that u satisfies a variational ineqality. Let $\phi \in \mathcal{K}$. By (2.5),

$$\begin{aligned}
&(\frac{\partial\phi}{\partial t} - g, \phi - u_\epsilon) + (\lambda^2 \nabla \frac{\partial \phi}{\partial t}, \nabla\phi - \nabla u_\epsilon) \\
&= (\frac{\partial\phi}{\partial t} - \frac{\partial u_\epsilon}{\partial t}, \phi - u_\epsilon) + \Big(\lambda^2(\nabla\frac{\partial\phi}{\partial t} - \nabla\frac{\partial u_\epsilon}{\partial t}), \nabla\phi - \nabla u_\epsilon\Big) \\
&\quad -\frac{1}{\epsilon}(\beta_0(u_\epsilon), \phi - u_\epsilon)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^s \Big[(\frac{\partial\phi}{\partial t} - g, \phi - u_\epsilon) + (\lambda^2\nabla\frac{\partial\phi}{\partial t}, \nabla\phi - \nabla u_\epsilon)\Big]\, dt \\
&\quad = -I_\epsilon + \int_0^s \frac{1}{2}\frac{\partial}{\partial t}(\phi - u_\epsilon, \phi - u_\epsilon)\, dt \\
&\qquad + \int_0^s \frac{\lambda^2}{2}\frac{\partial}{\partial t}(\nabla\phi - \nabla u_\epsilon, \nabla\phi - \nabla u_\epsilon)\, dt \\
&\quad = I_\epsilon + \Big[\frac{1}{2}\{\|\phi - u_\epsilon\|^2_{L^2(\Omega)} + \lambda^2\, \|\nabla\phi - \nabla u_\epsilon\|^2_{L^2(\Omega)}\}\Big]_0^s, \qquad (4.1)
\end{aligned}$$

where

$$I_\epsilon = \frac{1}{\epsilon}\int_0^s (\beta_0(u_\epsilon), \phi - u_\epsilon)\, dt\,.$$

Using the monotonicity of β_0, we have that, since $\phi \in \mathcal{K}$,

$$I_\epsilon = \frac{1}{\epsilon}\int_0^s (\beta_0(u_\epsilon), \phi - u_\epsilon)\, dt \le \frac{1}{\epsilon}\int_0^s (\beta_0(\phi), \phi - u_\epsilon)\, dt = 0\,.$$

Therefore,

$$\lim_{\epsilon\to 0} I_\epsilon \le 0 \quad . \qquad (4.2)$$

As a result, we have the following theorem.

Theorem 4.1 *Let* $g \in L^2(0,T;H^{-1}(\Omega))$ *and* $u_0 \in \mathcal{K}$*. Then, there exists a unique solution of the following variational inequality problem: find* $u \in \mathcal{K} \cap H^1(0,T;W_0^{1,2}(\Omega))$ *such that* $u(0) = u_0$ *and*

$$\begin{aligned}\int_0^s \Big[(\frac{\partial \phi}{\partial t} - g, \phi - u) + (\lambda^2 \nabla \frac{\partial \phi}{\partial t}, \nabla\phi - \nabla u)\Big]\, dt \\ \le \frac{1}{2}\|\phi(s) - u(s)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(s) - \nabla u(s)\|^2_{L^2(\Omega)} \\ -\frac{1}{2}\|\phi(0) - u(0)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(0) - \nabla u(0)\|^2_{L^2(\Omega)}\end{aligned} \tag{4.3}$$

$\forall\, \phi \in \mathcal{K} \cap H^1(0,T;W_0^{1,2}(\Omega))$.

Proof. By (4.1) and (4.2),

$$\begin{aligned}\int_0^s \Big[(\frac{\partial \phi}{\partial t} - g, \phi - u_\epsilon) + (\lambda^2 \nabla \frac{\partial \phi}{\partial t}, \nabla\phi - \nabla u_\epsilon)\Big]\, dt \\ \le \frac{1}{2}\|\phi(s) - u_\epsilon(s)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(s) - \nabla u_\epsilon(s)\|^2_{L^2(\Omega)} \\ -\frac{1}{2}\|\phi(0) - u_{0,\epsilon}\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(0) - \nabla u_{0,\epsilon}\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Letting $\epsilon \to 0$, we obtain (4.3).

To show the uniqueness of the solution, suppose u_1 and u_2 are two solutions of (4.3), i.e., for $i = 1, 2$,

$$\begin{aligned}\int_0^s \Big(\frac{\partial \phi}{\partial t} - g, \phi - u_i) + (\lambda^2 \nabla \frac{\partial \phi}{\partial t}, \nabla\phi - \nabla u_i)\Big]\, dt \\ \le \frac{1}{2}\|\phi(s) - u_i(s)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(s) - \nabla u_i(s)\|^2_{L^2(\Omega)} \\ -\frac{1}{2}\|\phi(0) - u(0)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(0) - \nabla u(0)\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Let $\phi = (u_1 + u_2)/2$ and add the two equations. Then, we obtain

$$\begin{aligned}0 &\ge \frac{1}{2}\Big(\|\frac{u_2(s) - u_1(s)}{2}\|^2_{L^2(\Omega)} + \|\frac{u_1(s) - u_2(s)}{2}\|^2_{L^2(\Omega)}\Big) \\ &\qquad + \frac{\lambda^2}{2}\Big(\|\nabla\frac{u_2(s) - u_1(s)}{2}\|^2_{L^2(\Omega)} + \|\nabla\frac{u_1(s) - u_2(s)}{2}\|^2_{L^2(\Omega)}\Big) \\ &= \frac{1}{4}\Big(\|u_1(s) - u_2(s)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{4}\|\nabla u_1(s) - \nabla u_2(s)\|^2_{L^2(\Omega)}\Big)\,.\end{aligned}$$

Therefore, $u_1 = u_2$. □

5. Semi-discrete approximation

We now consider a finite element discretization with respect to the spatial variables based on the regularized problem. Suppose $S^h \subset W_0^{1,4}(\Omega)$ is a finite dimensional subspace. Specifically, we may assume S^h is the standard continuous piecewise linear finite element space. For $u_0 \in \mathcal{K}$, the initial approximation u_0^h is defined as the piecewise linear interpolant $I^h u_0$ of u_0. By approximation properties, we have that

$$u_0^h \to u_0 \quad \text{in } W_0^{1,4}(\Omega) \quad \text{as } h \to 0\,.$$

Then, a weak formulation of an approximate solution is given by

$$(\frac{\partial u_\epsilon^h}{\partial t}, \phi) + \lambda^2(\nabla \frac{\partial u_\epsilon^h}{\partial t}, \nabla\phi) + \frac{1}{\epsilon}(\beta_0(u_\epsilon^h), \phi) = (g, \phi) \quad \forall\, \phi \in S^h\,. \tag{5.1}$$

In one dimension, we have that

$$|\frac{d}{dx}(I^h u_0)(x)| \le |\frac{d}{dx}u_0(x)| \quad \text{a.e. in } \Omega\,. \tag{5.2}$$

In addition, in two dimensions and for $p > 2$, there exists a constant $C > 0$ such that

$$\|I^h v - v\|_{W^{1,\infty}(\Omega)} \le C\, h^{1-2/p}\|v\|_{W^{2,p}(\Omega)} \quad \forall\, v \in W^{2,p}(\Omega)\,.$$

In particular,

$$\|\nabla u_0^h - \nabla u_0\|_{L^\infty(\Omega)} \le C\, h^{1-2/p}\|u_0\|_{W^{2,p}(\Omega)}.$$

Since $u_0 \in \mathcal{K}$,

$$|\nabla u_0^h|^2 \le (1 + C\, h^{1-2/p}\|u_0\|_{W^{2,p}(\Omega)})^2 \le 1 + \tilde{c}\, h^{1-2/p}\,.$$

Thus,

$$(|\nabla u_0^h|^2 - 1)^+ \le c\; h^{1-2/p}\,.$$

Now assume that the initial solution is smooth. Then, for any ϵ such that $h^{2-4/p}/\epsilon$ remains uniformly bounded,

$$\frac{1}{\epsilon}\|(|\nabla u_0^h|^2 - 1)^+\|^2_{L^2(\Omega)} \le C$$

for some constant C which is independent of ϵ and h. Then we can get uniform estimates for u_ϵ^h that are independent of h and ϵ by the method used in Sect. 3. Hence, as $h \to 0$, the sequence $\{u_\epsilon^h\}$ converges weakly to a limit u^* in $L^4(0,T; W_0^{1,4}(\Omega)) \cap H^1(0,T; W_0^{1,2}(\Omega))$. We prove in the next theorem that the limit u^* is the solution of the variational inequality (4.3).

Theorem 5.1 *Let $g \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in \mathcal{K}$. Also, assume, in the two-dimensional case, that $u_0 \in W^{2,p}(\Omega)$ for some $p > 2$. Then, the solution u_ϵ^h of the weak problem* (5.1) *converges weakly to the solution u of* (4.3) *in $L^4(0,T;W_0^{1,4}(\Omega)) \cap H^1(0,T;W_0^{1,2}(\Omega))$ as $h \to 0$ and $\epsilon \to 0$ if $h^{2-4/p}/\epsilon$ is uniformly bounded and $h/\epsilon \to 0$.*

Proof. By adding and sutracting $I^h\phi(t)$, $\nabla I^h\phi(t)$, $\partial I^h\phi(t)/\partial t$ and $\nabla(\partial I^h\phi(t)/\partial t)$, we have that, for any $\phi \in \mathcal{K} \cap C^\infty(\Omega)$,

$$\begin{aligned}
&\int_0^s \Big[(\frac{\partial \phi}{\partial t} - g, \phi - u_\epsilon^h) + (\lambda^2 \nabla \frac{\partial \phi}{\partial t}, \nabla\phi - \nabla u_\epsilon^h)\Big]\, dt \\
&\qquad -\Big\{\frac{1}{2}\|\phi(t) - u_\epsilon^h(t)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi(t) - \nabla u_\epsilon^h(t)\|^2_{L^2(\Omega)}\Big\}\Big|_0^s \\
&= \Big\{(\phi(t) - I^h\phi(t), u_\epsilon^h(t)) + \lambda^2(\nabla\phi(t) - \nabla I^h\phi(t), \nabla u_\epsilon^h(t))\Big\}\Big|_0^s \\
&\qquad + \int_0^s \Big[(\frac{\partial I^h\phi}{\partial t} - g, I^h\phi - u_\epsilon^h) + (\lambda^2\nabla\frac{\partial I^h\phi}{\partial t}, \nabla I^h\phi - \nabla u_\epsilon^h)\Big]\, dt \\
&\qquad -\Big\{\frac{1}{2}\|I^h\phi(t) - u_\epsilon^h(t)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla I^h\phi(t) - \nabla u_\epsilon^h(t)\|^2_{L^2(\Omega)}\Big\}\Big|_0^s \\
&\qquad - \int_0^s \Big[(g, \phi - I^h\phi) + (\frac{\partial(\phi - I^h\phi)}{\partial t}, u_\epsilon^h) \\
&\qquad + (\lambda^2\nabla\frac{\partial(I^h\phi - \phi)}{\partial t}, \nabla u_\epsilon^h)\Big]\, dt \\
&= \mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_3 - \mathbf{I}_4\,.
\end{aligned}$$

As $n \to \infty$, we have

$$\mathbf{I}_1 = \Big\{(\phi(t) - I^h\phi(t), u_\epsilon^h(t)) + \lambda^2(\nabla\phi(t) - \nabla I^h\phi(t), \nabla u_\epsilon^h(t))\Big\}\Big|_0^s \to 0$$

by a density argument and the uniform boundness of u_ϵ^h. Moreover,

$$\begin{aligned}
\mathbf{I}_2 - \mathbf{I}_3 = &\int_0^s \Big[(\frac{\partial I^h\phi}{\partial t} - g, I^h\phi - u_\epsilon^h) + (\lambda^2\nabla\frac{\partial I^h\phi}{\partial t}, \nabla I^h\phi - \nabla u_\epsilon^h)\Big]\, dt \\
&-\Big\{\frac{1}{2}\|I^h\phi(t) - u_\epsilon^h(t)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla I^h\phi(t) \\
&-\nabla u_\epsilon^h(t)\|^2_{L^2(\Omega)}\Big\}\Big|_0^s \\
= &\,\frac{1}{\epsilon}\Big(\beta_0(u_\epsilon^h), u_\epsilon^h - I^h(\phi)\Big) \geq \frac{1}{\epsilon}\Big(\beta_0(I^h\phi), u_\epsilon^h - I^h(\phi)\Big)\,.
\end{aligned}$$

The right hand side is identically zero in one dimension by (5.2), while in two dimension, we note that

$$\|(|\nabla I^h\phi|^2 - 1)^+\|_{L^\infty(\Omega)} \leq ch\|\phi\|_{W^{2,\infty}(\Omega)}\,.$$

Hence, we obtain

$$\begin{aligned}\mathbf{I}_2-\mathbf{I}_3 &\geq \frac{1}{\epsilon}\Big(\beta_0(I^h\phi), u_\epsilon^h - I^h(\phi)\Big)\\ &\geq -\frac{1}{\epsilon}\|(\mid \nabla I^h\phi \mid^2 -1)^+\|_{L^\infty(\Omega)}\|\nabla I^h\phi\|_{L^2(\Omega)}\|\nabla u_\epsilon^h\\ &\quad -\nabla I^h(\phi)\|_{L^2(\Omega)}\\ &\geq -C\frac{h}{\epsilon}\end{aligned}$$

so that

$$\lim_{h\to 0}\{\mathbf{I}_2-\mathbf{I}_3\}\geq 0$$

if $h/\epsilon \to 0$. Finally, the density argument and the uniform boundness again yield

$$\begin{aligned}\mathbf{I}_4 = \int_0^s \Big[&(g,\phi - I^h\phi) + (\frac{\partial(\phi - I^h\phi)}{\partial t}, u_\epsilon^h)\\ &+(\lambda^2\nabla\frac{\partial(I^h\phi-\phi)}{\partial t},\nabla u_\epsilon^h)\Big]\,dt \to 0\,.\end{aligned}$$

Therefore, u_ϵ^h converges weakly to the solution of (4.3) as $h\to 0$ and $\epsilon\to 0$ with $h^{2-4/p}/\epsilon$ being uniformly bounded and $h/\epsilon \to 0$. □

6. Fully discrete approximation

We again consider a finite element discretization in the spatial variables. There are various ways to implement the time discretization. For simplicity, let us consider the backward Euler scheme.

Given the time step size δt, let $g^n = g(t_n)$ for $t_n = n\delta t$, $n \geq 1$. The initial approximation u_h^0 may be obtained by

$$\int_\Omega (u_h^0 v_h + \lambda^2\nabla u_h^0\cdot\nabla v_h)\,d\Omega = \int_\Omega (\omega_0 - H_1)v_h\,d\Omega \quad \forall\, v_h\in S^h\,.$$

For $n \geq 1$, we have the discrete variational inequality:

find $u_h^n \in \mathcal{K}\cap S^h$ such that for any $\phi_h\in\mathcal{K}\cap S^h$

$$(u_h^n - u_h^{n-1} - \delta t g^n\,,\ \phi_h - u_h^n) + \lambda^2(\nabla(u_h^n - u_h^{n-1}), \nabla\phi_h - \nabla u_h^n) \geq 0\,.$$

Rather than working with the regularized problem as in the previous section, we consider a saddle point formulation for the variational inequalities using Lagrange multipliers.

Recall that our problem may be rewritten as:

$$u - \lambda^2 \Delta u = \omega \quad \text{in } \Omega \times (0,T)$$
$$\omega_t - \nabla \cdot (m \nabla u) = g \quad \text{in } \Omega \times (0,T)$$
$$\omega = \omega_0 - H_1 \quad \text{in } \Omega \quad \text{at } t = 0$$
$$u = 0 \quad \text{on } \Gamma \times (0,T)$$

where $g = \partial H_1 / \partial t$, along with the compatibility conditions

$$m \geq 0 \quad \text{and} \quad |\nabla u| - 1 \leq 0$$
$$m = 0 \quad \text{if} \quad |\nabla u| - 1 < 0 \,.$$

This may be viewed as a saddle point formulation of the variational inequality where the function m is the Lagrange multiplier.

In the discrete case, let Z^h be the space of piecewise nonnegative constants and let

$$J_n(v) = \frac{1}{2} \int_\Omega \left(|v|^2 + \lambda^2 |\nabla v|^2 - 2v(u_h^{n-1} + \delta t g^n) \right) \, d\Omega$$

and

$$\mathcal{L}(v,q) = J_n(v) + (q, |\nabla v|^2 - 1) \,.$$

Using the Lagrange multiplier theory, an equivalent saddle point formulation is given by:

find $(u_h^n, \mu_h^n) \in S^h \times Z^h$, such that

$$\mathcal{L}(u_n^h, q_h) \leq \mathcal{L}(u_h^n, \mu_h^n) \leq \mathcal{L}(v_h, \mu_h^n)$$

for any $(v_h, q_h) \in S^h \times Z^h$.

To find the solution, we employ the following Uzawa-type iteration:

1. Set $\nu^{(0)} = \mu_h^{n-1}$. For $k = 1, 2, 3, \ldots, K$,
   a. determine $Q^{(k)} \in S^h$ such that

$$\mathcal{L}(Q^{(k)}, \nu^{(k-1)}) = \min_{Q_h \in S^h} \mathcal{L}(Q_h, \nu^{(k-1)}) \,,$$

   i.e.,

$$\int_\Omega \left(Q^{(k)} v + (\lambda^2 + \delta t \, \nu^{(k-1)}) \nabla Q^{(k)} \cdot \nabla v \right) \, d\Omega$$
$$= \int_\Omega \left((u_h^{n-1} + \delta t g^n) v + \lambda^2 \nabla u_h^{n-1} \cdot \nabla v \right) \, d\Omega \qquad \forall \, v \in S^h \,;$$

b. set

$$\nu^{(k)} = \left[\nu^{(k-1)} + \rho(|\nabla Q^{(k)}|^2 - 1)\right]^+ \qquad \text{in } \Omega\,,$$

where $[f]^+ = \max(0, f)$ and ρ is a properly chosen relaxation parameter;

2. Set

$$u_h^n = Q^{(K)} \qquad \text{and} \qquad \mu_h^n = \nu^{(K)} \qquad \text{in } \Omega\,.$$

We now investigate the convergence of the iteration as $k \to \infty$ by using the idea presented in [8]. First, from the weak forms for u_h^n and $Q^{(k)}$, we can easily obtain

Lemma 6.1 *There exists a constant $C > 0$, independent of k, such that:*

$$\|Q^{(k)}\|_1 \leq C \qquad \textit{and} \qquad \|u_h^n\|_1 \leq C\,. \quad \square$$

The inverse inequality would then imply that:

Corollary 6.1 *There exists a constant $C > 0$, independent of k, such that*

$$\|Q^{(k)}\|_{1,\infty} \leq C \qquad \textit{and} \qquad \|u_h^n\|_{1,\infty} \leq C\,. \quad \square$$

By properties of the saddle point, we have that

$$\begin{aligned}
&(u_h^n - u_h^{n-1} - \delta t g^n\ ,\ \phi_h - u_h^n) \\
&\quad + \lambda^2(\nabla(u_h^n - u_h^{n-1}), \nabla\phi_h - \nabla u_h^n) \\
&\quad + (\mu_h^n, |\nabla\phi_h|^2 - |\nabla u_h^n|^2) \\
&\geq 0\,.
\end{aligned}$$

Meanwhile, by the definition of $Q^{(k)}$, we have

$$\begin{aligned}
(Q^{(k)} - u_h^{n-1} - \delta t g^n,\ \psi_h - Q^{(k)}) + \lambda^2(\nabla(Q^{(k)} - u_h^{n-1}), \nabla\psi_h - \nabla Q^{(k)}) \\
+ (\nu^{(k-1)}, |\nabla\psi_h|^2 - |\nabla Q^{(k)}|^2) \geq 0\,.
\end{aligned}$$

Letting $\phi_h = Q^{(k)}$ and $\psi_h = u_h^n$ in the above, and adding the resulting inequalities, we obtain

$$\begin{aligned}
(u_h^n - Q^{(k)}, u_h^n - Q^{(k)}) + \lambda^2(\nabla(u_h^n - Q^{(k)}), \nabla(u_h^n - Q^{(k)})) \\
+ (\nu^{(k-1)} - \mu_h^n, |\nabla u_h^n|^2 - |\nabla Q^{(k)}|^2) \leq 0\,.
\end{aligned}$$

Obviously, $|a^+ - b^+| \leq |a - b|$. Let $a = \nu^{(k-1)} + \rho(|\nabla Q^{(k)}|^2 - 1)$ and $b = \mu_h^n + \rho(|\nabla \mu_h^n|^2 - 1)$. Then, $a^+ = \nu^{(k)}$ and $b^+ = \mu_h^n$. Thus, we obtain

$$\begin{aligned}
\|\nu^{(k)} - \mu_h^n\|_0^2 &\leq \|\nu^{(k-1)} - \mu_h^n + \rho(|\nabla Q^{(k)}|^2 - |\nabla u_h^n|^2)\|_0^2 \\
&= \|\nu^{(k-1)} - \mu_h^n\|_0^2 + 2(\nu^{(k-1)} - \mu_h^n, \rho(|\nabla Q^{(k)}|^2 - |\nabla u_h^n|^2)) \\
&\quad + \rho^2\||\nabla Q^{(k)}|^2 - |\nabla u_h^n|^2\|_0^2\,.
\end{aligned}$$

Using the corollary, we get

$$\|\nu^{(k)} - \mu_h^n\|_0^2 \leq \|\nu^{(k-1)} - \mu_h^n\|_0^2 - 2\rho(u_h^n - Q^{(k)}, u_h^n - Q^{(k)}) \\ -2\rho\lambda^2(\nabla(u_h^n - Q^{(k)}), \nabla(u_h^n - Q^{(k)})) \\ +C_1\rho^2\|\nabla Q^{(k)} - \nabla u_h^n\|_0^2$$

for some contant C_1 which is dependent on C, but independent of k. Therefore, if we choose ρ suitably small, we can find a positive constant γ (independent of k), such that

$$\|\nu^{(k)} - \mu_h^n\|_0^2 + \gamma\|Q^{(k)} - u_h^n\|_1^2 \leq \|\nu^{(k-1)} - \mu_h^n\|_0^2\,.$$

Thus, we have the following convergence result.

Theorem 6.1 *There exists a properly chosen ρ such that*

$$\lim_{k\to\infty} \|Q^{(k)} - u_h^n\|_1 = 0\,. \qquad \square$$

Note that in the proof, we allow the constant C to depend on the discretization parameters such as δt and h. How to avoid such dependence will be studied in future work.

7. Numerical results

The numerical computations were performed using a finite element method based on a triangular mesh with continuous piecewise quadratic polynomials for the magnetic field and continuous piecewise linear polynomials for the Lagrange multiplier. Although the theorem in the previous section was proved under the assumption that the finite element space for the magnetic field is piecewise linear, it was necessary to use piecewise quadratic elements in order to compute the vorticity which involves the term ΔH. Obviously, the theorem is still true with the choice of piecewise quadratic polynomials.

In the first example, we took Ω to be the unit square. We chose a uniform mesh of size $1/14$ and $\delta t = 0.05$ for the time step size. For the boundary condition, we used, for given $T > 0$, $H_1(t) = t$ for $t \leq T$ and $H(t) = T$ for $t > T$, i.e. the external field is linearly ramped up until some time T and held constant after that. For the initial vorticity we chose $\omega_0 = 0$. We computed approximate solutions for various penetration depths λ. For $\lambda \leq 0.5$, the iterations converged successfully with an appropriately chosen parameter ρ. Here, we report results for $\lambda = 0.1$, 0.3, and 0.5 for which we used $\rho = 0.1$, 0.8, and 1.0, respectively. We noticed that after the external field H_1 became constant, i.e., for $t > T$, the magnetic field remained unchanged as well;

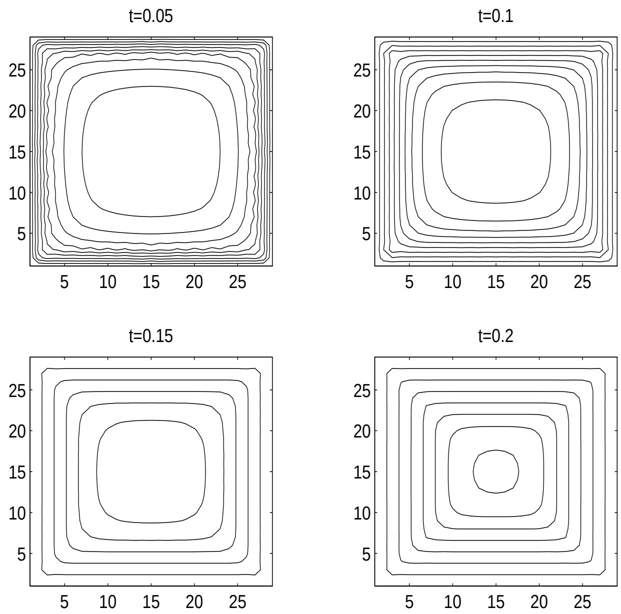

Fig. 1. Contour plots of the magnetic field for $\lambda = 0.1$

thus we only report results for $t \leq T$. For the three values of λ we chose $T = 0.2$, 0.4, and 0.7, respectively. Figures 1, 3, and 5 show contour plots of magnetic field with increasing time up to $t = T$ for $\lambda = 0.1,$ 0.3, and 0.5, repectively. Figures 2, 4, and 6 give the corresponding surface plots of the magnetic field. The surface plots show that the magnetic field forms pyramid as time increases. But as λ becomes larger, the formation rate gets slow. For example, the contour plot for $\lambda = 0.1$ at $t = 0.2$ can be compared with the ones for $\lambda = 0.3$ at $t = 0.4$ or for $\lambda = 0.5$ at $t = 0.7$. Figures 7, 8, and 9 show the corresponding contour plots of vorticity for each of the three values of λ.

Flux penetration into non-rectangular shapes is often more difficult to study by analytic methods. As the second example, we study the patterns of the magnetic field and vortex densities for a cross-shaped sample. In our calculation, we set the length of the arm to be half of its width, $\lambda = 0.1$ and $\rho = 0.3$. The set-up for $H_1(t)$ is similar to the that for the fist example. Specifically, we chose $H_1(t) = t$ for $t \leq 0.3$ and $H_1(t) = 0.3$ for $t > 0.3$. The time step size is taken to be 0.05. Figure 7 shows contour plots of magnetic field as time increases while Fig. 8 gives the corresponding contour plots of the vorticity field. The creation of vorticity occurs first near the re-entrant corners of the cross. For later times, in each arm, the magnetic field again forms half of an up-side-down pyramid and a full pyramid in the center of cross. Our two-dimensional model describes the behavior of the so-

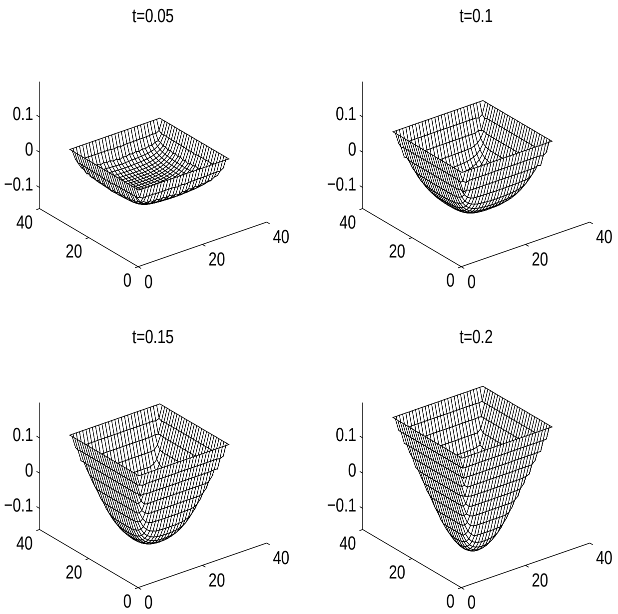

Fig. 2. Surface plots of the magnetic field for $\lambda = 0.1$

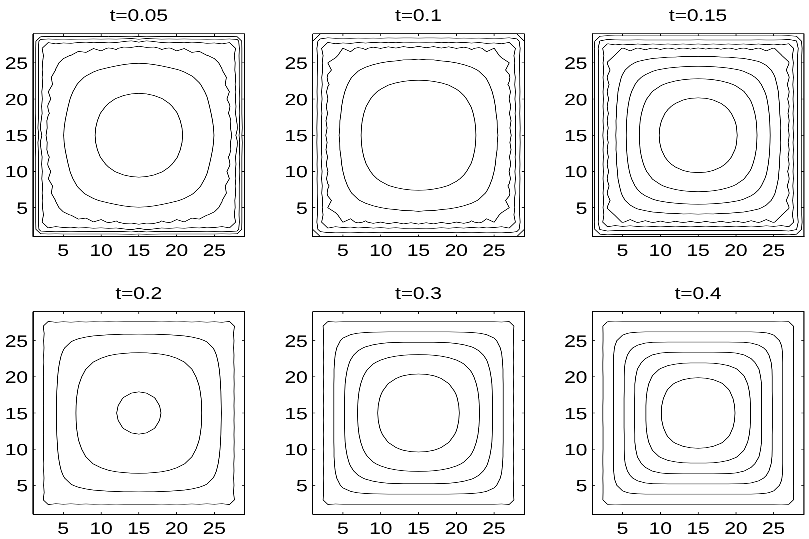

Fig. 3. Contour plots of the magnetic field for $\lambda = 0.3$

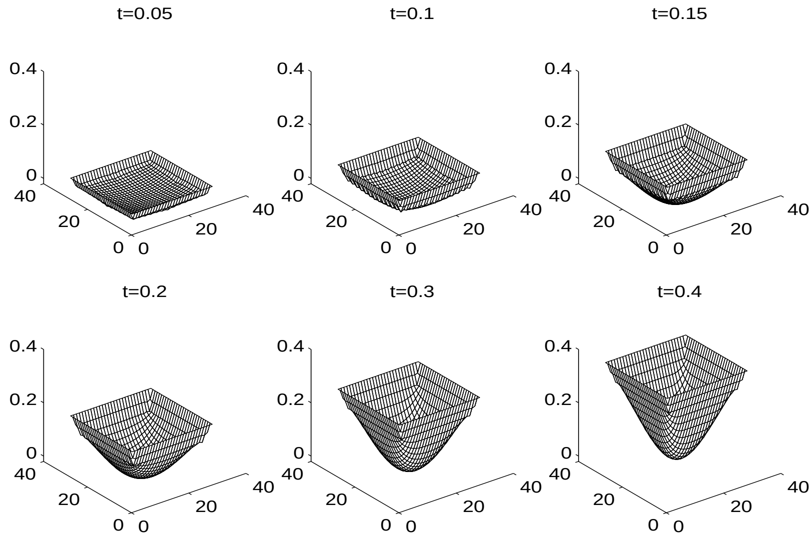

Fig. 4. Surface plots of the magnetic field for $\lambda = 0.3$

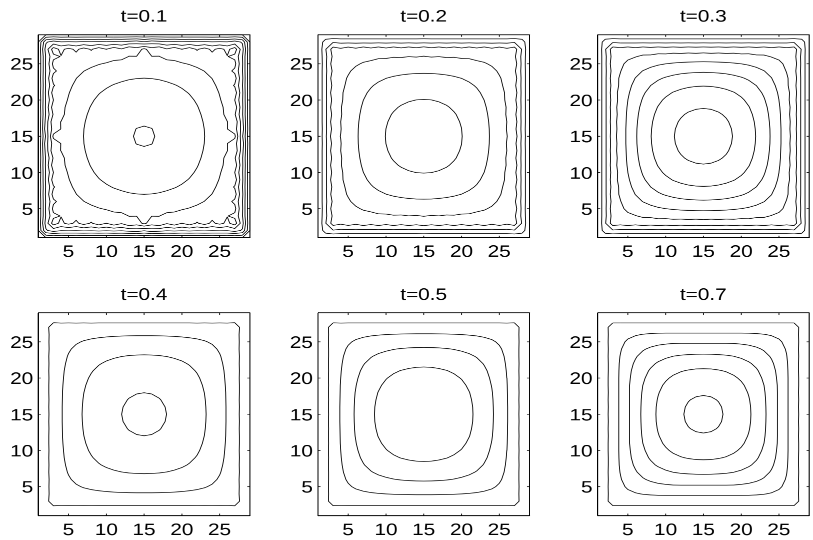

Fig. 5. Contour plots of the magnetic field for $\lambda = 0.5$

lution for a two-dimensional cross-section of an infinite, three-dimensional cylindrical domain. It is interesting to observe the differences between our calculation and the theoretical as well as experimental studies made in [11] for crossshaped thin films.

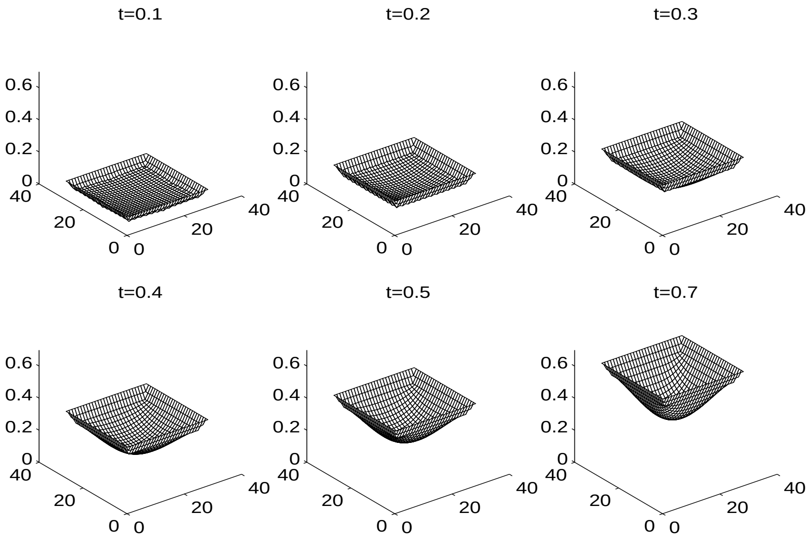

Fig. 6. Surface plots of the magnetic field for $\lambda = 0.5$

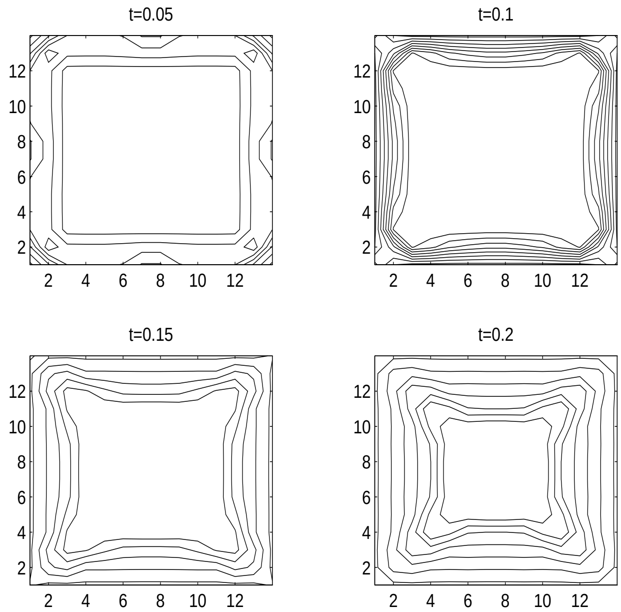

Fig. 7. Contour plots of the vorticity for $\lambda = 0.1$

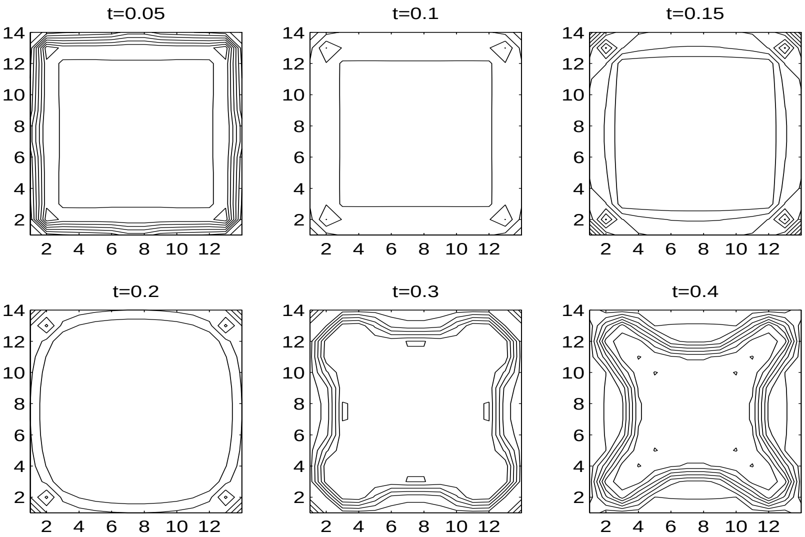

Fig. 8. Contour plots of the vorticity for $\lambda = 0.3$

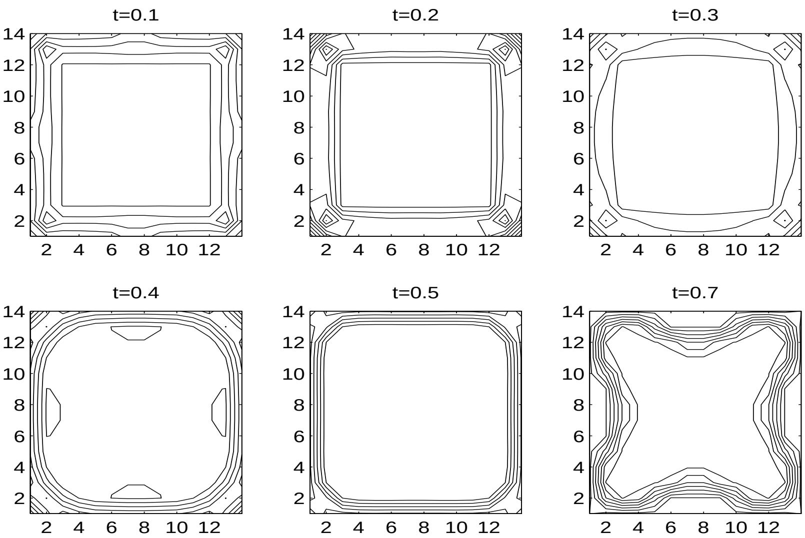

Fig. 9. Contour plots of the vorticity for $\lambda = 0.5$

8. Conclusion

In this paper, we analyzed a mean field model for superconductivity based on a variational inequality formulation and its discrete approximations. Preliminary numerical experiments on the flux penetration into square and cross shaped domains are performed. Various generalizations are to be studied in the future. On the modeling side, one may address issues such as anisotropy,

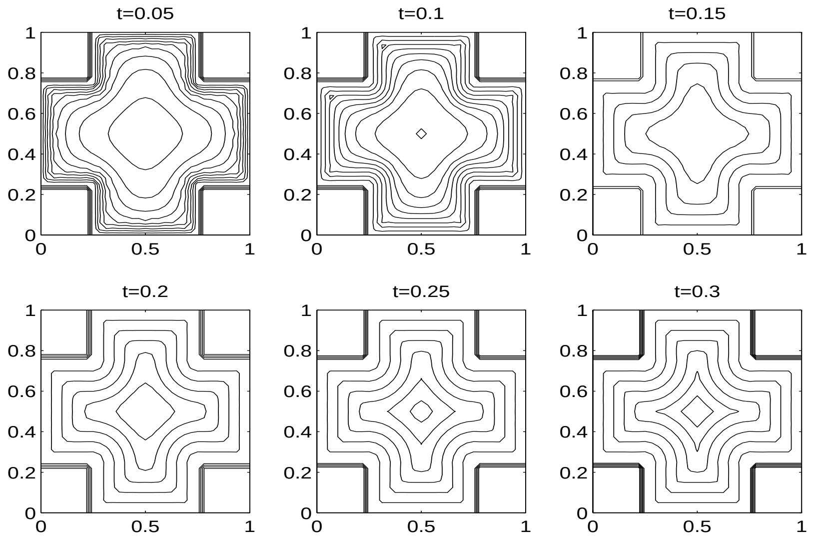

Fig. 10. Contour plots of the magnetic field for $\lambda = 0.1$

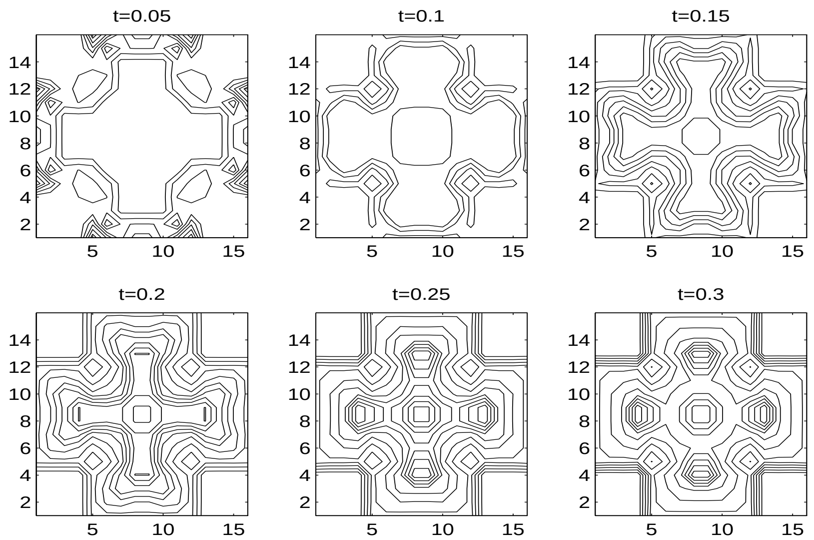

Fig. 11. Contour plots of the vorticity for $\lambda = 0.1$

the inclusion of normal materials, and applied currents. On the numerical analysis side, a more general convergence theory and error estimates are to be established. We also intend to carry out more extensive commputational studies.

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