SIAM J. NUMER. ANAL.
Vol. 27, No. 1, pp. 1-19, February 1990
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FINITE-ELEMENT APPROXIMATIONS OF A LADYZHENSKAYA MODEL FOR STATIONARY INCOMPRESSIBLE VISCOUS FLOW*

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Abstract. Some finite-element approximation procedures are presented for a model proposed by Ladyzhenskaya for stationary incompressible viscous flow. The approximate problems are proved to be well posed and stable under standard assumptions on the finite-element families. The solutions of the approximate problems converge to the solution of the original problem under minimum regularity assumptions. Some error estimates are derived. The optimal order of accuracy is assured with, or even without, using exact integration rules in the approximation procedure. Iterative methods for solving the discrete nonlinear problems and comments on some computational experiments are provided. Special attention is also paid to the common properties as well as differences between the approximation procedure presented here and the approximation for the stationary Navier-Stokes equations.

Key words. finite-element method, Ladyzhenskaya model, incompressible flow

AMS(MOS) subject classifications. 65N30, 76D07

0. Introduction. In [8]-[10], a model for the motion of ideal incompressible viscous flow has been proposed. Further studies are made in [4]. In this paper, we study computational aspects of one of the models for stationary flows.

The model we work with is as follows. Consider the motion of stationary ideal incompressible viscous fluids in a bounded domain Ω in $\mathbb{R}^n$ with Lipshitz boundary Γ ($n=2$ or 3), and let $\mathbf{u}$ denote the velocity field, p the pressure, and $\mathbf{f}$ the body force per unit mass. Then the model is given by (L):

$$\text{(L)}\quad \begin{aligned} &(0.1) && -\sum \partial_k(\mathscr{A}(\mathbf{u})\partial_k u_j)+\sum u_k\partial_k u_j+\partial_j p=f_j && \text{in } \Omega,\\ &(0.2) && \operatorname{div}\mathbf{u}=0 && \text{in } \Omega,\\ &(0.3) && \mathbf{u}=0 && \text{on } \Gamma \end{aligned}$$

where in (0.1), $j=1,2$, or $j=1,2,3$, $\mathscr{A}(\mathbf{u})$ is defined by

$$\mathscr{A}(\mathbf{u})=\nu_0+\nu_1|\nabla\mathbf{u}|^{q-2} \quad \text{with } \nu_0,\ \nu_1, \text{ and } q-2>0, \tag{0.4}$$

$$|\nabla\mathbf{u}|=\left[\sum_{i,j=1}^{n}(\partial_i u_j)^2\right]^{1/2} \qquad (n=2 \text{ or } 3). \tag{0.5}$$

To make comparisons, we also consider the stationary Navier-Stokes equations (NS):

$$\text{(NS)}\quad \begin{aligned} &(0.6) && -\nu_0\Delta u_j+\sum u_k\partial_k u_j+\partial_j p=f_j && \text{in } \Omega,\\ &(0.7) && \operatorname{div}\mathbf{u}=0 && \text{in } \Omega,\\ &(0.8) && \mathbf{u}=0 && \text{on } \Gamma. \end{aligned}$$

1. Notation, function spaces, and variational formulation. We now introduce some notation and function spaces. First, let us define

$$\mathscr{D}(\Omega):=\{\mathbf{f}\in C_0^\infty(\Omega)\} \tag{1.1}$$

* Received by the editors December 9, 1987; accepted for publication (in revised form) November 16, 1988. This work was supported by Air Force Office of Scientific Research grant AFOSR-83-0101.

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to be the space of (real-valued) smooth functions with compact support in the domain Ω. We will use standard notation for the usual Sobolev spaces, e.g., $\mathbf{L}^2(\Omega)$ denotes the space of (real valued) square integrable functions with respect to the Lebesgue measure over Ω. $\mathbf{L}_0^2(\Omega)$ denotes the space of functions in $\mathbf{L}^2(\Omega)$ with mean zero. For simplicity, the domain Ω is sometimes omitted, e.g., $\mathbf{L}^2 = \mathbf{L}^2(\Omega)$. We also define

$$\mathfrak{S} := \{\boldsymbol{\varphi} \in [\mathscr{D}(\Omega)]^n : \operatorname{div} \boldsymbol{\varphi} = 0\} \qquad (n = 2 \text{ or } 3), \tag{1.2}$$

$$\mathbf{V} := \text{Completion of } \mathfrak{S} \text{ in the } \mathbf{H}^1\text{-norm}, \tag{1.3}$$

$$\mathbf{V}_q := \text{Completion of } \mathfrak{S} \text{ in the } \mathbf{W}^{1,q}\text{-norm} \qquad (q > 2). \tag{1.4}$$

Note that the spaces $\mathbf{V}$ and $\mathbf{V}_q$ have the following equivalent definitions:

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{u} = 0\}, \tag{1.5}$$

$$\mathbf{V}_q = \{\mathbf{u} \in \mathbf{W}_0^{1,q}(\Omega) \mid \operatorname{div} \mathbf{u} = 0\}. \tag{1.6}$$

The spaces $\mathbf{L}^2(\Omega)$ and $\mathbf{V}$ are Hilbert spaces with corresponding inner products and norms

$$(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega), \tag{1.7}$$

$$\|\mathbf{u}\|_{0,2} := (\mathbf{u}, \mathbf{u})^{1/2}. \tag{1.8}$$

Similarly,

$$\langle \mathbf{u}, \mathbf{v} \rangle := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathbf{V}, \tag{1.9}$$

$$\|\mathbf{u}\|_{1,2} := \left[\int_\Omega |\nabla \mathbf{u}|^2 \, d\Omega \right]^{1/2}. \tag{1.10}$$

Also, $\mathbf{L}^q$ and $\mathbf{V}_q$ are reflexive Banach spaces, endowed with the following norms: for $\mathbf{u} \in \mathbf{L}^q$,

$$\|\mathbf{u}\|_{0,q} := \left[\int_\Omega |\mathbf{u}|^q \, d\Omega \right]^{1/q}, \tag{1.11}$$

and for $\mathbf{u} \in \mathbf{V}_q$

$$\|\mathbf{u}\|_{1,q} := \left[\int_\Omega |\nabla \mathbf{u}|^q \, d\Omega \right]^{1/q}. \tag{1.12}$$

In addition, $\mathbf{V}'$ and $\mathbf{W}'$ denote the dual spaces of $\mathbf{V}$ and $\mathbf{V}_q$, respectively, with the duality pair being equivalent to the $\mathbf{L}^2$ inner product. The induced norms are denoted by $\|\cdot\|_{\mathbf{V}'}$ and $\|\cdot\|_{\mathbf{W}'}$. For problem (L) we now give a weak form (LP):

(LP) For $\mathbf{f}$ given,

$$\mathbf{f} \in \mathbf{V}', \tag{1.13}$$

find $\mathbf{u} \in \mathbf{V}_q$ satisfying

$$(\mathscr{A}(\mathbf{u}) \nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{1.14}$$

for all $\mathbf{v} \in \mathbf{V}_q$, where

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}) \, d\Omega. \tag{1.15}$$

Similarly, for problem (NS) we have the weak formulation (NSP):

(NSP) For $\mathbf{f}$ given,

$$\mathbf{f} \in \mathbf{V}', \tag{1.16}$$

find $\mathbf{u} \in \mathbf{V}$ satisfying

$$\nu_0(\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{1.17}$$

for all $\mathbf{v} \in \mathbf{V}$.

The following lemmas can be found in [4] and in the references cited there.

LEMMA 1.1. *The trilinear form b has the following properties*:

(1.18) (i) $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$ *for* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w} \in \mathbf{V}$.
(ii) *For* $n \leqq 3$, *there is a constant* $c > 0$, *such that for any* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \leqq c(\|\mathbf{u}\|_{0,4})^2 \|\mathbf{v}\|_{1,2} \tag{1.19}$$

and for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}_q$, $q > 2$ and $q + q' = qq'$

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \leqq c(\|\mathbf{u}\|_{0,2q'})^2 \|\mathbf{v}\|_{1,q}. \tag{1.20}$$

(Note that we have $\|\mathbf{v}\|_{0,4} \leqq c\|\mathbf{u}\|_{1,2}$ for $\mathbf{v} \in \mathbf{V}$ and $\|\mathbf{u}\|_{0,2q'} \leqq c\|\mathbf{u}\|_{1,q}$ for $\mathbf{u} \in \mathbf{V}_q$.)

LEMMA 1.2. *There exist constants* $\lambda > 0$, $M > 0$, *and* $M_q > 0$ *such that*

$$\begin{aligned} &(\mathcal{A}(\mathbf{u})\nabla \mathbf{u}, \nabla \mathbf{u} - \nabla \mathbf{v}) - (\mathcal{A}(\mathbf{v})\nabla \mathbf{v}, \nabla \mathbf{u} - \nabla \mathbf{v}) \\ &\quad \geqq \nu_0 \|\mathbf{u} - \mathbf{v}\|_{1,2}^2 + \lambda \nu_1 \|\mathbf{u} - \mathbf{v}\|_{1,q}^q \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_q, \end{aligned} \tag{1.21}$$

$$\begin{aligned} &|(|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{u}, \nabla \mathbf{w}) - (|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{v}, \nabla \mathbf{w})| \\ &\quad \leqq M_q \|\mathbf{u} - \mathbf{v}\|_{1,q} \|\mathbf{w}\|_{1,q} (\|\mathbf{u}\|_{1,q} + \|\mathbf{v}\|_{1,q})^{q-2} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_q. \end{aligned} \tag{1.22}$$

Also,

$$\begin{aligned} &|(|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{u}, \nabla \mathbf{w}) - (|\nabla \mathbf{u}|^{q-2}\nabla \mathbf{v}, \nabla \mathbf{w})| \\ &\quad \leqq M \|\mathbf{u} - \mathbf{v}\|_{1,2} \|\mathbf{w}\|_{1,2} (\|\mathbf{u}\|_{1,\infty} + \|\mathbf{v}\|_{1,\infty})^{q-2} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,\infty}, \quad \mathbf{w} \in \mathbf{V}_q. \end{aligned} \tag{1.23}$$

Next, we define some constants and notation as follows:

$$C_f := \sup_{\substack{\mathbf{v} \in \mathbf{V} \\ \mathbf{v} \neq 0}} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\mathbf{v}\|_{1,2}}, \tag{1.24}$$

$$C_{fq} := \sup_{\substack{\mathbf{v} \in \mathbf{V}_q \\ \mathbf{v} \neq 0}} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\mathbf{v}\|_{1,q}}, \tag{1.25}$$

$$\gamma_q := \sup_{\substack{\mathbf{v} \in \mathbf{V}_q \\ \mathbf{v} \neq 0}} \frac{\|\mathbf{v}\|_{1,2}}{\|\mathbf{v}\|_{1,q}}, \tag{1.26}$$

$$N := \sup_{\substack{\mathbf{u},\mathbf{v} \in \mathbf{V} \\ \mathbf{u},\mathbf{v} \neq 0}} \frac{|b(\mathbf{u}, \mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\|_{1,2}^2 \|\mathbf{v}\|_{1,2}}, \tag{1.27}$$

$$N_q := \sup_{\substack{\mathbf{u},\mathbf{v} \in \mathbf{V}_q \\ \mathbf{u},\mathbf{v} \neq 0}} \frac{|b(\mathbf{u}, \mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\|_{1,2}^2 \|\mathbf{v}\|_{1,2}}. \tag{1.28}$$

Note that the constants above are well defined and obviously $N = N_q$. In [9] and [10], existence of the weak solution for problem [LP] has been shown. In [4], we also established the following theorem.

THEOREM 1.1. *For any weak solution* $\mathbf{u} \in \mathbf{V}_q$ of (LP), *we have*

$$(\|\mathbf{u}\|_{1,q})^{q-1} \leqq C_{fq}/\nu_1, \tag{1.29}$$

$$\|\mathbf{u}\|_{1,2} \leqq \Psi_q(C_f). \tag{1.30}$$

Here Ψ_q *is defined as the inverse function of* $\Phi_q : (0, +\infty) \to \mathbb{R}$

$$\Phi_q(x) := \nu_0 x + \nu_1 \gamma_q^{-q} x^{q-1} \quad \text{for } x > 0. \tag{1.31}$$

THEOREM 1.2 (Uniqueness theorem). *Assume that the following condition holds*:

$$N\Psi_q(C_f) \leqq \nu_0 \qquad [\text{or } C_f \leqq \Phi_q(\nu_0/N)]. \tag{1.32}$$

Then problem (LP) *has a unique solution.*

Concrete results obtained through theoretical analysis are very limited, especially when compared to the information needed for practical purposes. Thus, solving the problem by numerical methods becomes important. In the following discussion, we devote our attention toward finite-element approximations of the model problem described above.

Before we go into the details, let us look at problem (NSP). Its numerical approximations have been studied intensively in the past. See, e.g., [5] or [6]. Many questions have been raised, for example, the so-called div-stability (or inf-sup or LBB) condition for the approximation scheme. Because of the similarity between our model problem and standard Navier–Stokes equations, those questions are also likely to arise in our studies. Thus, we will examine conditions such that the approximation is well posed and provides accurate results. We show that families of finite-element spaces that are div-stable for the approximation of the corresponding Navier–Stokes equations will also yield a stable approximation for our model problem. We discuss the effect of numerical integration, which has more significance due to the presence of the additional nonlinear term. In addition, iterative solution methods for the discrete system and the results of some computational experiments are discussed.

2. Discretization. For simplicity, we assume that Ω is a polygonal domain. Let a regular finite-element triangulation Δ^h be given where h is a discretization parameter that tends to zero. We define two finite-dimensional spaces $\mathbf{X}^h$ and P^h such that

$$\mathbf{X}^h \subset \mathbf{W}_0^{1,q} \quad \text{and} \quad P^h \subset L_0^2$$

and $\bigcup_{h \downarrow 0} \mathbf{X}^h$ is dense in $\mathbf{W}_0^{1,q}$ and $\bigcup_{h \downarrow 0} P^h$ is dense in L_0^2. Typically, we assume that $\mathbf{X}_K$, the set of restrictions of the functions in $\mathbf{X}^h$ on a single simplex K in the triangulation, is a subset of the set containing all the polynomials of degree not exceeding k, while $\mathbf{X}_K$ itself contains all the polynomials defined on K up to degree l, i.e. $P_l \subset \mathbf{X}_K \subset P_k$. Other requirements on the discrete spaces will be described later.

We define the bilinear form $c(\mathbf{v}, q)$ by

$$c(\mathbf{v}, q) := (q, \operatorname{div} \mathbf{v}) \quad \text{for } (\mathbf{v}, q) \in \mathbf{H}_0^1 \times L_0^2. \tag{2.1}$$

Let us also define a subspace $\mathbf{V}^h$ of $\mathbf{X}^h$ by

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h \mid (q^h, \operatorname{div} \mathbf{v}^h) = 0, \forall q^h \in P^h\}. \tag{2.2}$$

We assume that $\mathbf{V}^h$ is nonempty. In fact, this is true under the hypotheses we will propose later. Note that most often $\mathbf{V}^h \not\subset \mathbf{V}$. Thus, we need an extension of the trilinear form $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of the following form. For $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in \mathbf{W}_0^{1,q}$,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} \int_\Omega (u_k(\partial_k v_j) w_j - u_k(\partial_k w_j) v_j)\, d\Omega. \tag{2.3}$$

By direct calculation, we can get that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in \mathbf{V}$, (2.3) gives

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (u_k(\partial_k v_j) w_j)\, d\Omega, \tag{2.4}$$

which coincides with the original definition. Hence the extension is well defined. The extension above enables us to pursue the discussion in the finite-dimensional subspace $\mathbf{V}^h$ without losing some symmetry properties. Now, let us define a bilinear form $a(\mathbf{u}, \cdot, \cdot)$ for every $\mathbf{u} \in \mathbf{W}_0^{1,q}$

$$a(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega \mathcal{A}(\mathbf{u})(\partial_k v_j)(\partial_k w_j)\, d\Omega \quad \text{for } \mathbf{v}, \mathbf{w} \in \mathbf{W}_0^{1,q}. \tag{2.5}$$

Then, we approximate (LP) by the following discrete problem:

(LPh) Find a pair $(\mathbf{u}^h, p^h) \in \mathbf{X}^h \times P^h$ such that

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) - c(\mathbf{w}^h, p^h) = (\mathbf{f}, \mathbf{w}^h), \tag{2.6}$$

for all $\mathbf{w}^h \in \mathbf{X}^h$, and for all $q^h \in P^h$

$$c(\mathbf{u}^h, q^h) = 0. \tag{2.7}$$

We can formulate another problem in $\mathbf{V}^h$ associated with (LPh):

(LQh) Find $\mathbf{u}^h \in \mathbf{V}^h$ such that for all $\mathbf{w}^h \in \mathbf{V}^h$,

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) = (\mathbf{f}, \mathbf{w}^h). \tag{2.8}$$

Next, we introduce some new constants. We define

$$N_h := \sup_{\substack{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}^h \\ \mathbf{u},\mathbf{v},\mathbf{w} \neq 0}} \frac{b(\mathbf{u}, \mathbf{v}, \mathbf{w})}{\|\mathbf{u}\|_{1,2} \|\mathbf{v}\|_{1,2} \|\mathbf{w}\|_{1,2}}, \tag{2.9}$$

$$C_{fh} := \sup_{\substack{\mathbf{u} \in \mathbf{V}^h \\ \mathbf{u} \neq 0}} \frac{|(\mathbf{u}, \mathbf{f})|}{\|\mathbf{u}\|_{1,2}}. \tag{2.10}$$

We refer to § 1 for the definitions of some other constants related to the new constants defined here. Finally, let us give the following lemma, which will be used later.

LEMMA 2.1 (Clarkson inequality). *For $\rho \geqq 2$ and $f, g \in L^\rho(\Omega)$,*

$$\int_\Omega \left| \frac{1}{2}(f+g) \right|^\rho d\Omega + \int_\Omega \left| \frac{1}{2}(f-g) \right|^\rho d\Omega \leqq \frac{1}{2} \left[\int_\Omega |f|^\rho\, d\Omega + \int_\Omega |g|^\rho\, d\Omega \right]. \tag{2.11}$$

3. The well-posedness of the discrete problem. In this section, we show that both [LPh] and [LQh] are well posed. First, a standard fixed-point argument gives us the following theorem.

THEOREM 3.1. *Problem* (LQh) *has at least one solution in* $\mathbf{V}^h$. *Moreover, if* $\mathbf{u}^h$ *is a solution, then it satisfies*

$$\|\mathbf{u}^h\|_{1,q} \leqq [\gamma_q[\nu_1]^{-1} C_{fh}]^{1/(q-1)}, \tag{3.1}$$

$$\|\mathbf{u}^h\|_{1,2} \leqq \Psi_q(C_{fh}) \tag{3.2}$$

where the function Ψ_q *is defined as in* § 1.

Proof. For all $\mathbf{v} \in \mathbf{V}^h$, let us define a mapping $\mathfrak{T}: \mathbf{V}^h \to \mathbf{V}^h$ satisfying

$$(\mathfrak{T}(\mathbf{v}), \mathbf{w}) := a(\mathbf{v}, \mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{v}, \mathbf{w}) - (\mathbf{f}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}^h. \tag{3.3}$$

Taking $\mathbf{w} = \mathbf{v}$, we get

$$(\mathfrak{T}(\mathbf{v}), \mathbf{v}) = a(\mathbf{v}, \mathbf{v}, \mathbf{v}) - (\mathbf{f}, \mathbf{v}) \geqq \|\mathbf{v}\|_{1,2}\{\Phi_q(\|\mathbf{v}\|_{1,2}) - C_{fh}\} \tag{3.4}$$

where the function Φ_q is defined by (1.31). So,

$$(\mathfrak{T}(\mathbf{v}), \mathbf{v}) > 0 \quad \text{for } \|\mathbf{v}\|_{1,2} < \Psi_q(C_{fh}). \tag{3.5}$$

By a fixed-point theorem (see [5]), there exists an element $\mathbf{u}^h \in \mathbf{V}^h$ such that

$$(\mathfrak{T}(\mathbf{u}^h), \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}^h, \tag{3.6}$$

i.e., $\mathbf{u}^h$ solves (LQh). Estimates (3.1) and (3.2) can be obtained directly from (2.8). □

Remark 3.1. From above, we see that the discrete solutions $\{\mathbf{u}^h\}$ are bounded, independent of *h*. Hence, we can get some existence results for the continuous problem by extracting a subsequence and passing to the limit.

Corresponding to the continuous case (see Theorem 1.2), the uniqueness of the solution of (LQh) can be derived under certain conditions. Its proof is similar to the one used in [4] for the continuous problem.

THEOREM 3.2. *Problem* (LQh) *has a unique solution* $\mathbf{u}^h$ *in* $\mathbf{V}^h$, *provided that*

$$N_h \Psi_q(C_{fh}) \leqq \nu_0. \tag{3.7}$$

Next, we discuss the well-posedness of problem (LPh). To do this, the usual inf-sup condition (or the div-stability condition) is introduced.

DEFINITION 3.1. The pair of finite-dimensional subspaces $\mathbf{X}^h$ and P^h satisfies the inf-sup condition, if and only if there exists a constant $\beta > 0$, independent of *h*, such that

$$\inf_{\substack{q \in P^h \\ q \neq 0}} \sup_{\substack{\mathbf{u} \in \mathbf{X}^h \\ \mathbf{u} \neq 0}} \frac{c(\mathbf{u}, q)}{\|q\|_{0,2}\|\mathbf{u}\|_{1,2}} \geqq \beta. \tag{3.8}$$

The definition above plays an important role in studies of the finite-element approximation of the Navier–Stokes equations. Usually, it is taken as a criterion of whether or not the families of finite-element spaces yield stable approximations. As we stated earlier, some details are omitted to avoid tedious repetition of known results.

THEOREM 3.3. *Assume that the pair of finite-dimensional spaces* $\mathbf{X}^h$ *and* P^h *satisfies the* inf-sup *condition* (3.8). *Then* $\mathbf{V}^h \neq \emptyset$ *and for every solution* $\mathbf{u}^h$ *of problem* (LQh), *there exists a unique element* $p^h \in P^h$ *such that the pair* $(\mathbf{u}^h, p^h)$ *satisfies problem* (LPh).

The above theorem is a simple consequence of the Lax–Milgram Theorem. Consequently, we can give a corollary regarding the uniqueness of the solution of problem (LPh).

COROLLARY 3.1. *The problem* (LPh) *has a unique solution whenever* (LQh) *has a unique solution and the* inf-sup *condition* (3.8) *holds.*

It is desirable that whenever certain properties of the continuous problem are known, the corresponding conclusions for discrete problems can also be drawn, at least if we use sufficiently fine approximation spaces. Indeed, we can discuss some of those matters under minor modifications of the known results in [5], where hypotheses similar to the following ones have been introduced.

Hypothesis 3.1. There exists a mapping $\Pi_h \in \mathscr{L}(\mathbf{W}_0^{1,q} \cap \mathbf{H}^2, \mathbf{X}^h)$ such that for some constant $c_1 > 0$ and $c_2 > 0$:

$$(q^h, \operatorname{div}(\mathbf{v} - \Pi_h \mathbf{v})) = 0 \quad \forall q^h \in P^h, \tag{3.9}$$

and for $1 \leqq m \leqq l$,

$$\|(\mathbf{v} - \Pi_h \mathbf{v})\|_{1,2} \leqq c_1 h^m \|\mathbf{v}\|_{m+1,2} \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,q} \cap \mathbf{H}^{m+1}, \tag{3.10}$$

$$\|\Pi_h \mathbf{v}\|_{1,\theta} \leqq c_2 \|\mathbf{v}\|_{1,\theta} \quad \text{for } \theta \in \{q, +\infty\} \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,\infty} \cap \mathbf{H}^{m+1}. \tag{3.11}$$

Hypothesis 3.2. The orthogonal projection operator Λ_h on the space P^h satisfies that for some constant $c > 0$,

$$\|(q - \Lambda_h q)\|_{0,2} \leqq c h^m \|q\|_{m,2} \quad \forall q \in L_0^2 \cap \mathbf{H}^m. \tag{3.12}$$

Under the hypotheses above, we can prove the following lemma by density arguments.

LEMMA 3.1.

$$\lim_{h \to 0} N_h = N, \qquad \lim_{h \to 0} C_{fh} = C_f. \tag{3.13}$$

As a consequence of the lemma above and Theorem 3.2, we have the following theorem.

THEOREM 3.4. *Assume that*

$$N \Psi_q(C_f) < \nu_0 \tag{3.14}$$

holds; *then when h is small enough, problem* (LQh) *has a unique solution.*

In fact, by Lemma 3.1, there exists a constant $\delta > 0$ such that for h small, we have

$$\nu_0^{-1} N_h \Psi_q(C_{fh}) \leqq 1 - \delta < 1. \tag{3.15}$$

Next, we prove a convergence result under minimal regularity assumptions on the solution $\mathbf{u}$ of the continuous problem.

THEOREM 3.5. *Assume* $\mathbf{u}$ *satisfies* (LP) *and* $\mathbf{u}^h$ *solves* (LQh). *If* (3.14) *holds, then*

$$\lim_{h \to 0} \|\mathbf{u} - \mathbf{u}^h\|_{1,q} = 0. \tag{3.16}$$

Proof. We see from (3.1) that $\{\mathbf{u}^h\}$ are uniformly bounded in $\mathbf{W}_0^{1,q}$. The space $\mathbf{W}_0^{1,q}$ is reflexive, so, there exists a subsequence of $\{\mathbf{u}^h\}$ that converges weakly to $\mathbf{u}$ as $h \to 0$ in the space $\mathbf{W}_0^{1,q}$. Since $\bigcup_{h \downarrow 0} \mathbf{X}^h$ is dense in $\mathbf{W}_0^{1,q}$, we can easily show that the limit $\mathbf{u} \in \mathbf{W}_0^{1,q}$ is the only solution of the continuous problem under assumption (3.14) and does not depend on the subsequence we have chosen. So,

$$w\text{-}\lim_{h \to 0} \mathbf{u}^h = \mathbf{u}. \tag{3.17}$$

Furthermore, by the weak form, we have

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{u}^h) = (\mathbf{f}, \mathbf{u}^h), \tag{3.18}$$

$$a(\mathbf{u}, \mathbf{u}, \mathbf{u}) = (\mathbf{f}, \mathbf{u}). \tag{3.19}$$

Since

$$(\mathbf{f}, \mathbf{u}^h) \to (\mathbf{f}, \mathbf{u}) \quad \text{as } h \to 0, \tag{3.20}$$

we get

$$\lim_{h\to 0} a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{u}^h) = a(\mathbf{u}, \mathbf{u}, \mathbf{u}). \tag{3.21}$$

Let us denote

$$\mathbf{w}^h = (\mathbf{u}^h - \mathbf{u})/2, \tag{3.22}$$

$$\mathbf{z}^h = (\mathbf{u}^h + \mathbf{u})/2. \tag{3.23}$$

Then, by Lemma 2.1

$$a(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) + a(\mathbf{z}^h, \mathbf{z}^h, \mathbf{z}^h) \leqq \tfrac{1}{2}[a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{u}^h) + a(\mathbf{u}, \mathbf{u}, \mathbf{u})]. \tag{3.24}$$

By construction

$$w\text{-}\lim_{h\to 0} \mathbf{z}^h = \mathbf{u}, \tag{3.25}$$

and $a(\mathbf{v}, \mathbf{v}, \mathbf{v})$ is a convex functional of $\mathbf{v}$ in the space $\mathbf{W}_0^{1,q}$. So,

$$a(\mathbf{u}, \mathbf{u}, \mathbf{u}) \leqq \lim_{h\to 0} a(\mathbf{z}^h, \mathbf{z}^h, \mathbf{z}^h). \tag{3.26}$$

Combining the above results, we obtain

$$\lim_{h\to 0} a(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) \leqq 0. \tag{3.27}$$

But $a(\mathbf{v}, \mathbf{v}, \mathbf{v}) \geqq 0$, therefore,

$$\lim_{h\to 0} a(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) = 0. \tag{3.28}$$

Now, (3.16) follows immediately. □

Finally, we give some remarks on the hypothesis made above and the stability of the approximation procedure.

Remark 3.2. We point out that the essential assumption we had in Hypothesis 3.1 is the approximation property (3.10). In most practical cases, once we have (3.10), (3.11) will be valid without further assumptions. For a recent discussion on this issue, we refer to [3] in which many such results are derived even without assuming the regularity of the finite-element triangulation.

Remark 3.3. From our discussion, we can note that (3.8) may also be applied to our problem as a criterion for the stability of the finite-element approximations. This means that in the sense of (3.8), any pair of finite-element spaces used in the approximation scheme (LQh) will provide stable approximations, if in the standard discretization scheme of (NSP), they can assure the stability of the approximations (see [5]).

4. An error estimate. We will work on the case where (3.14) holds. Consequently, (3.15) is also valid for h small.

THEOREM 4.1. *Assume that the solution* $(\mathbf{u}, p)$ *of problem* (L) *satisfies that* $p \in \mathbf{H}^m(\Omega)$ *and* $\mathbf{u} \in \mathbf{W}_0^{1,\infty}(\Omega) \cap \mathbf{H}^{m+1}(\Omega)$ *for some integer* $m \leqq l$. *Then for h sufficiently small, there exists a constant* $c = c(\|\mathbf{u}\|_{1,\infty}) > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,2} \leqq ch^m\{\|\mathbf{u}\|_{m+1,2} + \|p\|_{m,2}\}. \tag{4.1}$$

Proof. Let $\mathbf{v}^h \in \mathbf{X}^h$ be given; in fact, we take $\mathbf{v}^h = \Pi_h \mathbf{u}$. Let $\mathbf{w}^h = \mathbf{u}^h - \mathbf{v}^h$ and define

$$\mathscr{E}_h := a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) - a(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h) - b(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h); \tag{4.2}$$

then, by Lemma 1.2

$$\begin{aligned} \mathscr{E}_h &= a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) - a(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h) + b(\mathbf{w}^h, \mathbf{u}^h, \mathbf{w}^h) \\ &\geqq \nu_0 \|\mathbf{u}^h - \mathbf{v}^h\|_{1,2}^2 + \lambda \nu_1 \|\mathbf{u}^h - \mathbf{v}^h\|_{1,q}^q - N_h \|\mathbf{w}^h\|_{1,2}^2 \|\mathbf{u}^h\|_{1,2} \\ &\geqq \lambda \nu_1 \|\mathbf{w}^h\|_{1,q}^q + (\nu_0 - N_h \Psi_q(C_{fh})) \|\mathbf{w}^h\|_{1,2}^2 \\ &\geqq \lambda \nu_1 (\gamma_q)^{-q} \|\mathbf{w}^h\|_{1,2}^q + \nu_0 \delta \|\mathbf{w}^h\|_{1,2}^2. \end{aligned} \tag{4.3}$$

Next, let us give an upper bound for $\mathscr{E}_h$. Since

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) = (\mathbf{f}, \mathbf{w}^h), \tag{4.4}$$

then

$$\begin{aligned} \mathscr{E}_h &= (\mathbf{f}, \mathbf{w}^h) - a(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h) - b(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h) \\ &= a(\mathbf{u}, \mathbf{u}, \mathbf{w}^h) + b(\mathbf{u}, \mathbf{u}, \mathbf{w}^h) - c(\mathbf{w}^h, p) - a(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h) - b(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h). \end{aligned} \tag{4.5}$$

We know that for all $q^h \in P^h$,

$$c(\mathbf{u} - \mathbf{u}^h, q^h) = 0 \quad \text{and} \quad c(\mathbf{u} - \Pi_h \mathbf{u}, q^h) = 0. \tag{4.6}$$

So,

$$\begin{aligned} |c(\mathbf{w}^h, p)| &= |c(\mathbf{w}^h, p - q^h)| \\ &\leqq c \|p - q^h\|_{0,2} \|\mathbf{w}^h\|_{1,2}, \end{aligned} \tag{4.7}$$

and,

$$\begin{aligned} |b(\mathbf{u}, \mathbf{u}, \mathbf{w}^h) - b(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h)| &= |b(\mathbf{u}, \mathbf{u} - \mathbf{v}^h, \mathbf{w}^h) - b(\mathbf{u} - \mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h)| \\ &\leqq N_h \|\mathbf{w}^h\|_{1,2} \|\mathbf{u} - \mathbf{v}^h\|_{1,2} (\|\mathbf{u}\|_{1,2} + \|\mathbf{v}^h\|_{1,2}). \end{aligned} \tag{4.8}$$

Also,

$$\begin{aligned} |a(\mathbf{u}, \mathbf{u}, \mathbf{w}^h) - a(\mathbf{v}^h, \mathbf{v}^h, \mathbf{w}^h)| \leqq\ & c\nu_0 \|\mathbf{w}^h\|_{1,2} \|\mathbf{u} - \mathbf{v}^h\|_{1,2} \\ &+ M \|\mathbf{u} - \mathbf{v}^h\|_{1,2} \|\mathbf{w}^h\|_{1,2} (\|\mathbf{u}\|_{1,\infty} + \|\mathbf{v}^h\|_{1,\infty})^{q-2}. \end{aligned} \tag{4.9}$$

Hence,

$$\begin{aligned} \mathscr{E}_h \leqq\ & c \|p - q^h\|_{0,2} \|\mathbf{w}^h\|_{1,2} + c\nu_0 \|\mathbf{w}^h\|_{1,2} \|\mathbf{u} - \mathbf{v}^h\|_{1,2} \\ &+ \|\mathbf{w}^h\|_{1,2} \|\mathbf{u} - \mathbf{v}^h\|_{1,2} [N_h (\|\mathbf{u}\|_{1,2} + \|\mathbf{v}^h\|_{1,2}) + M (\|\mathbf{u}\|_{1,\infty} + \|\mathbf{v}^h\|_{1,\infty})^{q-2}]. \end{aligned} \tag{4.10}$$

We then get

$$\begin{aligned} &\lambda \nu_1 (\gamma_q)^{-q} (\|\mathbf{w}^h\|_{1,2})^{q-1} + \nu_0 \delta \|\mathbf{w}^h\|_{1,2} \\ &\quad \leqq c \|p - q^h\|_{0,2} + \|\mathbf{u} - \mathbf{v}^h\|_{1,2} [c\nu_0 + N_h (\|\mathbf{u}\|_{1,2} + \|\mathbf{v}^h\|_{1,2}) \\ &\qquad + M (\|\mathbf{u}\|_{1,\infty} + \|\mathbf{v}^h\|_{1,\infty})^{q-2}]. \end{aligned} \tag{4.11}$$

Now, (4.1) follows naturally from the estimates above. □

Remark 4.1. The regularity assumptions on the solution usually cannot be easily verified. Thus, the significance of proving some convergence results under minimal regularity assumptions is clearly exhibited. We have already given such a result in the last section.

Remark 4.2. Here our attention is restricted to the uniqueness case. The generalization to the nonuniqueness case may be considered using similar approaches to that in [5].

5. Effect of numerical integrations. Part I. The convergence proof and the error estimates given above are based on an exact implementation of the approximation procedure. However, due to the appearance of the nonlinear term, it is of practical interest to investigate the effect of the application of some numerical integration rules. The related issue for the Navier–Stokes equations is discussed in [7]. It is shown that the same accuracy can be retained, without the need to integrate the nonlinear term $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ exactly. Here, we intend to discuss this matter, i.e., integrating $a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ exactly while applying numerical integration rule to terms such as $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ and show how this will not affect the accuracy of the approximate solution. The extra term appearing in the Ladyzhenskaya model brings us into a more complicated situation, as we would expect. But in § 6 we separate that issue in the second part of our discussion.

Standard theory on the effects of numerical integration for linear problems can be found in [2], which also serves as a source of most of the notation we use in our discussion. For example, let

$$I(\vartheta) := \int_k \vartheta(x)\, dx, \tag{5.1}$$

and according to a certain quadrature rule, $I(\vartheta)$ is approximated by

$$I^h(\vartheta) := \sum_{1 \leqq i \leqq L} \omega_{i,k} \vartheta(\zeta_{i,k}) \tag{5.2}$$

where $\omega_{i,k}$ are some positive constant weights and $\zeta_{i,k}$ are some specified points distributed in the integration area. Generally, (5.2) is established in a certain reference simplex, then formulated for the finite-element triangulation by affine mappings. In the sequel, we use the convention that $\mathcal{F}^h(\mathbf{w}^h)$ and $b^h(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h)$ denote the approximations of the integrals $(\mathbf{f}^h, \mathbf{w}^h)$ and $b(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h)$, respectively, resulting from the numerical integration. We first note that $b^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ still has the antisymmetry property with respect to its last two arguments, i.e.,

$$b^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) = -b^h(\mathbf{u}^h, \mathbf{w}^h, \mathbf{v}^h) \quad \text{for all } \mathbf{u}^h, \mathbf{v}^h, \quad \text{and} \quad \mathbf{w}^h \in \mathbf{X}^h. \tag{5.3}$$

Then we can formulate our modified discrete problem.

(LPh1) Find $(\mathbf{u}^h, p^h) \in \mathbf{X}^h \times P^h$ such that for any $(\mathbf{w}^h, q^h) \in \mathbf{X}^h \times P^h$,

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) - c(\mathbf{w}^h, p^h) = \mathcal{F}^h(\mathbf{w}^h), \tag{5.4}$$

$$c(\mathbf{u}^h, q^h) = 0. \tag{5.5}$$

Correspondingly we have

(LQh1) Find $\mathbf{u}^h \in \mathbf{V}^h$ such that for any $\mathbf{w}^h \in \mathbf{V}^h$

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) = \mathcal{F}^h(\mathbf{w}^h). \tag{5.6}$$

The well-posedness of the above problems can be verified under the following hypothesis on the numerical quadrature rules. (See also [2], [7].)

Hypothesis 5.1. The quadrature formula (5.2) satisfies the following requirements:

(i) The weights $\omega_{i,k}$ are positive and the set of quadrature points $\{\zeta_{i,k}\}$ contains a P_{k-1}-unisolvent subset.

(ii) There exists an integer $s \geqq k-1$ such that (5.2) is exact for all polynomials of degree up to s.

Let us denote the integer $s-k+1$ by ρ. As in [7], we also need the following refined hypothesis on the approximation subspaces. In practice, it will be a direct consequence of hypotheses given in earlier sections.

Hypothesis 5.2. Assume that the mapping Π_h defined previously also satisfies that for $0 \leqq \mu \leqq m+1$, there exists a constant $c > 0$ such that for all $\mathbf{u} \in \mathbf{W}^{m+1,\alpha}(\Omega)$, we have

$$(5.7) \qquad \left(\sum_{K \in \Delta^h} \|\mathbf{u} - \Pi_h \mathbf{u}\|_{\mu,\alpha,K}^{\alpha} \right)^{1/\alpha} \leqq ch^{m+1-\mu} \|\mathbf{u}\|_{m+1,\alpha}$$

where Δ^h is the triangulation of Ω defined in § 2, K is any simplex in Δ^h, and $0 \leqq m \leqq l$, $\alpha(m+1) - n > 0$, $n = 2$ or 3, is the space dimension.

Now we define that

$$(5.8) \qquad \mathcal{N}_h := \sup_{\substack{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}^h \\ \mathbf{u},\mathbf{v},\mathbf{w} \neq 0}} \frac{b^h(\mathbf{u}, \mathbf{v}, \mathbf{w})}{\|\mathbf{u}\|_{1,2} \|\mathbf{v}\|_{1,2} \|\mathbf{w}\|_{1,2}},$$

$$(5.9) \qquad \mathscr{C}_{fh} := \sup_{\substack{\mathbf{u} \in \mathbf{V}^h \\ \mathbf{u} \neq 0}} \frac{|\mathscr{F}^h(\mathbf{u})|}{\|\mathbf{u}\|_{1,2}}.$$

As in previous discussions, we have the existence and uniqueness results for the solution of problem (LQh1).

THEOREM 5.1. *Problem* (LQh1) *has at least one solution in* $\mathbf{V}^h$. *Moreover, if* $\mathbf{u}^h$ *is a solution, it satisfies*

$$(5.10) \qquad \|\mathbf{u}^h\|_{1,q} \leqq [\gamma_q [\nu_1]^{-1} \mathscr{C}_{fh}]^{1/(q-1)},$$

and

$$(5.11) \qquad \|\mathbf{u}^h\|_{1,2} \leqq \Psi_q(\mathscr{C}_{fh}).$$

If, in addition, we have

$$(5.12) \qquad \mathcal{N}_h \Psi_q(\mathscr{C}_{fh}) \leqq \nu_0,$$

then the solution is unique.

Proof. The proof is similar to that of the continuous case. □

Next, we give a result in [7] that is obtained from standard inverse estimates.

LEMMA 5.1. *Under the hypotheses we have made, there exists a constant* $c > 0$ *such that for all* $\mathbf{u}^h$, $\mathbf{v}^h$, *and* $\mathbf{w}^h \in \mathbf{V}^h$

$$(5.13) \qquad |b(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) - b^h(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h)| \leqq ch^{1-n/4} \|\mathbf{u}^h\|_{1,2} \|\mathbf{v}^h\|_{1,2} \|\mathbf{w}^h\|_{1,2}$$

where $n = 2$ *or* 3 *is the space dimension.*

Hence we immediately get Lemma 5.2.

LEMMA 5.2.

$$(5.14) \qquad \lim_{h \to 0} \mathcal{N}_h = N, \qquad \lim_{h \to 0} \mathscr{C}_{fh} = C_f.$$

Consequently we have Theorem 5.2.

THEOREM 5.2. *Assume* (3.14) *holds, then there exists a constant* $\delta > 0$ *such that for* h *small enough,*

$$(5.15) \qquad \nu_0^{-1} \mathcal{N}_h \Psi_q(\mathscr{C}_{fh}) \leqq 1 - \delta < 1,$$

and the problem (LQh1) *has a unique solution* $\mathbf{u}^{*h} \in \mathbf{V}^h$. *Moreover, if the solution pair* $(\mathbf{u}, p)$ *of* (L) *satisfies that* $\mathbf{u} \in \mathbf{W}_0^{1,\infty}(\Omega) \cap \mathbf{H}^{m+1}(\Omega)$ *and* $p \in \mathbf{H}^m(\Omega)$ *for some integer* $m \leqq l$, *then there exist constants* $c_1 > 0$ *and* $c^* = c^*(\|\mathbf{u}\|_{1,\infty}) > 0$ *such that*

$$(5.16) \qquad \begin{aligned} \|\mathbf{u} - \mathbf{u}^{*h}\|_{1,2} \leqq c^* h^m \{\|\mathbf{u}\|_{m+1,2} + \|p\|_{m,2}\} + c_1 \sup_{\mathbf{w}^h \in \mathbf{V}^h} \frac{|(\mathbf{f}, \mathbf{w}^h) - \mathscr{F}^h(\mathbf{w}^h)|}{\|\mathbf{w}^h\|_{1,2}} \\ + c_1 \sup_{\mathbf{w}^h \in \mathbf{V}^h} \frac{|b(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h) - b^h(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h)|}{\|\mathbf{w}^h\|_{1,2}}. \end{aligned}$$

Proof. The existence and uniqueness of the solution $\mathbf{u}^{*h}$ can be verified in a manner similar to that used in previous discussions. As far as (5.16) is concerned, an identical proof to that for (4.1) can be applied with only the change of adding the residual terms. □

From (7) we have Lemma 5.3.

LEMMA 5.3. *Let* $\mathbf{f} \in \mathbf{W}^{\rho+1,\alpha}(\Omega)$, *where* $\alpha(\rho+1) > n$, $\alpha \geqq 2$, $l-1 \geqq \rho \geqq 0$, *and* $\alpha^{-1} + \beta^{-1} = 1$. *Then there exists constant* $c_2 > 0$ *such that for all* $\mathbf{w}^h \in \mathbf{V}^h$

$$|(\mathbf{f}, \mathbf{w}^h) - \mathscr{F}^h(\mathbf{w}^h)| \leqq c_2 h^{\rho+1} \|\mathbf{f}\|_{\rho+1,\alpha} \|\mathbf{w}^h\|_{1,\beta}. \tag{5.17}$$

Also, for $\mathbf{u} \in H^{\rho+2}(\Omega)$, *there exists a constant* $c_3 > 0$, *such that for all* $\mathbf{w}^h \in \mathbf{V}^h$

$$|b(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h) - b^h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h)| \leqq c_3 h^{\rho+1} (\|\mathbf{u}\|_{\rho+2,2})^2 \|\mathbf{w}^h\|_{1,2}. \tag{5.18}$$

Finally, we combine the results above to yield Theorem 5.3.

THEOREM 5.3. *Under the hypotheses in the above theorems and lemmas, there exists a constant* $\gamma_\beta > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^{*h}\|_{1,2} \leqq c^* h^m \{\|\mathbf{u}\|_{m+1,2} + \|p\|_{m,2}\} + c_1 h^{\rho+1} \{c_2 \gamma_\beta \|\mathbf{f}\|_{\rho+1,\alpha} + c_3 (\|\mathbf{u}\|_{\rho+2,2})^2\}. \tag{5.19}$$

Also, we have a special case of Theorem 5.3.

COROLLARY 5.1. *Let* $m = l$ *and* $\rho = l-1$ (*i.e.*, $s = k+l-2$); *then we have*

$$\|\mathbf{u} - \mathbf{u}^{*h}\|_{1,2} \leqq c^* h^l \{\|\mathbf{u}\|_{l+1,2} + \|p\|_{l,2} + c_1 [c_2 \gamma_\beta \|\mathbf{f}\|_{l,\alpha} + c_3 (\|\mathbf{u}\|_{l+1,2})^2]\}. \tag{5.20}$$

Remark 5.1. We can see that in the case where the quadrature formula (5.2) applied to the terms $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ and $(\mathbf{f}, \mathbf{w}^h)$ is exact for all polynomials of degree less than or equal to $k+l-2$, the order of accuracy of the finite-element method remains the same as when using exact integration on the terms $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ and $(\mathbf{f}, \mathbf{w}^h)$.

6. Effect of numerical integrations. Part II. In the previous section, our discussion has shown that the application of some proper quadrature rules on the terms $b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ and $(\mathbf{f}, \mathbf{w}^h)$ alone will not lower the order of accuracy of the approximation method. In fact, as we have observed, if the finite-element subspaces consist of piecewise kth-degree polynomials, then to retain the best accuracy, the quadrature rule should be exact at least for all the polynomials of degree up to $2k-2$. This, by the way, implies that we should integrate the term $(\nabla\mathbf{u}^h, \nabla\mathbf{w}^h)$ exactly. This is also a well-known fact in the theory of the approximation of other problems including the Navier-Stokes equations. However, we still need to consider, as one of the differences with the Navier-Stokes case, the extra nonlinear term in the Ladyzhenskaya model.

To begin the discussion, we give the formulation of the new problem. Let $a^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$ denote the numerical value of the term $a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h)$, evaluated by the quadrature rule (5.2). We then define the problem:

(LQh2) Find $\mathbf{u}^h$ in $\mathbf{V}^h$ such that for any $\mathbf{w}^h \in \mathbf{V}^h$

$$a^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) + b^h(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) = \mathscr{F}^h(\mathbf{w}^h). \tag{6.1}$$

We now prove that the new formulation inherits some type of "monotone" properties of the original form. First, we give the following theorem.

THEOREM 6.1. *Assume that* $s \geqq 2k-2$ *as in Hypothesis* 5.1; *then there exists a constant* $\eta > 0$, *independent of h, such that for all* $\mathbf{u}, \mathbf{v} \in \mathbf{V}^h$

$$a^h(\mathbf{u}, \mathbf{u}, \mathbf{u}-\mathbf{v}) - a^h(\mathbf{v}, \mathbf{v}, \mathbf{u}-\mathbf{v}) \geqq \eta \nu_0 \|\mathbf{u}-\mathbf{v}\|_{1,2}^2. \tag{6.2}$$

Proof. Let us first define a function as follows:

$$\psi := |\nabla\mathbf{u}|^{q-2} \nabla\mathbf{u} \cdot \nabla(\mathbf{u}-\mathbf{v}) - |\nabla\mathbf{v}|^{q-2} \nabla\mathbf{v} \cdot \nabla(\mathbf{u}-\mathbf{v}), \tag{6.3}$$

Obviously,

$$\psi \geqq 0 \quad \text{a.e.} \tag{6.4}$$

Noting that the weights appearing in (5.2) are all positive, we have

$$I^h(\psi) \geqq 0. \tag{6.5}$$

Now, define

$$f := \nabla(\mathbf{u}-\mathbf{v}) \cdot \nabla(\mathbf{u}-\mathbf{v}). \tag{6.6}$$

Since (5.2) is exact for polynomials of degree up to $2k-2$, by a result in [2, § 4.1, p. 187], there exists a constant $\eta > 0$, independent of h and such that

$$I^h(f) \geqq \eta \|\mathbf{u}-\mathbf{v}\|_{1,2}^2. \tag{6.7}$$

Hence (6.2) follows immediately from (6.5) and (6.7). □

With the conclusion of the theorem above, similar results to Theorem 5.1 can be derived. We do not repeat the exact statement of the conclusion.

Next, to show the uniqueness of the solution for problem (LQh2) under condition (3.14) when h is small, it remains to show that the value of η will not be much less than 1. To do that, let us define a constant:

$$\eta_h := \inf_{\mathbf{u}\in \mathbf{V}^h} \frac{I^h(\nabla \mathbf{u} \cdot \nabla \mathbf{u})}{\|\mathbf{u}\|_{1,2}^2}. \tag{6.8}$$

The following has been shown in [7].

LEMMA 6.1.

$$\liminf_{h\to 0} \eta_h \geqq 1. \tag{6.9}$$

As a consequence, we have Theorem 6.2.

THEOREM 6.2. *Assume* (3.14) *holds; then there exists a constant* $\delta > 0$ *such that for h small enough,*

$$\mathcal{N}_h \Psi_q(\mathscr{C}_{fh}) \leqq \eta_h \tag{6.10}$$

and problem (LQh2) *has a unique solution* $\hat{\mathbf{u}}^h$ *in* $\mathbf{V}^h$. *Moreover, if the solution pair* $(\mathbf{u}, p)$ *of* (L) *satisfies* $\mathbf{u} \in \mathbf{W}_0^{1,\infty}(\Omega) \cap \mathbf{H}^{m+1}(\Omega)$ *and* $p \in \mathbf{H}^m(\Omega)$ *for some integer* $m \leqq l$, *then there exist constants* $c_1 > 0$ *and* $c^* = c^*(\|\mathbf{u}\|_{1,\infty}) > 0$ *such that*

$$\begin{aligned} \|\mathbf{u}-\hat{\mathbf{u}}^h\|_{1,2} \leqq c^* h^m \{\|\mathbf{u}\|_{m+1,2} + \|p\|_{m,2}\} &+ c_1 \sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{|(\mathbf{f}, \mathbf{w}^h) - \mathscr{F}^h(\mathbf{w}^h)|}{\|\mathbf{w}^h\|_{1,2}} \\ &+ c_1 \sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{|b(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h) - b^h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h)|}{\|\mathbf{w}^h\|_{1,2}} \\ &+ c_1 \sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{|a(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h) - a^h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h)|}{\|\mathbf{w}^h\|_{1,2}}. \end{aligned} \tag{6.11}$$

Now, estimating the term $|a(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h) - a^h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h)|$ is the only issue remaining. This will only necessitate an application of the Bramble-Hilbert Lemma and the Sobolev imbedding result. We take the case where $q = 4$ as an example to illustrate the main idea, since by [10] this is also a physically interesting case.

LEMMA 6.2. *Let* $q = 4$ *and* $\mathbf{u} \in \mathbf{H}^{\rho+2}(\Omega)$, *where* $l-1 \geqq \rho \geqq 0$; *then there exists a constant* $c_0 > 0$ *such that for all* $\mathbf{w}^h \in \mathbf{V}^h$

$$\begin{aligned} &|a(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h) - a^h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}, \mathbf{w}^h)| \\ &\qquad \leqq c_0 h^{\rho+1} \{\|\mathbf{u}\|_{\rho+2,2} + (\|\mathbf{u}\|_{\rho+2,2})^3\} \|\mathbf{w}^h\|_{1,2}. \end{aligned} \tag{6.12}$$

Proof. Let $\mathbf{v} := \Pi_h \mathbf{u}$, and for any function f

$$E^h(f) := I(f) - I^h(f).$$

By our hypotheses,

$$I^h(\pi_\rho[\nabla \mathbf{v}] \cdot \nabla \mathbf{w}^h) = I(\pi_\rho[\nabla \mathbf{v}] \cdot \nabla \mathbf{w}^h). \tag{6.13}$$

Also,

$$I^h(\pi_\rho[|\nabla \mathbf{v}|^2 \nabla \mathbf{v}] \cdot \nabla \mathbf{w}^h) = I(\pi_\rho[|\nabla \mathbf{v}|^2 \nabla \mathbf{v}] \cdot \nabla \mathbf{w}^h) \tag{6.14}$$

where π_ρ denotes the projection operator onto the space of all the piecewise polynomials of degree up to ρ. So,

$$\begin{aligned} &|a(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h) - a^h(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h)| \\ &\quad = E^h(\{\nabla \mathbf{v} - \pi_\rho[\nabla \mathbf{v}]\} \cdot \nabla \mathbf{w}^h) + E^h(\{|\nabla \mathbf{v}|^2 \nabla \mathbf{v} - \pi_\rho[|\nabla \mathbf{v}|^2 \nabla \mathbf{v}]\} \cdot \nabla \mathbf{w}^h). \end{aligned} \tag{6.15}$$

From the standard mathematical analysis on the effect of numerical integration (see [2]), we can transform the above quantities into functional forms that can be analyzed by the Bramble-Hilbert Lemma. Then,

$$E^h(\{\nabla \mathbf{v} - \pi_\rho[\nabla \mathbf{v}]\} \cdot \nabla \mathbf{w}^h) \leqq c h^{\rho+1} \left(\sum_{K \in \Delta^h} \|\nabla \mathbf{v}\|^2_{\rho+1,2,K} \right)^{1/2} \|\mathbf{w}^h\|_{1,2}, \tag{6.16}$$

$$E^h(\{|\nabla \mathbf{v}|^2 \nabla \mathbf{v} - \pi_\rho[|\nabla \mathbf{v}|^2 \nabla \mathbf{v}]\} \cdot \nabla \mathbf{w}^h) \leqq c h^{\rho+1} \left(\sum_{K \in \Delta^h} \| |\nabla \mathbf{v}|^2 \nabla \mathbf{v}\|^2_{\rho+1,2,K} \right)^{1/2} \|\mathbf{w}^h\|_{1,2}. \tag{6.17}$$

Under Hypothesis 5.2 and assumptions on ρ and on $\mathbf{u}$, we can show that

$$\left(\sum_{K \in \Delta^h} \| |\nabla \mathbf{v}|^2 \nabla \mathbf{v}\|^2_{\rho+1,2,K} \right)^{1/2} \leqq c(\|\mathbf{u}\|_{\rho+2,2})^3. \tag{6.18}$$

Hence, the conclusion of this lemma follows. □

COROLLARY 6.1. *If $s \geqq 2k-2$ in Hypothesis 5.1, then for all $\mathbf{w}^h \in \mathbf{V}^h$*

$$|a(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h) - a^h(\Pi_h \mathbf{u}, \Pi_h \mathbf{u}, \mathbf{w}^h)| \leqq c_0 h^{\rho+1} (\|\mathbf{u}\|_{\rho+2,2})^3 \|\mathbf{w}^h\|_{1,2}. \tag{6.19}$$

Proof. Note that when $s \geqq 2k-2$, we get

$$I^h(\nabla \mathbf{v} \cdot \nabla \mathbf{w}^h) = I(\nabla \mathbf{v} \cdot \nabla \mathbf{w}^h). \tag{6.20}$$

So, (6.19) follows from (6.17) and (6.18). □

Combining the above results, we conclude the following.

THEOREM 6.3. *Under the hypotheses of Theorems 6.1 and 6.2 and Lemma 6.2 there exists a constant $\gamma_\beta > 0$ such that*

$$\begin{aligned} \|\mathbf{u} - \hat{\mathbf{u}}^h\|_{1,2} &\leqq c^* h^m \{\|\mathbf{u}\|_{m+1,2} + \|p\|_{m,2}\} \\ &\quad + c_1 h^{\rho+1} \{c_2 \gamma_\beta \|\mathbf{f}\|_{\rho+1,\alpha} + c_3(\|\mathbf{u}\|_{\rho+2,2})^2 + c_0(\|\mathbf{u}\|_{\rho+2,2})^3\}. \end{aligned} \tag{6.21}$$

Next, we give a consequence of this theorem for the case where the quadrature rule (5.2) evaluates the term $(\nabla \mathbf{u}^h, \nabla \mathbf{w}^h)$ exactly. The estimate shows that the same order of accuracy as in the exact integration case can be obtained even when a nonexact quadrature rule is applied to all the rest of the terms involved in the approximation scheme.

COROLLARY 6.2. *Let $m = l$ and $\rho = l-1$ (i.e., $s = k+l-2$); then we have*

$$\begin{aligned} \|\mathbf{u} - \hat{\mathbf{u}}^h\|_{1,2} &\leqq c^* h^l \{\|\mathbf{u}\|_{l+1,2} + \|p\|_{l,2} \\ &\quad + c_1[c_2 \gamma_\beta \|\mathbf{f}\|_{l,\alpha} + c_3(\|\mathbf{u}\|_{l+1,2})^2 + c_0(\|\mathbf{u}\|_{l+1,2})^3]\}. \end{aligned} \tag{6.22}$$

To conclude the discussion on numerical integration, we remark as follows.

Remark 6.1. The discussion can, of course, be carried out for the model that we have considered with some other choices of the exponential index q. Results similar to some of the theorems above may also be obtained.

7. Remarks on iterative methods and computational experiments. We have formulated the discrete approximation problems for our model in previous sections. In addition, many properties of the approximation procedures have been discussed. After the establishment of the existence of discrete approximate solutions, the question that remains to be answered is how to find those solutions. In fact, the discrete formulation may be converted into a system of nonlinear algebraic equations by explicitly choosing bases for $\mathbf{V}^h$ and P^h. Hence, in this section, our task would be to discuss some iterative methods for the nonlinear systems resulting from the discretization. For simplicity, we pay special attention to problem (LQh), which usually is of the most importance. We also drop the use of the superscript h to denote discrete functions.

There are many known methods for solving the nonlinear algebraic equations (or systems). For a discussion of these, we refer to [11] and [12]. Some of those methods, e.g., Newton's method and the "chord" method, have been applied to solving the finite-element approximation problems of the Navier–Stokes equation. To develop such an iterative method, typically, we start with an initial guess, then use an approximate linear equation (or system) as an iterative scheme to get a sequence of functions that is expected to converge to the solution of the original problem.

In our case, the first method we consider simply lags all the nonlinearities in the approximate system. Let $\mathbf{u}^{(0)}\in\mathbf{V}^h$ be given; then we define the sequence $\mathbf{u}^{(n)}\in\mathbf{V}^h$ for $n=1,2,3,\cdots$, to be the solution of the following linear discrete system:

(SLh) Find $\mathbf{u}^{(n)}\in\mathbf{V}^h$ such that for all $\mathbf{w}\in\mathbf{V}^h$

$$a(\mathbf{u}^{(n-1)},\mathbf{u}^{(n)},\mathbf{w})+b(\mathbf{u}^{(n-1)},\mathbf{u}^{(n)},\mathbf{w})=(\mathbf{f},\mathbf{w}). \tag{7.1}$$

This method is called a simple iteration method. Generally, it is at most linearly convergent. Fast convergence rates can be obtained by applying Newton's method or a modified Newton method. To do that, we have to formally compute the Gâteaux derivatives of the nonlinear functionals involved in the discrete formulation. In the sequel, we again take the case $q=4$ as an example.

For fixed $\mathbf{u}\in\mathbf{V}^h$, the following map defines a continuous linear functional on $\mathbf{V}^h$:

$$\mathbf{v}\to a(\mathbf{u},\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})-(\mathbf{f},\mathbf{v})\quad \forall\mathbf{u}\in\mathbf{V}^h. \tag{7.2}$$

Hence, (7.2) defines a continuous function $\mathcal{G}:\mathbf{V}^h\to\mathbf{V}^h$ such that for any $\mathbf{u}\in\mathbf{V}^h$

$$\langle\mathcal{G}[\mathbf{u}],\mathbf{v}\rangle := a(\mathbf{u},\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})-(\mathbf{f},\mathbf{v}). \tag{7.3}$$

Then, the solution $\mathbf{u}$ of the problem (LQh) will be a solution of

$$\mathcal{G}[\mathbf{u}]=0 \tag{7.4}$$

or

$$\langle\mathcal{G}[\mathbf{u}],\mathbf{v}\rangle=0\quad\forall\mathbf{v}\in\mathbf{V}^h. \tag{7.5}$$

Let $\mathcal{J}_{\mathbf{u}}\mathcal{G}[\mathbf{w}]$ denote the Gâteaux derivative of $\mathcal{G}$ evaluated at $\mathbf{u}$, in the direction $\mathbf{w}$. The Newton's method for solving (7.5) will be as follows.

For a given initial guess $\mathbf{u}^{(0)}\in\mathbf{V}^h$, find a sequence $\{\mathbf{u}^{(n)}\}\in\mathbf{V}^h$ such that for $n=1,2,\cdots,\mathbf{u}^{(n)}$ satisfies

$$\langle\mathcal{J}_{\mathbf{u}^{(n-1)}}\mathcal{G}[\mathbf{u}^{(n)}-\mathbf{u}^{(n-1)}],\mathbf{v}\rangle := -\langle\mathcal{G}[\mathbf{u}^{(n-1)}],\mathbf{v}\rangle\quad\forall\mathbf{v}\in\mathbf{V}^h. \tag{7.6}$$

Let us now calculate $\mathcal{J}_{\mathbf{u}}\mathcal{G}[\mathbf{w}]$. By definition,

$$\begin{aligned}\langle \mathcal{J}_{\mathbf{u}}\mathcal{G}[\mathbf{w}], \mathbf{v}\rangle := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{[a(\mathbf{u}+\varepsilon\mathbf{w}, \mathbf{u}+\varepsilon\mathbf{w}, \mathbf{v})+b(\mathbf{u}+\varepsilon\mathbf{w}, \mathbf{u}+\varepsilon\mathbf{w}, \mathbf{v})-(\mathbf{f}, \mathbf{v})]\\ -[a(\mathbf{u}, \mathbf{u}, \mathbf{v})+b(\mathbf{u}, \mathbf{u}, \mathbf{v})-(\mathbf{f}, \mathbf{v})]\}\end{aligned} \tag{7.7}$$

$$= a(\mathbf{u}, \mathbf{w}, \mathbf{v})+2\nu_1([\nabla\mathbf{u}\cdot\nabla\mathbf{w}]\nabla\mathbf{u}, \nabla\mathbf{v})+b(\mathbf{u}, \mathbf{w}, \mathbf{v})+b(\mathbf{w}, \mathbf{u}, \mathbf{v}). \tag{7.8}$$

Substituting this result in (7.6), we find that the Newton method for problem (LQh) has the following linearized system.

(NLh) Find $\mathbf{u}^{(n)}\in\mathbf{V}^h$ such that for all $\mathbf{v}\in\mathbf{V}^h$,

$$\begin{aligned}a(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{v})+b(\mathbf{u}^{(n)}, \mathbf{u}^{(n-1)}, \mathbf{v})+b(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{v})\\ +2\nu_1(\{\nabla\mathbf{u}^{(n-1)}\cdot\nabla\mathbf{u}^{(n)}\}\nabla\mathbf{u}^{(n-1)}, \nabla\mathbf{v})\\ =2\nu_1(|\nabla\mathbf{u}^{(n-1)}|^2\nabla\mathbf{u}^{(n-1)}, \nabla\mathbf{v})+b(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n-1)}, \mathbf{v})+(\mathbf{f}, \mathbf{v}).\end{aligned} \tag{7.9}$$

The smoothness of the function assures that, locally, Newton's method has a quadratic convergence rate. However, at this time, we do not have a good estimate on the convergence radius.

In practice, the discrete pressure term is solved together with the velocity term. The above iterative schemes can be adapted to do that very easily. Usually, we need not specify an initial value for the pressure. For example, the version of the simple iteration scheme for problem (LPh) has the following linearized system:

(SLPh) Find $(\mathbf{u}^{(n)}, p^{(n)})\in\mathbf{X}^h\times P^h$ such that for all $(\mathbf{w}, q)\in\mathbf{X}^h\times P^h$,

$$a(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{w})+b(\mathbf{u}^{(n-1)}, \mathbf{u}^{(n)}, \mathbf{w})-c(\mathbf{w}, p^{(n)})=(\mathbf{f}, \mathbf{w}), \tag{7.10}$$

$$c(\mathbf{u}^{(n)}, q)=0. \tag{7.11}$$

Next, let us remark on the numerical implementation procedures. Some small-scale numerical experiments have been performed, particularly in the case where $q=4$. Our principal concerns are to verify some of the predictions made in our analysis and to see how easily the approximation scheme can be implemented.

First, there may be some question on whether it is difficult to implement the approximation procedure discussed throughout this chapter, due to the appearance of the more complicated terms. The truth is that we will not need any major effort to complete a programmer's work. In fact, in our experiments we modified an existing code for the finite-element approximation procedure for the Navier-Stoke equation. The modification comes basically in the assembly of the iteration matrices, where the addition of a few terms are the only changes necessary.

In our computations, we intentionally take the value of the Reynolds number to be small so that the uniqueness of the solution of the corresponding Navier-Stokes equation can be clearly verified at that level. To suit the physical model, the second viscosity coefficient ν_1 is also chosen to be relatively small, compared with ν_0. We take the standard Taylor-Hood finite-element pair (see [5]) as our discrete approximation subspaces. Moreover, in the implementation of the iterative scheme, we either first use several steps of a simple iteration scheme and then switch to the Newton iteration, or apply Newton's method directly, while using a Gaussian elimination procedure to solve the linear systems resulting from the iterative methods. Continuation methods on both parameters ν_0 and ν_1 are also used mainly to provide good initial guesses for the iterative scheme.

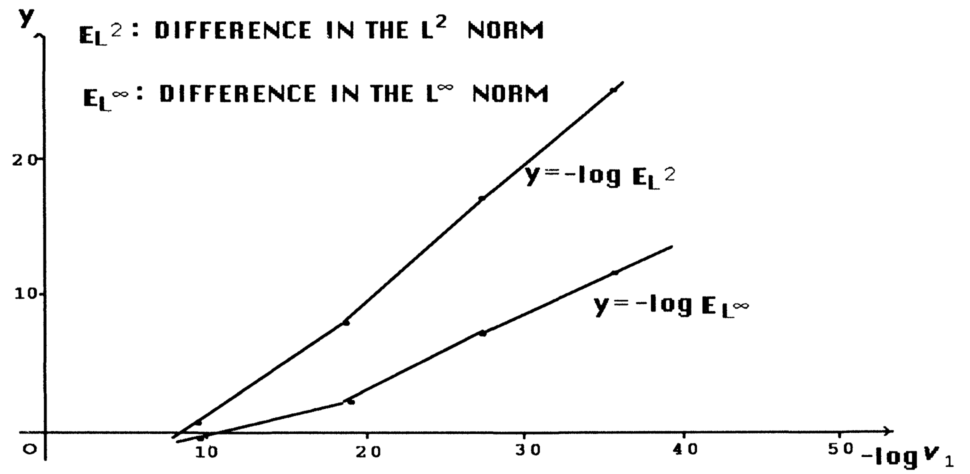

FIG. 7.1. *Differences between the approximate solution of the Ladyzhenskaya equations and the Navier-Stokes equations. Reynolds number* = 1.0, $q = 4$. *Number of triangles* = 50.

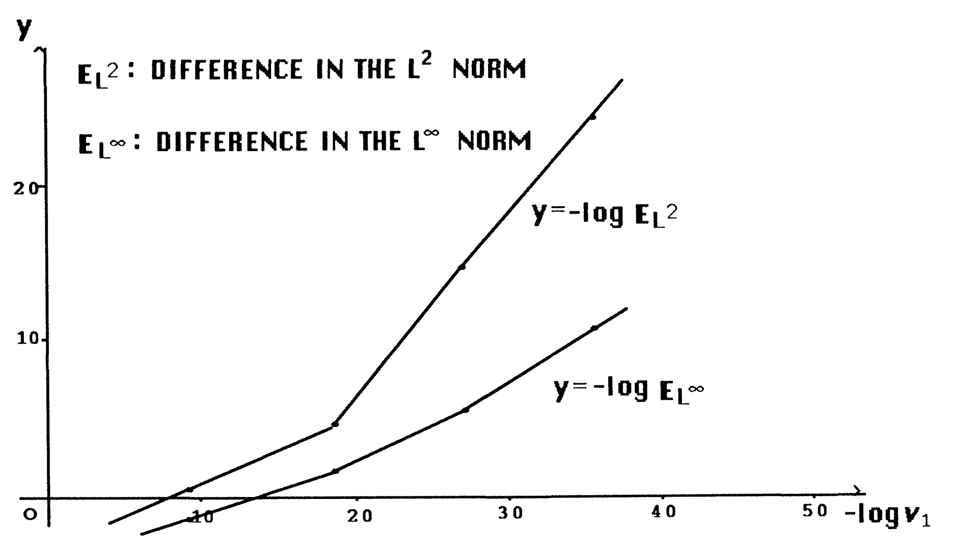

FIG. 7.2. *Differences between the approximate solution of the Ladyzhenskaya equations and the Navier-Stokes equations. Reynolds number* = 1.0, $q = 4$. *Number of triangles* = 98.

Our computations show that we do get a stable approximation of the unique solution for the Ladyzhenskaya model. It also shows that the continuation method on the parameter ν_1 works well, so that the quadratic convergence property of Newton's method is evident in the computational output. In addition, we show some figures based on numerical computations for the driven cavity problem in the two-dimensional box $[0, 1] \times [0, 1]$. The problem is solved using both the Navier-Stokes model and the Ladyzhenskaya model.

The numerical programs were performed for a series of different choices of coefficient ν_1 as well as different sizes of triangulations. Each time, we evaluate the difference between the solution of the Navier-Stokes equations and the solution of the Ladyzhenskaya equations. Then, we interpolate the results from a few cases to

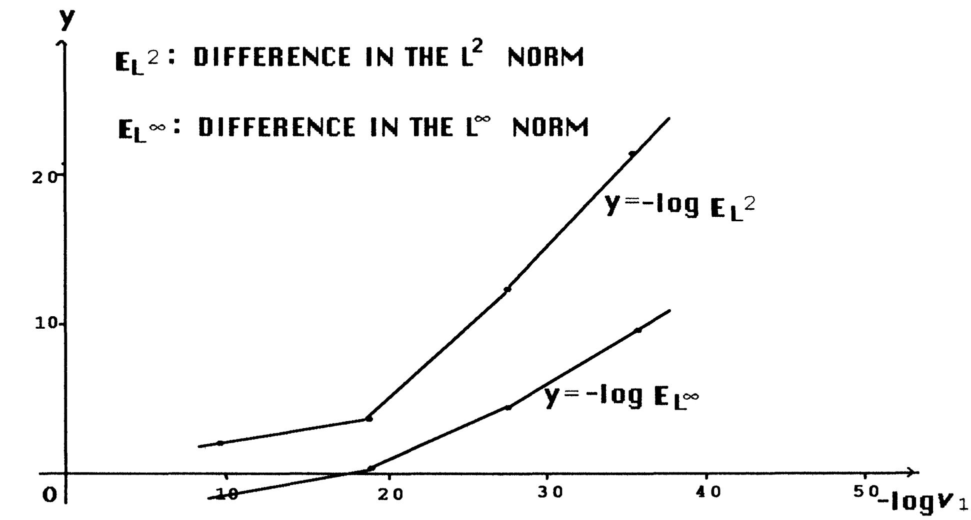

FIG. 7.3. *Differences between the approximate solution of the Ladyzhenskaya equations and the Navier-Stokes equations. Reynolds number* = 100, $q = 4$. *Number of triangles* = 50.

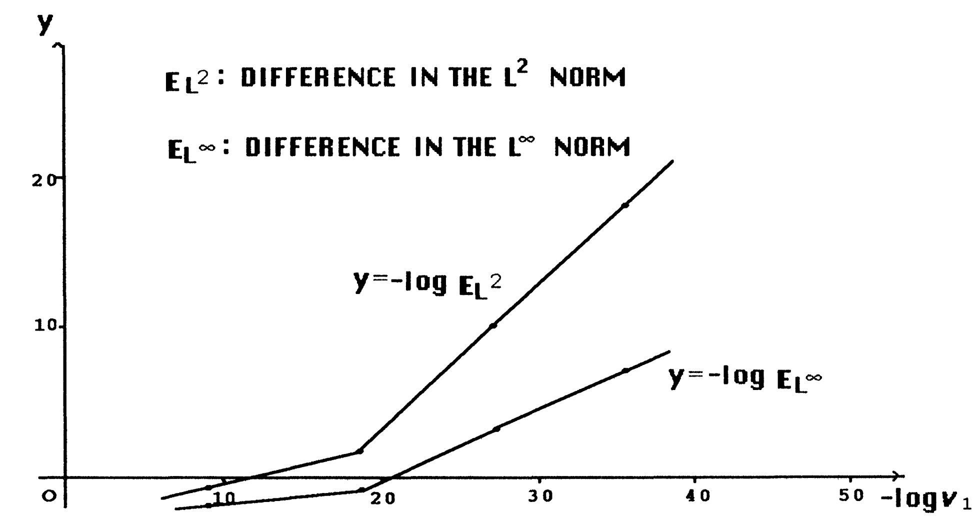

FIG. 7.4. *Differences between the approximate solution of the Ladyzhenskaya equations and the Navier-Stokes equations. Reynolds number* = 100, $q = 4$. *Number of triangles* = 98.

obtain the graphs in Figs. 7.1–7.4. The graphs are produced in the logarithmic coordinate system so that we can see more clearly the fact that the difference in the discrete L^2 norm between solutions does tend to zero as ν_1 diminishes. At the same time, discrete L^∞ differences are also evaluated.

Finally, we make some comments to end our discussion. There are many related meaningful topics on our model problem that we have not discussed or mentioned yet in this exposition. Examples include the approximation for the model problem in the time-dependent case, and the discussion of the case where the domain has a nonsmooth boundary or the forcing function has certain singularities. Also, some large-scale computation may provide deeper insight into physical phenomena, especially when the Reynolds number is large. Furthermore, there are many other more complicated models for viscous incompressible flows that usually have more nonlinearities. Our discussion may also lead to the study of their properties as well as the study of the relations between them. We hope these subjects may be taken as topics for our study in the future.

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