JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **155**, 21–45 (1991)

Analysis of a Ladyzhenskaya Model for Incompressible Viscous Flow*

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Submitted by Ivar Stakgold

Received March 1, 1989

In this paper we analyze a model for the motion of incompressible viscous flows proposed by Ladyzhenskaya. We establish a new *a priori* estimate for the nonstationary model and then obtain more general existence and uniqueness results. We also present some uniqueness results for the stationary model. The relationship between fluid motion governed by the Ladyzhenskaya model and that governed by the Navier–Stokes equations is addressed. © 1991 Academic Press, Inc.

0. INTRODUCTION

Mathematical models for fluid motion play an important role in theoretical and computational studies in the aeronautical, meteorological, and oceanographic sciences, in plasma physics, in the chemical and petroleum industries, etc. However, in many situations it is still not clear which models are most appropriate, especially in the case of turbulent flows. We refer to, e.g., [5] for studies of the physical background.

The Navier–Stokes equations are generally accepted as providing an accurate model for the incompressible motion of viscous fluids in many

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* This work was supported by the Air Force Office of Scientific Research under Grants AFOSR-90-0179 and AFOSR-88-0197.

0022-247X/91 $3.00

practical situations. There are many who believe this to be true even for turbulent flows. On the other hand, there are various reasons why scientists abandon the Navier–Stokes model in favor of models employing nonlinear constitutive laws. For example, for flows of polymers or of visco-elastic or visco-plastic fluids, one generally has to use nonlinear stress-rate of strain relations. However, the models introduced by Ladyzhenskaya in [11–13] address a different issue, namely that the *linear* constitutive law used in the derivation of the Navier–Stokes equations presumes that derivatives of the components of the velocity are small.

Here we consider one particular model introduced by Ladyzhenskaya. The study of this model may be justified through a variety of physical and mathematical arguments. In the first place, for certain values of a parameter appearing in the model, e.g., for $r = 2$ in (0.12) below, the model still conforms with the definition of a fluid as given by Stokes; see [16]. For the incompressible flow of a viscous fluid, the laws of conservation of mass and momentum, which no one questions, provide an underdetermined system of partial differential equations for the velocity, pressure, and stress fields; in order to close the system a constitutive law relating the stress to the velocity must be provided. The particular form this constitutive relation takes depends on what kind of fluid one is dealing with. Stokes introduced a series of requirements which together serve to define an "ordinary" fluid, e.g., water or air; see [16] for a lucid account. The Stokes hypotheses which define our fluid lead to a specific mathematical form for the nonlinear relation between the stress and the velocity fields; again see [16] for details. If, *in addition*, one requires that the relation between the stress and velocity fields be *linear*, then one arrives at the Navier–Stokes equations. However, if one retains the Stokes hypotheses defining a fluid and then retains some of the nonlinear terms in the general constitutive relation which a Stokesian fluid must satisfy, then one arrives at the Ladyzhenskaya model considered here with, e.g., $r = 2$; see [12, 13]. In other words, the Ladyzhenskaya model is derived by combining the principles of conservation of mass and momentum with the rules which define a Stokesian fluid and then retaining some of the nonlinear terms in the resulting constitutive law. The Navier–Stokes model is derived by first invoking exactly the same assumptions *plus* the assumption that the constitutive relation is a linear one. Thus, from a *modeling* standpoint, the Navier–Stokes equations are a special case of the Ladyzhenskaya equations considered here. This leads to the obvious conclusion that any flow which can be accurately described by solutions of the Navier–Stokes equations can be at least as accurately described by solutions of the Ladyzhenskaya equations. Incidentally, Ladyzhenskaya also gives a partial justification, based on kinetic theory arguments, for why one should retain the nonlinear terms she chooses to include in the constitutive relation.

A second justification for the study of the Ladyzhenskaya model considered here comes from the field of turbulence modeling. A scenario played in that field is to first assume that the variables which describe the flow, e.g., the velocity and pressure, may be decomposed into the sum of mean, or averaged, and fluctuating quantities. The averaged form of the Navier–Stokes equations cannot by themselves determine the averaged quantities; one must also provide a relation between the fluctuating and averaged quantities which determine how energy is transferred from the small scales to the larger scales in the flow. The various methods for relating the mean velocity field and the "Reynolds stresses" arising from the fluctuating velocity field are known collectively as *turbulence closure models*. See, e.g., [3] or [9] for details. One class of such models is known as *algebraic* or *zero-equation models*. The main feature of these models is that the Reynolds stresses are related to the derivatives of the components of the velocity through algebraic relations, as opposed to more complex models for which this relation involves one or more partial differential equations. Of course, no turbulence closure model has gained anything close to universal acceptance; however, due to their simplicity and the fact that more complicated models do not always yield better results, algebraic turbulence models are very popular in everyday engineering calculations. Again, see [3] for a detailed discussion. It turns out, again for certain values of the parameter r such as $r = 2$, that the Ladyzhenskaya equations considered here are identical to those of a popular algebraic turbulence model. Thus, from a *practical engineering* point of view, the study of the Ladyzhenskaya equations and of properties of their solutions is of substantial interest.

There are also some *a posteriori* reasons why the Ladyzhenskaya equations are of interest. First, Ladyzhenskaya herself has shown, in the above cited references, that solutions of the equations considered here, in the non-stationary case and in three space dimensions, are globally unique in time. The analogous result for the Navier–Stokes equations has not been proved and is not believed to be true. Indeed, from a mathematical point of view, this was a motivation for Ladyzhenskaya's exploration of alternate models.

Another point of interest is that the condition derived in Section 7 below which guarantees the uniqueness of solutions of the stationary Ladyzhenskaya model is, in some sense, less pessimistic than the analogous condition for the Navier–Stokes model. Basically, these conditions imply that the stationary problem has a unique solution whenever the Reynolds number for the flow is "sufficiently small." However, when these conditions are not met, mathematically nothing can be implied about the uniqueness, or lack thereof, of the flow. However, it is known from experiments that in many physical situations stationary flows are unique for much higher values of the Reynolds number than that required for mathematically guaranteeing the uniqueness of solutions of the stationary Navier–Stokes. On the other hand, the analogous condition for the stationary Ladyzhenskaya model generally guarantees uniqueness for higher values of the Reynolds number than that predicted for the Navier–Stokes model and thus is not so pessimistic compared to actual physical situations.

From a practical point of view, and in view of the whole of the above discussion, perhaps the most important *a posteriori* reason for studying the Ladyzhenskaya model is that an existing code for computing approximate solutions of the Navier–Stokes equations may be amended in a *trivial* manner in order to handle the Ladyzhenskaya equations as well. This is shown, in the case of finite element discretizations, in [8]. Thus for the same physical situation, i.e., for the same data, if one owns a working Navier–Stokes code, numerical solutions for both models may be easily computed and compared.

The physical phenomenon that we consider here is the incompressible motion of a viscous fluid in a bounded domain Ω in $\mathbb{R}^n$ with boundary Γ, where $n = 2$ or 3 in most of our discussions. Let $\mathbf{u}$ denote the velocity field, p the pressure, and $\mathbf{f}$ the body force per unit mass. The Navier–Stokes equations with homogeneous boundary conditions are given as

$$[\mathrm{NS1}] \qquad \begin{array}{lll} \mathbf{u}_t - \nu_0 \Delta \mathbf{u} + \Sigma u_i \partial_i \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega & (0.1) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega & (0.2) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma & (0.3) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, & (0.4) \end{array}$$

where the constant density has been absorbed into the pressure and ν_0 is the kinematic viscosity. For the mathematical treatment of Navier–Stokes equations, one may see, e.g., [10, 11, 14, 18, 19].

For later consideration we also state stationary Navier–Stokes equations:

$$[\mathrm{NS2}] \qquad \begin{array}{lll} -\nu_0 \Delta \mathbf{u} + \Sigma u_i \partial_i \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega & (0.5) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega & (0.6) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma. & (0.7) \end{array}$$

Ladyzhenskaya has proposed some alternate mathematical models for viscous incompressible flows which feature nonlinear constitutive equations. For all of her new models, she was able to show that the solutions are globally unique in time for any value of the Reynolds number. The particular model we consider here is given by

$$[\mathrm{L1}]\quad \begin{array}{lll} u_{jt} - \Sigma \partial_k(\mathscr{A}(\mathbf{u})\, \partial_k u_j) + \Sigma u_k \partial_k u_j + \partial_j p = f_j & \text{in } \Omega & (0.8) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega & (0.9) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma & (0.10) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, & (0.11) \end{array}$$

where in (0.8), $j = 1, 2$, or $j = 1, 2, 3$, and $\mathscr{A}(\mathbf{u})$ is defined by

$$\mathscr{A}(\mathbf{u}) = \nu_0 + \nu_1 \, |\nabla \mathbf{u}|^r \qquad \text{with } r > 0 \tag{0.12}$$

and

$$|\nabla \mathbf{u}| = \left[\sum_{i,j=1}^{n} (\partial_i u_j)^2 \right]^{1/2}. \tag{0.13}$$

The boundary condition (0.10) has been chosen only for the sake of simplifying the exposition. For the same reasons we also assume that $\nu_0 > 0$ and $\nu_1 > 0$ are constants. We also have the steady state version of the Ladyzhenskaya equations:

$$[\mathrm{L2}]\quad \begin{array}{lll} -\Sigma \partial_k(\mathscr{A}(\mathbf{u})\, \partial_k u_j) + \Sigma u_k \partial_k u_j + \partial_j p = f_j & \text{in } \Omega & (0.14) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega & (0.15) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma & (0.16) \end{array}$$

In [12, 13], Ladyzhenskaya showed that weak solutions for [L1] are globally unique in time for any Reynolds number and any exponent $r \geqslant 1/2$. She also mentioned that [L2] has at least one solution for any $\nu_0 > 0$, $\nu_1 \geqslant 0$, and $r > 0$, regardless of the size of $\mathbf{f}$. Here, we will establish some new *a priori* estimates for weak solutions of [L1], then show that weak solutions for [L1] are globally unique in time for any Reynolds number and any exponent $r \geqslant 1/5$, thus generalizing her original result. We will also show that [L2] has a unique weak solution under certain conditions on the Reynolds number. Furthermore, it is of importance to notice that the conditions we derive are less restrictive when compared to the standard ones for the stationary Navier–Stokes equations.

Formally, if we set $\nu_1 = 0$, equations [L1] and [L2] reduce to the equations [NS1] and [NS2], respectively. We will make rigorous this formal reduction. Therefore, even when $\nu_1 \neq 0$, we retain the usual meaning for ν_0 and also as usual, the Reynolds number is still propotional to $(\nu_0)^{-1}$. Indeed, if the physical variables have been nondimensionalized, $(\nu_0)^{-1}$ is the Reynolds number. Information concerning the constant ν_1 and the exponent r may be gleaned from the kinetic theory of gases and the Stokes hypotheses defining a fluid; see [12, 13]. Generally, $\nu_1 \ll \nu_0$ and two natural choices for the exponent r are $r=1$ or 2. Comparing with equations [NS1] or [NS2], the newly added term $\nu_1 |\nabla \mathbf{u}|^r$ clearly has a "stabilizing" effect by increasing the "viscosity." In situations where a loss of stability is indicated, the solutions of [L1] or [L2] are more likely to be damped than are solutions of the Navier–Stokes equations.

1. NOTATION AND FUNCTION SPACES

We start our discussion by introducing some familiar function spaces and some notation which will be used in the sequel. First, define

$$\mathscr{D}(\Omega) := C_0^\infty(\Omega) \tag{1.1}$$

to be the space of (real-valued) smooth functions with compact support in the domain Ω, and let us use the standard notation (see [1]) for the usual Sobolev spaces, e.g., $L^2(\Omega)$ denotes the space of (real-valued) functions which are square integrable over Ω with respect to the Lebegue measure. We also define the following:

$$\mathfrak{S} := \{\boldsymbol{\varphi} \in [\mathscr{D}(\Omega)]^n \colon \operatorname{div} \boldsymbol{\varphi} = 0\} \qquad (n = 2 \text{ or } 3). \tag{1.2}$$

$$\mathbf{H} := \text{Completion of } \mathfrak{S} \text{ in the } \mathbf{L}^2\text{-norm}. \tag{1.3}$$

$$\mathbf{H}_q := \text{Completion of } \mathfrak{S} \text{ in the } \mathbf{L}^q\text{-norm} \qquad (q > 2). \tag{1.4}$$

$$\mathbf{V} := \text{Completion of } \mathfrak{S} \text{ in the } \mathbf{H}^1\text{-norm}. \tag{1.5}$$

$$\mathbf{V}_q := \text{Completion of } \mathfrak{S} \text{ in the} \mathbf{W}^{1,q}\text{-norm} \qquad (q > 2). \tag{1.6}$$

The spaces $\mathbf{H}$ and $\mathbf{V}$ are Hilbert spaces with corresponding inner products and norms

$$(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega \qquad \text{for} \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}, \tag{1.7}$$

and

$$\|\mathbf{u}\|_{0,2} := (\mathbf{u}, \mathbf{u})^{1/2}. \tag{1.8}$$

Similarly, for $\mathbf{u}$, $\mathbf{v} \in \mathbf{V}$,

$$\langle \mathbf{u}, \mathbf{v} \rangle := \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega, \tag{1.9}$$

$$\|\mathbf{u}\|_{1,2} := \left[\int_\Omega |\nabla \mathbf{u}|^2 \, d\Omega \right]^{1/2}. \tag{1.10}$$

The spaces $\mathbf{H}_q$ and $\mathbf{V}_q$ are reflexive Banach spaces, endowed with the following norms, for $\mathbf{u} \in \mathbf{H}_q$,

$$\|\mathbf{u}\|_{0,q} := \left[\int_\Omega |\mathbf{u}|^q \, d\Omega\right]^{1/q} \tag{1.11}$$

and for $\mathbf{u} \in \mathbf{V}_q$,

$$\|\mathbf{u}\|_{1,q} := \left[\int_\Omega |\nabla\mathbf{u}|^q \, d\Omega\right]^{1/q}. \tag{1.12}$$

In addition, $\mathbf{V}'$ and $\mathbf{W}'$ are used to denote the dual spaces of $\mathbf{V}$ and $\mathbf{V}_q$, respectively, with the duality pair being equivalent to the $\mathbf{L}^2$ inner product. The induced norms are denoted by $\|\cdot\|_{V'}$ and $\|\cdot\|_{W'}$.

Remark 1.1. Primarily, we focus our discussion on real-valued functions. However, if complex-valued functions are considered, the above definitions should all take their proper complex extensions. For simplicity, in such circumstances, we use the same notation without any further distinction.

Spaces of functions depending on both time and space variables are defined as usual; e.g.,

$$L^1[\mathbb{R}; \mathbf{H}] := \left\{\mathbf{f}(\cdot, t) \in \mathbf{H}, \mathbf{f} \text{ is Bochner integrable, } \int_{\mathbf{R}} \|\mathbf{f}(\cdot, \tau)\|_{0,2} \, d\tau < \infty\right\} \tag{1.13}$$

and is equipped with the norm

$$|||\mathbf{f}|||_{1;0,2} := \int_{\mathbf{R}} \|\mathbf{f}(\cdot, \tau)\|_{0,2} \, d\tau. \tag{1.14}$$

Let $\mathscr{X}$ be a Banach space; for a σth Bochner-integrable $\mathscr{X}$-valued function $\mathbf{u} \in L^\sigma[\mathbb{R}; \mathscr{X}]$, we denote its Fourier transform by

$$\mathbf{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}} \exp(-i\mathbf{x}\xi) \, \mathbf{u}(\mathbf{x}) \, d\mathbf{x}. \tag{1.15}$$

Finally, we define the following spaces: given $T > 0$, $\alpha > 0$, and Hilbert spaces $\mathbf{S}_0$ and $\mathbf{S}_1$, then

$$\mathbf{H}^\alpha(\mathbb{R}; \mathbf{S}_1) := \left\{\mathbf{u} \in L^2[\mathbb{R}; \mathbf{S}_1]; \int_{\mathbf{R}} |\tau|^{2\alpha} \|\hat{\mathbf{u}}(\tau)\|^2_{\mathbf{S}_1} \, d\tau < \infty\right\}; \tag{1.16}$$

$$\mathbf{H}^\alpha(\mathbb{R}; \mathbf{S}_0, \mathbf{S}_1) := L^2[\mathbb{R}; \mathbf{S}_0] \cap \mathbf{H}^\alpha(\mathbb{R}; \mathbf{S}_1); \tag{1.17}$$

and

$$\mathbf{H}^{\alpha}((0,T);\mathbf{S}_0,\mathbf{S}_1)$$
$$:=\{\mathbf{u};\mathbf{u}=\mathbf{w}|_{(0,T)}\text{ for some }\mathbf{w}\in\mathbf{H}^{\alpha}(\mathbb{R};\mathbf{S}_0,\mathbf{S}_1)\}. \tag{1.18}$$

We use $|||\mathbf{f}|||_{\alpha;\mathbf{S}_0,\mathbf{S}_1}$ to denote the norm of $\mathbf{f}$ in the space $\mathbf{H}^{\alpha}((0,T);\mathbf{S}_0,\mathbf{S}_1)$. Spaces like those above have been studied in [15]. In fact, similar spaces can be defined even in the general Banach space setting by interpolation theory.

2. VARIATIONAL FORMULATION OF THE NONSTATIONARY MODEL

We define a trilinear form as follows: for $\mathbf{u}$, $\mathbf{w}\in\mathbf{H}$ and $\mathbf{v}\in\mathbf{V}$,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}):=\int_{\Omega}(\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w})\,d\Omega. \tag{2.1}$$

LEMMA 2.1. *The above trilinear form has the following properties*:

(i) $b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v})$ *for* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}\in\mathbf{V}$. (2.2)

(ii) *For* $n\leqslant 3$, *there is a constant* $c>0$, *such that for any* $\mathbf{u}$, $\mathbf{v}\in\mathbf{V}$,

$$|b(\mathbf{u},\mathbf{u},\mathbf{v})|\leqslant c(\|\mathbf{u}\|_{0,4})^2\,\|\mathbf{v}\|_{1,2} \tag{2.3}$$

and for any $\mathbf{u}$, $\mathbf{v}\in\mathbf{V}_q$, $q>2$,

$$|b(\mathbf{u},\mathbf{u},\mathbf{v})|\leqslant c(\|\mathbf{u}\|_{0,2q'})^2\,\|\mathbf{v}\|_{1,q}. \tag{2.4}$$

Proof. Part (i) can be easily established for the elements in $\mathfrak{S}$ and follows by continuity for elements in $\mathbf{V}$ (see [19]). In [18], one can also find the proof for (2.3). Both (2.3) and (2.4) are consequences of the Cauchy–Schwartz inequality. ▌

Associated with the trilinear form b, a bounded operator $B\colon\mathbf{V}\to\mathbf{V}'$ is defined in the following lemma by using the Riesz representation theorem and Lemma 2.1 (see [18, 19]).

LEMMA 2.2. *For* $\mathbf{u}\in\mathbf{V}$, *there exists a* $B\mathbf{u}\in\mathbf{V}'$ *such that*

$$(B\mathbf{u},\mathbf{v}):=b(\mathbf{u},\mathbf{u},\mathbf{v})\qquad\text{for all}\quad\mathbf{v}\in\mathbf{V}.$$

Moreover, there is a constant $c>0$ *such that for any* $\mathbf{u}\in\mathbf{V}$,

$$\|B\mathbf{u}\|_{\mathbf{V}'}\leqslant c(\|\mathbf{u}\|_{0,4})^2, \tag{2.5}$$

and for any $\mathbf{u} \in \mathbf{V}_q$,

$$\|B\mathbf{u}\|_{\mathbf{W}'} \leqslant c(\|\mathbf{u}\|_{0,2q'})^2. \tag{2.6}$$

We now present the weak formulation for problem [L1] which can be obtained through the standard procedure, e.g., multiplying the original equations by test functions and integrating by parts:

[L3]

for $T > 0$, $\mathbf{f}$ and $\mathbf{u}_0$ given,

$$\mathbf{f} \in L^2[(0, T); \mathbf{V}'], \tag{2.7}$$

$$\mathbf{u}_0 \in \mathbf{H}, \tag{2.8}$$

find $\mathbf{u} \in C[(0, T); \mathbf{H}] \cap L^q[(0, T); \mathbf{V}_q]$ satisfying

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + (\mathscr{A}(\mathbf{u}) \nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \tag{2.9}$$

for all $\mathbf{v} \in \mathbf{V}_q$, and

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}). \tag{2.10}$$

The above weak formulation is analogous to the standard weak form of Navier–Stokes equation [NS3]:

[NS3]

For $T > 0$, $\mathbf{f}$ and $\mathbf{u}_0$ given,

$$\mathbf{f} \in L^2[(0, T); \mathbf{V}'], \tag{2.11}$$

$$\mathbf{u}_0 \in \mathbf{H}, \tag{2.12}$$

find $\mathbf{u} \in C[(0, T); \mathbf{H}] \cap L^2[(0, T); \mathbf{V}]$ satisfying

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + \nu_0\langle \mathbf{u}, \mathbf{v} \rangle + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \tag{2.13}$$

for all $\mathbf{v} \in \mathbf{V}$, and

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}). \tag{2.14}$$

3. New A Priori Estimates for the Nonstationary Problem

In general, a priori estimates are very important tools in analysis. New estimates usually give sharper insight toward the behavior of solutions. In [12, 13], by essentially applying certain a priori estimates, Ladyzhenskaya was able to show the existence of a globally unique solution for her models, assuming that $2r \geqslant 1$. However, her direct calculations resulted in the use of a generalized Gronwall inequality, which involves many complicated terms, in order to complete the estimates. These complications, to a certain degree, cause some loss of sharpness when one tries to see the effects of the added "viscosity term." In our discussion, by applying Fourier analysis, we establish some new estimates which change in a more explicit way when the new viscosity varies or diminishes.

Let us formally take $\mathbf{v}=\mathbf{u}(t)$ in (2.9) to begin with. Since $b(\mathbf{u}, \mathbf{u}, \mathbf{u})=0$ by (2.2), we have

$$\tfrac{1}{2}(\|\mathbf{u}\|_{0,2}^2)_t + \langle \mathscr{A}(\mathbf{u})\,\mathbf{u}, \mathbf{u}\rangle = (\mathbf{f}, \mathbf{u}). \tag{3.0}$$

After integrating the above equation with respect to time t and applying the Cauchy–Schwartz inequality to the right-hand side, one can formally derive the estimates obtained by Ladyzhenskaya. For convenience, we restate them in Lemma 3.1. As one may notice, those estimates are easy to derive and are only based on an "energy" type inequality.

LEMMA 3.1 (Ladyzhenskaya). *Assume that* $\mathbf{f}\in L^2[(0, T); \mathbf{H}]$, $q=2+r$, *and* $\mathbf{u}_0\in \mathbf{V}$. *Let* $\mathbf{u}$ *be a solution of* [L3]; *then for* $t\in(0, T)$,

$$\|\mathbf{u}(\cdot, t)\|_{0,2} \leqslant \|\mathbf{u}_0\|_{0,2} + \int_0^t \|\mathbf{f}(\mathbf{x}, \tau)\|_{0,2}\, d\tau \tag{3.1}$$

and

$$\|\mathbf{u}(\cdot, t)\|_{0,2}^2 + 2\left\{\int_0^t [\nu_0 \|\mathbf{u}\|_{0,2}^2 + \nu_1 \|\mathbf{u}\|_{0,q}^q]\, d\tau\right\}$$
$$\leqslant 2\|\mathbf{u}_0\|_{0,2}^2 + 3\left[\int_0^t \|\mathbf{f}(x, \tau)\|_{0,2}\, d\tau\right]^2 = c_1(t). \tag{3.2}$$

Next, we restate another estimate in [11]. More sophisticated Sobolev imbedding results are needed in its derivation.

LEMMA 3.2 (Ladyzhenskaya). *Assume that* $\mathbf{f}\in L^2[(0, T); \mathbf{H}]$, $q=2+r$, $5r\geqslant 2$, *and* $\mathbf{u}_0\in \mathbf{V}_q$. *Let* $\mathbf{u}$ *be a solution of* [L3]; *then for* $t\in(0, T)$,

$$\|\mathbf{u}(\cdot, t)\|_{1,q}^q + \frac{q}{2\nu_1}\int_0^t \|\mathbf{u}_t(\mathbf{x}, \tau)\|_{0,2}^2\, d\tau \leqslant c_2(t), \tag{3.3}$$

where $c_2(t)$ *only depends on* $\mathbf{f}$, $\mathbf{u}_0$, ν_0, ν_1, *and* r.

As we pointed out before, due to the application of the generalized Gronwall type inequality, $c_2(t)$ has an unpleasant form, especially including the product of several exponential functions of the constants ν_1 and q. For instance, one of the terms looks like $\exp\{c_r \nu_1^{1-4/q}(t^{2-4/q})\, c_1(t)^{4/q-1}\}$, where c_r is a constant depending on r and the function $c_1(t)$ is given in (3.2) (see [12, p. 99] for details). In certain situations, this results in major difficulties, as when we pursue the investigation of the behavior of solutions as ν_1 diminishes. However, we only need some weaker, but more simplified results with a more explicit dependence on the viscosity coefficients. To get such results, one may adopt Ladyzhenskaya's approach if $q \geqslant 4$. In case $q < 4$, it is evident that tremendous effort has to be made to sharpen the known imbedding results and inequalities. Hence, to complete the discussion, we herein present an alternative: the method of Fourier analysis.

As mentioned in Remark 1.1, when necessary, we extend our notation to complex-valued functions. To begin our discussion, let us state a result on the continuity of Fourier transformations, which is a generalization of the Hausdorff–Young theorem.

LEMMA 3.3. *Let the space* $\mathscr{X} = \mathbf{L}^2(\Omega)$ *be given; the Fourier transform is a continuous linear mapping from* $L^s[\mathbb{R}; \mathscr{X}]$ *to* $L^{s'}[\mathbb{R}; \mathscr{X}]$, *where* $1 \leqslant s \leqslant 2$ *and* $s^{-1} + s'^{-1} = 1$.

The original Hausdorf–Young theorem is only concerned with $\mathscr{X} = \mathbb{R}$, i.e., spaces of scalar-valued functions (see [7, 17]). It is a direct consequence of the Riesz–Thorin convexity theorem in interpolation theory and the fact that the Fourier transform is a continuous mapping both from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ and from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. However, since the Riesz–Thorin convexity theorem has been established in more general settings (see [4]), including the case $\mathscr{X} = \mathbf{L}^2(\Omega)$ (see [2]), we only need to verify that the Fourier transform is a bounded linear mapping both from $L^2[\mathbb{R}; \mathscr{X}]$ to $L^2[\mathbb{R}; \mathscr{X}]$ and from $L^1[\mathbb{R}; \mathscr{X}]$ to $L^\infty[\mathbb{R}; \mathscr{X}]$ in order to prove Lemma 3.3. Fortunately, the former claim follows directly from Parseval's Identity, and the latter follows from Jensen's inequality. So the validity of Lemma 3.3 is assured.

We now formally state the main theorem of this section. This result will also be used to derived uniform boundedness of the approximate solutions in the standard Galerkin procedure of proving the existence of the weak solutions.

THEOREM 3.1. *Assume that* $\mathbf{f} \in L^2[(0, T); \mathbf{V}']$, $q = 2 + r$, *and* $\mathbf{u}_0 \in \mathbf{H}$. *Let* $\mathbf{u}$ *be a solution of* [L3]; *then* $\mathbf{u} \in H^\alpha((0, T); \mathbf{V}, \mathbf{H})$ *for any* α *such that* $0 \leqslant \alpha < 1/4$. *Moreover, we have the estimates*

$$|||\mathbf{u}|||^2_{\alpha;\, \mathbf{V}, \mathbf{H}} \leqslant c(T) \left\{ \nu_1 \left[\int_0^T \|\mathbf{u}\|^q_{1,2}\, dt \right]^{1/q} + 1 \right\}, \tag{3.4}$$

where the constant $c(T)$ *depends on* $\mathbf{f}$, $\mathbf{u}_0$, ν_0, *and* r, *but not on* ν_1.

Proof. First, let us extend $\mathbf{u}(t)$ by zero outside the interval $[0, T]$; denote the extension by $\mathbf{u}^*(t)$. By the weak form, for $\mathbf{v} \in \mathscr{D}(\Omega)$, we have

$$\begin{aligned} &\frac{d}{dt}(\mathbf{u}^*(t), \mathbf{v}) + (\mathscr{A}(\mathbf{u}^*)\nabla\mathbf{u}^*, \nabla\mathbf{v}) + b(\mathbf{u}^*, \mathbf{u}^*, \mathbf{v}) \\ &\quad = (\mathbf{f}, \mathbf{v}) + (\mathbf{u}_0, \mathbf{v})\,\delta(0) - (\mathbf{u}(T), \mathbf{v})\,\delta(T) \qquad \text{in } \mathscr{D}'(\mathbb{R}). \end{aligned} \tag{3.5}$$

Next, we take the Fourier transform of the above equation and use (2.5) to yield

$$\begin{aligned} &-i\tau(\hat{\mathbf{u}}^*(\tau), \mathbf{v}) + ([\mathscr{A}(\mathbf{u}^*)\nabla\hat{\mathbf{u}}^*], \nabla\mathbf{v}) + ([\widehat{B\mathbf{u}}^*], \mathbf{v}) \\ &\quad = (\hat{\mathbf{f}}, \mathbf{v}) + (\mathbf{u}_0, \mathbf{v}) - (\mathbf{u}(T), \mathbf{v})\,e^{-i\tau T}. \end{aligned} \tag{3.6}$$

Setting $\mathbf{v} = (\mathbf{u}^*)^\wedge = \hat{\mathbf{u}}^*$ in (3.6) then gives

$$\begin{aligned} &-i\tau\;\|\hat{\mathbf{u}}^*(\tau)\|_{0,2}^2 + \nu_0(\nabla\hat{\mathbf{u}}^*, \nabla\hat{\mathbf{u}}^*) + ([\widehat{B\mathbf{u}}^*], \hat{\mathbf{u}}^*) \\ &\quad + \nu_1([|\nabla\mathbf{u}^*|^r\,\nabla\mathbf{u}^*]^\wedge, \nabla\hat{\mathbf{u}}^*) = (\hat{\mathbf{f}}, \hat{\mathbf{u}}^*) \\ &\quad + (\mathbf{u}_0, \hat{\mathbf{u}}^*) - (\mathbf{u}(T), \hat{\mathbf{u}}^*)\,e^{-i\tau T}. \end{aligned} \tag{3.7}$$

Let us take the imaginary part of (3.7); notice that

$$\mathscr{I}m[(\nabla\hat{\mathbf{u}}^*, \nabla\hat{\mathbf{u}}^*)] = 0. \tag{3.8}$$

Then

$$\begin{aligned} \tau\;\|\hat{\mathbf{u}}^*(\tau)\|_{0,2}^2 \leqslant{} & |\nu_1([|\nabla\mathbf{u}^*|^r\,\nabla\mathbf{u}^*]^\wedge, \nabla\hat{\mathbf{u}}^*)| \\ & + |([B\mathbf{u}^*]^\wedge, \hat{\mathbf{u}}^*)| + |(\hat{\mathbf{f}}, \hat{\mathbf{u}}^*)| \\ & + |(\mathbf{u}_0, \hat{\mathbf{u}}^*)| + |(\mathbf{u}(T), \hat{\mathbf{u}}^*)|. \end{aligned} \tag{3.9}$$

Now, we need to evaluate the right-hand side of (3.9) term by term. By our assumptions and Lemma 3.1, we have $\mathbf{f} \in L^2[\mathbb{R}; \mathbf{V}']$ and $\mathbf{u}^* \in L^\infty[\mathbb{R}; \mathbf{H}]$. Let $\Psi = |\nabla\mathbf{u}^*|^r\,\nabla\mathbf{u}^*$; then $\Psi \in L^\beta[\mathbb{R}; \mathbf{H}]$ for $\beta = q/(q-1) > 0$. Recall that $q = r + 2 > 2$; then $\beta < 2$. Moreover,

$$\left\{\int_{\mathbf{R}} \|\Psi\|_{0,2}^\beta\,dt\right\}^{1/\beta} \leqslant c\left\{\int_{\mathbf{R}} \|\mathbf{u}^*\|_{1,2}^q\,dt\right\}^{1/q}. \tag{3.10}$$

By Lemma 3.3, since $q^{-1} + \beta^{-1} = 1$, we have

$$\left\{\int_{\mathbf{R}} \|\hat{\Psi}\|_{0,2}^q\,d\tau\right\}^{1/q} \leqslant c\left\{\int_0^T \|\mathbf{u}\|_{1,2}^q\,dt\right\}^{1/q},$$

and according to (2.6),

$$\left\{\int_{\mathbf{R}} \|B\mathbf{u}\|_{\mathbf{V}'}\,d\tau\right\} \leqslant c\left\{\int_0^T \|\mathbf{u}\|_{1,2}^2\,d\tau\right\}. \tag{3.12}$$

So, for any τ,

$$\|[\widehat{B\mathbf{u}}](\tau)\|_{\mathbf{V}'} \leqslant c\int_0^T \|\mathbf{u}\|_{1,2}^2\,d\tau \leqslant c\nu_0^{-1}c_1(T). \tag{3.13}$$

Hence, (3.7) gives

$$\begin{aligned} |\tau|\ \|\hat{\mathbf{u}}^*(\tau)\|_{0,2}^2 \leqslant{} & \{\nu_1\ \|\hat{\Psi}(\tau)\|_{0,2} \\ & + \|[\widehat{B\mathbf{u}^*}](\tau)\|_{\mathbf{V}'} + \|\hat{\mathbf{f}}(\tau)\|_{\mathbf{V}'}\}\ \|\hat{\mathbf{u}}^*(\tau)\|_{1,2} \\ & + \|\hat{\mathbf{u}}^*(\tau)\|_{0,2}\ \{\|\mathbf{u}_0\|_{0,2} + \|\mathbf{u}(T)\|_{0,2}\} \\ \leqslant{} & \{\nu_1\ \|\hat{\Psi}(\tau)\|_{0,2} + c\nu_0^{-1}c_1(T) + C\}\ \|\hat{\mathbf{u}}^*(\tau)\|_{1,2} \quad \text{for any } \tau. \end{aligned} \tag{3.14}$$

Taking $\alpha < 1/4$, we have for $|\tau| \leqslant 1$

$$\int_{|\tau|\leqslant 1} |\tau|^{2\alpha}\ \|\hat{\mathbf{u}}^*\|_{0,2}^2\,d\tau \leqslant c\int_{\mathbf{R}} \|\mathbf{u}^*\|_{0,2}^2\,dt \leqslant C\nu_0^{-1}c_1(T). \tag{3.15}$$

For $|\tau| \geqslant 1$, multiply both sides of (3.14) by $|\tau|^{2\alpha-1}$; then

$$\begin{aligned} & \int_{|\tau|\geqslant 1} |\tau|^{2\alpha}\ \|\hat{\mathbf{u}}^*\|_{0,2}^2\,d\tau \\ & \quad \leqslant \nu_1 \int_{|\tau|\geqslant 1} |\tau|^{2\alpha-1}\ \|\hat{\Psi}(\tau)\|_{0,2}\ \|\hat{\mathbf{u}}^*(\tau)\|_{1,2}\,d\tau \\ & \qquad + [C + c\nu_0^{-1}c_1(T)] \int_{|\tau|\geqslant 1} |\tau|^{2\alpha-1}\ \|\hat{\mathbf{u}}^*\|_{1,2}\,d\tau. \end{aligned} \tag{3.16}$$

Since $\alpha < 1/4$, there is a constant C such that

$$\int_{|\tau|\geqslant 1} |\tau|^{(4\alpha q - 2q)/(q-2)}\,d\tau \leqslant C < \infty, \tag{3.17}$$

and

$$\int_{|\tau|\geqslant 1} |\tau|^{4\alpha-2}\,d\tau \leqslant C < \infty. \tag{3.18}$$

So, the first term on the right-hand side of (3.16) gives us

$$\begin{aligned}
&\nu_1 \int_{|\tau| \geqslant 1} |\tau|^{2\alpha - 1} \, \|\hat{\Psi}(\tau)\|_{0,2} \, \|\hat{\mathbf{u}}^*(\tau)\|_{1,2} \, d\tau \\
&\leqslant \nu_1 \left\{ \int_{|\tau| \geqslant 1} \|\hat{\Psi}(\tau)\|_{0,2}^{q} \, d\tau \right\}^{1/q} \\
&\quad \cdot \left\{ \int_{|\tau| \geqslant 1} |\tau|^{(2\alpha q - q)/(q - 1)} \, \|\hat{\mathbf{u}}^*(\tau)\|_{1,2}^{\beta} \, d\tau \right\}^{1/\beta} \\
&\leqslant c\nu_1 \left\{ \int_0^T \|\mathbf{u}\|_{1,2}^{q} \, dt \right\}^{1/q} \left\{ \int_{|\tau| \geqslant 1} |\tau|^{(4\alpha q - 2q)/(q - 2)} \, d\tau \right. \\
&\quad \left. + \int_{|\tau| \geqslant 1} \|\hat{\mathbf{u}}^*(\tau)\|_{1,2}^{2} \, d\tau \right\}^{1 - 1/q} \\
&\leqslant c\nu_1 \left\{ \int_0^T \|\mathbf{u}\|_{1,2}^{q} \, dt \right\}^{1/q} \left\{ C + \int_{\mathbf{R}} \|\mathbf{u}^*(t)\|_{1,2}^{2} \, dt \right\}^{1 - 1/q}. \qquad (3.19)
\end{aligned}$$

In deriving (3.19) we have used the inequality $ab < C[a^r + b^{r'}]$ with $r'\beta = 2$. A similar development of the second term gives

$$\begin{aligned}
\int_{|\tau| \geqslant 1} |\tau|^{2\alpha - 1} \, \|\hat{\mathbf{u}}^*\|_{1,2}^{2} \, d\tau &\leqslant C + \int_{|\tau| \geqslant 1} \|\hat{\mathbf{u}}^*(\tau)\|_{1,2}^{2} \, d\tau \\
&\leqslant C + \int_{\mathbf{R}} \|\mathbf{u}^*(t)\|_{1,2}^{2} \, dt. \qquad (3.20)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\int_{\mathbf{R}} |\tau|^{2\alpha} \, \|\hat{\mathbf{u}}^*\|_{0,2}^{2} \, d\tau &\leqslant \left\{ C'\nu_1 \left[\int_0^T \|\mathbf{u}\|_{1,2}^{q} \, dt \right]^{1/q} + C' \right\} \\
&\quad \cdot \left\{ \int_{\mathbf{R}} \|\mathbf{u}^*(t)\|_{1,2}^{2} \, dt + C \right\} + c. \qquad (3.21)
\end{aligned}$$

Combining with Lemma 3.1 yields the desired results. ▌

4. COMPACTNESS RESULTS AND APPLICATION OF THE NEW A PRIORI ESTIMATES

In this section we apply the new a priori estimates and some compactness arguments to establish a broader existence theorem for the time-dependent model. In addition, we rigorously investigate the behavior of the solution of the Ladyzhenskaya model when the coefficient of the "added viscosity" term ν_1 goes to zero.

To deal with the nonlinearity in the partial differential equations, the usual procedure first applies a Galerkin method to get a sequence of approximate solutions. Then, after obtaining certain a priori estimates, one needs to use some sort of compactness arguments to get a convergent subsequence. Finally, one will show that the limit of the convergent subsequence is the weak solution of the original problem.

For our problem, the nonlinearity appears in two places, namely the "added viscosity" term and the term $B\mathbf{u}$. To make sure that the arguments of passage to the limit are valid in our discussion, the latter can be treated by the following standard result. Then, the former term can be resolved by using a monotonicity argument after one gets certain regularity results on the weak limit.

THEOREM 4.1. *Assume that the Hilbert spaces* H_0, H_1, *and* H_2 *are given, such that*

$$H_1 \hookrightarrow H_2 \qquad (\textit{continuous imbedding}) \tag{4.1}$$

and

$$H_0 \hookrightarrow\hookrightarrow H_1 \qquad (\textit{compact imbedding}). \tag{4.2}$$

Let $\alpha > 0$, $T > 0$ *be some given constants; then*

$$H^\alpha((0, T); H_0, H_2) \hookrightarrow\hookrightarrow L^2((0, T); H_1). \tag{4.3}$$

We refer to [15, 19] for similar results. As a corollary, we have that:

COROLLARY 4.1. *Let constants* $\alpha > 0$, $T > 0$ *be given, then*

$$H^\alpha((0, T); \mathbf{V}, H) \hookrightarrow\hookrightarrow L^2((0, T); H). \tag{4.4}$$

Now, let us consider the case where $\nu_1 \in [0, \zeta]$. By the estimates of the pervious section, formally, we have the uniform boundedness of the following functions in their corresponding spaces, without dependence on the value of ν_1:

$$\mathbf{u} \in H^\alpha((0, T); \mathbf{V}, \mathbf{H}); \tag{4.5}$$

$$\mathbf{u} \in L^\infty((0, T); \mathbf{H}); \tag{4.6}$$

$$\Delta\mathbf{u} \in L^2((0, T); \mathbf{V}'), \qquad \text{where } \mathbf{V}' \text{ is the dual space of } \mathbf{V}; \tag{4.7}$$

$$B\mathbf{u} \in L^2((0, T); \mathbf{V}') \qquad \text{for} \quad n = 2; \tag{4.8}$$

$$B\mathbf{u} \in L^{4/3}((0, T); \mathbf{V}') \qquad \text{for} \quad n = 3; \tag{4.9}$$

$$\nu_1^{1/q}\mathbf{u} \in L^q((0, T); \mathbf{V}_q); \tag{4.10}$$

$$\nu_1^{1/q}\nabla\mathbf{u}\in L^q((0,T);\mathbf{H}_q); \tag{4.11}$$

$$\mathbf{g}[\mathbf{u}]=\nu_1^{q/(q-1)}\,|\nabla\mathbf{u}|^{q-2}\,\nabla\mathbf{u}\in L^{q/(q-1)}((0,T);\mathbf{H}_{q/(q-1)}); \tag{4.12}$$

$$\nabla(\nu_1^{q/(q-1)}\,|\nabla\mathbf{u}|^{q-2}\,\nabla\mathbf{u})\in L^{q/(q-1)}((0,T);\mathbf{W}'), \tag{4.13}$$

where $\mathbf{W}'$ is the dual of $\mathbf{V}_q$. Hence, from the weak form, we can further verify that

$$\boldsymbol{u}_t\in\mathbf{L}^{q/(q-1)}((0,T);\mathbf{W}')+\mathbf{L}^s((0,T);\mathbf{V}'), \tag{4.14}$$

where $s=2$ for $n=2$ and $s=4/3$ for $n=3$.

Moreover, if we allow the bounds to depend on ν_1, we can get a better estimate than (4.14). First of all, we have the following imbedding results (see [12]):

Lemma 4.1. *Let* $n=3$, *if* $3>q>1$, $\theta>0$, *and*

$$3/\theta=\alpha(3-q)/q+3/2(1-\alpha)\qquad \textit{for}\quad \alpha\in[0,1]; \tag{4.15}$$

then there exists a constant $c>0$ *such that for any* $\mathbf{u}\in\mathbf{W}^{1,q}(\Omega)$

$$\|\mathbf{u}\|_{0,\theta}\leqslant c\,\|\mathbf{u}\|_{1,q}^{\alpha}\,\|\mathbf{u}\|_{0,2}^{1-\alpha}. \tag{4.16}$$

Note that the above result is also valid when $q\geqslant 3$ and $\theta=2/(1-\alpha)$ for $\alpha\in[0,1]$. Next, recall that by (2.6), we have $\|B\mathbf{u}\|_{\mathbf{W}'}\leqslant c(\|\mathbf{u}\|_{0,2q'})^2$ where q' satisfies that $1<q'<+\infty$ and $q+q'=qq'$. By (4.16),

$$\|\mathbf{u}\|_{0,2q'}\leqslant c\,\|\mathbf{u}\|_{1,q}^{\beta}\,\|\mathbf{u}\|_{0,2}^{1-\beta}\qquad\text{for}\quad\beta=3/(5q-6). \tag{4.17}$$

Thus we have

$$\|B\mathbf{u}\|_{\mathbf{W}'}\leqslant c\,\|\mathbf{u}\|_{1,q}^{2\beta}\,\|\mathbf{u}\|_{0,2}^{2-2\beta}. \tag{4.18}$$

By the a priori estimates $\|\mathbf{u}\|_{1,q}\in L^q(0,T)$ (depending on ν_1) and $\|\mathbf{u}\|_{0,2}\in L^2(0,T)$. It follows that

$$\|B\mathbf{u}\|_{\mathbf{W}'}\in L^p(0,T)\qquad\text{for}\quad 6p\leqslant q(5q-6). \tag{4.19}$$

Now, if we have $(5q-6)\geqslant 6/(q-1)$ or $q\geqslant 11/5$, i.e.,

$$5r\geqslant 1, \tag{4.20}$$

where $r=q-2$, then

$$B\mathbf{u}\in L^{q/(q-1)}[(0,T),\mathbf{W}']. \tag{4.21}$$

Hence, formally, we have

$$\mathbf{u}'\in L^{q/(q-1)}[(0,T),\mathbf{W}']; \tag{4.22}$$

(4.22) gives enough regularity so that

$$\mathbf{u} \in C[(0, T), \mathbf{H}] \tag{4.23}$$

and

$$\frac{d}{dt}(\mathbf{u}, \mathbf{u}) = 2(\mathbf{u}', \mathbf{u}) \qquad \text{a.e.} \tag{4.24}$$

Thus, the energy equality will hold for weak solutions.

The above estimates together with compactness results enable us to get more general results on existence and uniqueness of the weak solution for [L3]. We only need assume that $5r \geqslant 1$, i.e., (4.20) holds, rather than assuming the conditions $5r \geqslant 2$ for existence and $2r \geqslant 1$ for uniqueness required in [12]. As mentioned before, we first construct a sequence of approximate solutions $\{\mathbf{u}_m\}$ $(m = 1, 2, 3,...)$ by the standard Galerkin procedure. The above estimates will hold for all $\mathbf{u}_m$ uniformly. When passing to limit, if $\mathbf{u}$ is the weak limit of a subsequence of $\mathbf{u}_m$, then due to the compactness lemma, we have that in the weak sense, $B\mathbf{u}_m \to B\mathbf{u}$. The nonlinear term $\nabla(|\nabla \mathbf{u}|^{q-2} \nabla \mathbf{u})$ can be handled by the standard monotonicity argument, i.e., using that for any $\mathbf{v} \in \mathbf{V}_q$,

$$(|\nabla \mathbf{u}_m|^{q-2} \nabla \mathbf{u}_m, \nabla(\mathbf{u}_m - \mathbf{v})) - (|\nabla \mathbf{v}|^{q-2} \nabla \mathbf{v}, \nabla(\mathbf{u}_m - \mathbf{v})) \geqslant 0 \tag{4.25}$$

and the energy equality to conclude that the weak limit of $\nabla(|\nabla \mathbf{u}_m|^{q-2} \nabla \mathbf{u}_m)$ is $\nabla(|\nabla \mathbf{u}|^{q-2} \nabla \mathbf{u})$ (see [12]). Therefore, we have the following theorem.

THEOREM 4.2. *Assume that* $n = 3$, $T > 0$, $\mathbf{f} \in L^2[(0, T); \mathbf{H}]$, $\mathbf{u}_0 \in \mathbf{H}$, $0 \leqslant \alpha < 1/4$, *and* $q = 2 + r$ *for* $5r \geqslant 1$. *Then the problem* [L3] *has a unique weak solution*

$$\mathbf{u} \in H^\alpha((0, T); \mathbf{V}, \mathbf{H}) \cap L^q[(0, T); \mathbf{V}_q] \text{ and } \mathbf{u}' \in L^{q/(q-1)}[(0, T); \mathbf{W}_{q/(q-1)}].$$

Remark 4.1. A much simpler argument can be applied to the two-dimensional case.

Next, estimates independent of ν_1 enable us to look into the behavior of the solutions as ν_1 tends to zero. Let us use $\mathbf{u}(\varepsilon)$ to denote the solution of [L3] in the case $\nu_1 = \varepsilon$ and $\mathbf{u}^* = \mathbf{u}(0)$ to denote the solution of [NS3].

THEOREM 4.3. *Assume that* $T > 0$, $\mathbf{f} \in L^2[(0, T); \mathbf{H}]$, $\mathbf{u}_0 \in \mathbf{H}$, *and* $q = 2 + r$. *Let us also assume that* $r \geqslant 1/5$ *if* $n = 3$. *Then, for any sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \to 0$, $\exists$ *subsequence* $\{\varepsilon_{nk}\}$ *such that* $\mathbf{u}(\varepsilon_{nk}) \to \mathbf{u}^*$ *weakly in space* $H_\alpha((0, T); \mathbf{V}, \mathbf{H})$ *as* $\varepsilon_{nk} \to 0$.

Proof. First of all, the space $H^{\alpha}((0, T); \mathbf{V}, \mathbf{H})$ is reflexive. By (4.5), after extracting a subsequence, there exists a weak limit, denoted by $\mathbf{u}^*$, for $\{\mathbf{u}(\varepsilon_{nk})\}$. Later in this proof, the technique of extracting subsequences will be implicitly used and be referred to using the same notation as $\{\mathbf{u}(\varepsilon_{nk})\}$. We need to establish that, as $\nu_1 \to 0$,

$$\nu_1^{1/q}\mathbf{g}[\mathbf{u}(\nu_1)] \to 0 \qquad \text{weakly in } L^{q/(q-1)}((0, T); \mathbf{H}_{q/(q-1)}) \tag{4.26}$$

for $\nu_1 \in \{\varepsilon_{nk}\}$. However, by (4.12), we can claim that $\exists \mathbf{g}^*$ such that

$$\mathbf{g}[\mathbf{u}(\varepsilon_{nk})] \to \mathbf{g}^* \qquad \text{weakly in } L^{q/(q-1)}((0, T); \mathbf{H}_{q/(q-1)}). \tag{4.27}$$

Obviously,

$$(\varepsilon_{nk})^{1/q}\,\mathbf{g}^* \to 0 \qquad \text{strongly in } L^{q/(q-1)}((0, T); \mathbf{H}_{q/(q-1)}). \tag{4.28}$$

Hence,

$$(\varepsilon_{nk})^{1/q}\,\mathbf{g}[\mathbf{u}(\varepsilon_{nk})] \to 0 \qquad \text{weakly in } L^{q/(q-1)}((0, T); \mathbf{H}_{q/(q-1)}). \tag{4.29}$$

Also, by Corollary 4.1, we have

$$\mathbf{u}(\varepsilon_{nk}) \to \mathbf{u}^* \qquad \text{strongly in } L^2((0, T); \mathbf{H}). \tag{4.30}$$

Now, we multiply the weak form (2.9) by a test function, then integrate the equation with respect to time. The conclusion in the theorem is obtained by the standard procedure of passage to the limit. ▌

The above theorem has shown some relationship between the Ladyzhenskaya equation and the Navier–Stokes equations. In fact, we obtained an existence theorem for the Navier–Stokes equations.

THEOREM 4.4. *Assume that* $T > 0$, $\mathbf{f} \in L^2[(0, T); \mathbf{H}]$, $\mathbf{u}_0 \in \mathbf{H}$. *Then,* [NS3] *has at least one weak solution in* $H^{\alpha}((0, T); \mathbf{V}, \mathbf{H})$.

In the case that the uniqueness of the solution of the Navier–Stokes equation holds, the weak limit $\mathbf{u}^*$ in Theorem 4.3 will not depend on the choice of the subsequence, so we have:

COROLLARY 4.2. *Assume that Theorem* 4.1 *holds and the problem* [NS3] *has a unique solution* $\mathbf{u}^* \in H^{\alpha}((0, T); \mathbf{V}, \mathbf{H})$. *Then, for any sequence* $\{\mathbf{u}(\varepsilon_n)\}$, *we have*

$$\mathbf{u}(\varepsilon_n) \to \mathbf{u}^* \qquad \textit{weakly in } H^{\alpha}((0, T); \mathbf{V}, \mathbf{H}), \textit{ as } \varepsilon_n \to 0, \tag{4.31}$$

and consequently,

$$\mathbf{u}(\varepsilon_n) \to \mathbf{u}^* \qquad \textit{strongly in } L^2((0, T); \mathbf{H}) \textit{ as } \varepsilon_n \to 0. \tag{4.32}$$

5. WEAK FORMULATIONS OF STATIONARY PROBLEMS

In Section 2, we presented the weak form for the time-dependent Ladyzhenskaya model. We can establish the weak form for the steady state equation in a similar way:

[L4]

For $\mathbf{f}$ given,

$$\mathbf{f} \in \mathbf{V}', \tag{5.1}$$

find $\mathbf{u} \in \mathbf{V}_q$ satisfying

$$(\mathscr{A}(\mathbf{u})\,\nabla\mathbf{u}, \nabla\mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{5.2}$$

for all $\mathbf{v} \in \mathbf{V}_q$.

Correspondingly, let us give the weak formulation of the stationary Navier–Stokes equations [NS2]:

[NS4]

For $\mathbf{f}$ given,

$$\mathbf{f} \in \mathbf{V}', \tag{5.3}$$

find $\mathbf{u} \in \mathbf{V}$ satisfying

$$\nu_0\langle \mathbf{u}, \mathbf{v}\rangle + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{5.4}$$

for all $\mathbf{v} \in \mathbf{V}$.

We define some constants and notation as

$$C_f := \sup_{\substack{\mathbf{v}\in\mathbf{V}\\ \mathbf{v}\neq\mathbf{0}}} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\mathbf{v}\|_{1,2}} \tag{5.5}$$

$$C_{fq} := \sup_{\substack{\mathbf{v}\in\mathbf{V}_q\\ \mathbf{v}\neq\mathbf{0}}} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\mathbf{v}\|_{1,q}} \tag{5.6}$$

$$\gamma_q := \sup_{\substack{\mathbf{v}\in\mathbf{V}_q\\ \mathbf{v}\neq\mathbf{0}}} \frac{\|\mathbf{v}\|_{1,2}}{\|\mathbf{v}\|_{1,q}} \tag{5.7}$$

$$N := \sup_{\substack{\mathbf{u},\mathbf{v}\in\mathbf{V}\\ \mathbf{u},\mathbf{v}\neq\mathbf{0}}} \frac{|b(\mathbf{u}, \mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\|_{1,2}^2\,\|\mathbf{v}\|_{1,2}} \tag{5.8}$$

$$N_q := \sup_{\substack{\mathbf{u},\mathbf{v}\in\mathbf{V}_q\\ \mathbf{u},\mathbf{v}\neq\mathbf{0}}} \frac{|b(\mathbf{u}, \mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\|_{1,2}^2\,\|\mathbf{v}\|_{1,2}}. \tag{5.9}$$

By the assumptions on $\mathbf{f}$, b, and Ω, the above constants are well-defined. In fact, we have:

LEMMA 5.1. *$N = N_q$.*

Proof. Obviously, $N \geqslant N_q$; On the other hand, $\mathbf{V}_q$ is dense in $\mathbf{V}$. So, $\forall \mathbf{u}, \mathbf{v} \in \mathbf{V}$, there are sequences of elements $\{\mathbf{u}_i\}$, $\{\mathbf{v}_i\} \in \mathbf{V}_q$ such that $\mathbf{u}_i \to \mathbf{u}$ and $\mathbf{v}_i \to \mathbf{v}$ in $\mathbf{V}$ as $i \to \infty$. Thus,

$$N_q \geqslant \lim_{i \to \infty} \frac{|\mathbf{b}(\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}_i)|}{\|\mathbf{u}_i\|_{1,2}^2 \, \|\mathbf{v}_i\|_{1,2}} \geqslant \frac{|\mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\|_{1,2}^2 \, \|\mathbf{v}\|_{1,2}}. \tag{5.10}$$

Hence, $N_q \geqslant N$. Thus $N = N_q$. ∎

6. PROPERTIES OF THE WEAK FORM AND A PRIORI ESTIMATES

Let $\mathbf{u}$ be a weak solution for the problem [L4]. Then, setting $\mathbf{v} = \mathbf{u}$ in (5.2), we obtain

$$\nu_0 \, \|\mathbf{u}\|_{1,2}^2 + \nu_1 \, \|\mathbf{u}\|_{1,q}^q = (\mathbf{f}, \mathbf{u}). \tag{6.1}$$

We then have the following a priori estimates.

THEOREM 6.1. *For any weak solution $\mathbf{u} \in \mathbf{V}_q$ of [L4], we have*

$$\|\mathbf{u}\|_{1,q}^{q-1} \leqslant C_{fq}/\nu_1, \tag{6.2}$$

and

$$\|\mathbf{u}\|_{1,2} \leqslant \Psi_q(C_f). \tag{6.3}$$

Here Ψ_q is defined as the inverse function of $\Phi_q\colon (0, +\infty) \to \mathbb{R}$

$$\Phi_q(x) := \nu_0 x + \nu_1 \gamma_q^{-q} x^{q-1}, \qquad \textit{for} \quad x > 0. \tag{6.4}$$

Proof. Notice that for any $x > 0$

$$\Phi_q'(x) = \nu_0 + \nu_1 (q-1)\, \gamma_q^{-q} x^{q-2} > 0;$$

thus the existence of the function Ψ_q is assured. Now, (6.2) and (6.3) follow immediately from Eq. (6.1) and the definitions of Ψ_q, C_f, and C_{fq}. ∎

Remark 6.1. For $q = 3$, an explicit expression of Ψ_3 can be obtained as

$$\Psi_3(y) = -\tfrac{1}{2} \nu_1^{-1} \gamma_3^3 [\nu_0 - (\nu_0^2 + 4\nu_1 \gamma_3^{-3} y)^{1/2}], \tag{6.5}$$

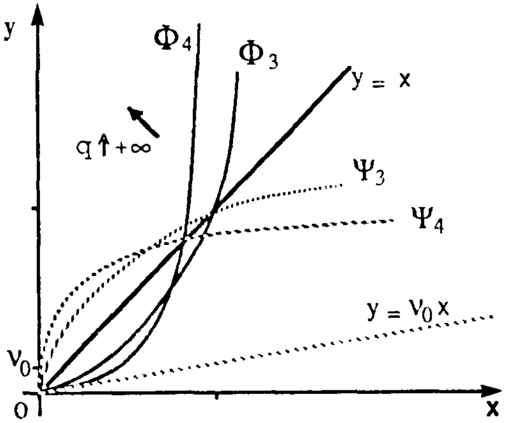

FIG. 6.1. Graphs of Φ_q and Ψ_q for $q = 3, 4$.

or,

$$\Psi_3(y) = 2y/[\nu_0 + (\nu_0^2 + 4\nu_1\gamma_3^{-3}y)^{1/2}]. \tag{6.6}$$

In general, there is no explicit expression for the function Ψ_q. We roughly illustrate some graphs of the functions defined above in Fig. 6.1.

We give an inequality concerning the function Ψ_q in the next lemma.

LEMMA 6.1. *For any $q > 2$ and $y > 0$, we have*

$$\Psi_q(y) < y/\nu_0. \tag{6.7}$$

Proof. From the definition of the function Φ_q, one can see that

$$\nu_0 y < \Phi_q(y) \tag{6.8}$$

for any $y > 0$. So, (6.7) follows from the monotonicity of Φ_q. ▌

We give another lemma concerning some important properties of the term $(\mathscr{A}(\mathbf{u})\nabla\mathbf{u}, \nabla\mathbf{v})$.

LEMMA 6.2. *There exist constants $\lambda > 0$, $M > 0$, and $M_q > 0$ such that*

$$(\mathscr{A}(\mathbf{u})\nabla\mathbf{u}, \nabla\mathbf{u} - \nabla\mathbf{v}) - (\mathscr{A}(\mathbf{v})\nabla\mathbf{v}, \nabla\mathbf{u} - \nabla\mathbf{v})$$

$$\geqslant \nu_0 \|\mathbf{u} - \mathbf{v}\|_{1,2}^2 + \lambda\nu_1 \|\mathbf{u} - \mathbf{v}\|_{1,q}^q \qquad \textit{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}_q \tag{6.9}$$

and

$$
\begin{aligned}
&|(|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{u}, \nabla \mathbf{w}) - (|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{v}, \nabla \mathbf{w})| \\
&\quad \leqslant M_q \, \|\mathbf{u}-\mathbf{v}\|_{1,q} \, \|\mathbf{w}\|_{1,q} \, (\|\mathbf{u}\|_{1,q} + \|\mathbf{v}\|_{1,q})^{q-2}, \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_q. \qquad (6.10)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&|(|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{u}, \nabla \mathbf{w}) - (|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{v}, \nabla \mathbf{w})| \\
&\quad \leqslant M \, \|\mathbf{u}-\mathbf{v}\|_{1,2} \, \|\mathbf{w}\|_{1,2} \, (\|\mathbf{u}\|_{1,\infty} + \|\mathbf{v}\|_{1,\infty})^{q-2}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,\infty}, \mathbf{w} \in \mathbf{V}_q. \\
&\qquad (6.11)
\end{aligned}
$$

This result is very closely related to studies of "monotone operators." It plays an essential role inour analysis. A detailed discussion is contained in [6] where one can find a complete proof of the similar result in two space dimensions. Moreover, as is mentioned there, the proof needs more technicalities in higher space dimensions, but nevertheless, the conclusion remains valid.

7. UNIQUENESS RESULTS OF THE STATIONARY MODEL AND RELATIONS WITH THE NAVIER–STOKES EQUATIONS

The discussions of the above sections enable us to prove the following result.

THEOREM 7.1. *Assume that both* $\mathbf{u}$ *and* $\mathbf{u}'$ *are solutions of* [L4]. *Let* $\mathbf{w} = \mathbf{u} - \mathbf{u}'$, *then either* $\mathbf{w} = \mathbf{0}$ *almost everywhere or*

$$\|\mathbf{w}\|_{1,2}^{q-2} \leqslant (\lambda \nu_1)^{-1} \gamma_q^q [N\Psi_q(C_{f2}) - \nu_0]. \qquad (7.1)$$

Proof. By assumption, we have for all $\mathbf{v} \in \mathbf{V}_q$

$$(\mathscr{A}(\mathbf{u})\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad (7.2)$$

$$(\mathscr{A}(\mathbf{u}')\nabla \mathbf{u}', \nabla \mathbf{v}) + b(\mathbf{u}', \mathbf{u}', \mathbf{v}) = (\mathbf{f}, \mathbf{v}). \qquad (7.3)$$

Let $\mathbf{v} = \mathbf{w} = \mathbf{u} - \mathbf{u}'$ and substract (7.3) from (7.2); then

$$
\begin{aligned}
&(\mathscr{A}(\mathbf{u})\nabla \mathbf{u}, \nabla \mathbf{u} - \nabla \mathbf{u}') - (\mathscr{A}(\mathbf{u}')\nabla \mathbf{u}', \nabla \mathbf{u} - \nabla \mathbf{u}') \\
&\quad = -b(\mathbf{u}, \mathbf{u}, \mathbf{u} - \mathbf{u}') + b(\mathbf{u}', \mathbf{u}', \mathbf{u} - \mathbf{u}'). \qquad (7.4)
\end{aligned}
$$

By (2.2)

$$b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = b(\mathbf{u}', \mathbf{u}', \mathbf{u}') = 0 \qquad (7.5)$$

so that

$$\begin{aligned}
&-b(\mathbf{u}, \mathbf{u}, \mathbf{u}-\mathbf{u}') + b(\mathbf{u}', \mathbf{u}', \mathbf{u}-\mathbf{u}') \\
&\quad = b(\mathbf{u}, \mathbf{u}, \mathbf{u}') + b(\mathbf{u}', \mathbf{u}', \mathbf{u}) \\
&\quad = b(\mathbf{u}, \mathbf{u}, \mathbf{u}') - b(\mathbf{u}', \mathbf{u}, \mathbf{u}') = b(\mathbf{w}, \mathbf{u}, \mathbf{u}') \\
&\quad = b(\mathbf{w}, \mathbf{u}', \mathbf{u}') + b(\mathbf{w}, \mathbf{w}, \mathbf{u}') = b(\mathbf{w}, \mathbf{w}, \mathbf{u}'). \qquad (7.6)
\end{aligned}$$

Then, (7.4) becomes

$$\begin{aligned}
&(\mathscr{A}(\mathbf{u})\nabla\mathbf{u}, \nabla\mathbf{u}-\nabla\mathbf{u}') - (\mathscr{A}(\mathbf{u}')\nabla\mathbf{u}', \nabla\mathbf{u}-\nabla\mathbf{u}') \\
&\quad = b(\mathbf{w}, \mathbf{w}, \mathbf{u}'). \qquad (7.7)
\end{aligned}$$

Now, we apply Lemma 6.2 to the left-hand side of the above equation to yield

$$\nu_0 \|\mathbf{u}-\mathbf{v}\|_{1,2}^2 + \lambda\nu_1 \|\mathbf{u}-\mathbf{v}\|_{1,q}^q \leqslant b(\mathbf{w}, \mathbf{w}, \mathbf{u}'). \qquad (7.8)$$

By (5.7) and (5.9), we have

$$\nu_0 \|\mathbf{w}\|_{1,2}^2 + \lambda\nu_1\gamma_q^{-q} \|\mathbf{w}\|_{1,2}^q \leqslant N \|\mathbf{w}\|_{1,2}^2 \|\mathbf{u}'\|_{1,2}. \qquad (7.9)$$

From the a-priori estimate (6.3) in Theorem 6.1, $\|\mathbf{u}'\|_{1,2} \leqslant \Psi_q(C_f)$. Hence, we get either $\|\mathbf{w}\|_{1,2} = 0$, i.e., $\mathbf{w} = 0$ a.e., or

$$\lambda\nu_1\gamma_q^{-q} \|\mathbf{w}\|_{1,2}^{q-2} \leqslant N\Psi_q(C_f) - \nu_0, \qquad (7.10)$$

which is equivalent to (7.1). ∎

The next result is a simple consequence of the above theorem:

THEOREM 7.2. (Uniqueness Theorem). *Assume that the following condition holds*:

$$N\Psi_q(C_f) \leqslant \nu_0 \qquad [\text{or } C_f \leqslant \Phi_q(\nu_0/N)]. \qquad (7.11)$$

Then the problem [L4] *has a unique solution.*

Remark 7.1. It is well known that problem [NS3] has a unique solution whenever

$$NC_f/\nu_0 < 1. \qquad (7.12)$$

By (6.7), we see that whenever (7.11) holds, so does (7.12). On the other hand, (7.11) is less restrictive for ν_0 than (7.12). Some known experiments have shown that condition (7.12) is very pessimistic regarding physical flows. Thus, condition (7.11) becomes significant, since, to a certain degree, it makes the gap between the experimental evidence and the theoretical estimates smaller.

Next, we briefly discuss the behavior of the solutions of problem [L4] as ν_1 goes to zero. Similar to the nonstationary case, all we need is some a priori estimates and compactness arguments. But contrary to the previous case, both of them are straightforward, simply because we know $\mathbf{u}$ is bounded in $\mathbf{V}$ (independent of ν_1) while $\nu_1^{1/q}\mathbf{u}$ is bounded in $\mathbf{V}_q$. We let $\mathbf{u}(\varepsilon)$ denote the solution for [L4] corresponding to the case $\nu_1 = \varepsilon$ and use $\mathbf{u}^*$ to denote the solution of [NS4]; then

$$\mathbf{g}[\mathbf{u}] = \nu_1^{(q-1)/q} |\nabla \mathbf{u}|^{q-2} \nabla \mathbf{u} \in \mathbf{H}_{q/(q-1)} \qquad \text{is uniformly bounded,} \tag{7.13}$$

i.e., its norm is independent of ν_1, as long as $\nu_1 \in [0, \zeta]$ for some $\zeta > 0$. So,

$$\nu_1^{1/q}\mathbf{g}[\mathbf{u}(\nu_1)] \to 0 \qquad \text{weakly in } \mathbf{H}_{q/(q-1)} \text{ as } \nu_1 \to 0. \tag{7.14}$$

Using the argument of passage to the limit, we can conclude that:

THEOREM 7.3. *Assume that* $\mathbf{f} \in \mathbf{H}$ *and* $q = 2 + r$. *Then, for any sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \to 0$, $\exists$ *subsequence* $\{\varepsilon_{nk}\}$ *such that* $\mathbf{u}(\varepsilon_{nk}) \to \mathbf{u}^*$ *weakly in* $\mathbf{V}$.

Similar to the nonstationary case, we then have:

COROLLARY 7.1. *Assume that* $\mathbf{f} \in \mathbf{H}$ *and* $q = 2 + r$. *Assume also that the corresponding problem* [NS4] *has a unique solution. Then, for any sequence* $\{\varepsilon_n\}$,

$$\mathbf{u}(\varepsilon_n) \to \mathbf{u}^* \qquad \textit{weakly in } \mathbf{V} \textit{ as } \varepsilon_n \to 0.$$

We conclude by noting that in [8] a finite element method to solve the stationary Ladyzhenskaya model is proposed and analyzed.

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