

A GRADIENT METHOD APPROACH TO OPTIMIZATION-BASED MULTIDISCIPLINARY SIMULATIONS AND NONOVERLAPPING DOMAIN DECOMPOSITION ALGORITHMS*

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Abstract. It has been shown recently that optimization-based nonoverlapping domain decomposition algorithms are connected to many well-known algorithms. Using a gradient-type iterative strategy for the optimization problem, we present further discussion on how to develop various algorithms that can integrate subdomain solvers into a solver for the problem in the whole domain. In particular, the algorithms we discuss can be used to develop efficient solvers of multidisciplinary problems which are constructed using existing subdomain solvers without the need for making changes in the latter.

Key words. multidisciplinary simulations and optimization, domain decomposition, nonoverlapping subdomains, optimization, convergence, finite element, parallel computation

AMS subject classifications. 65N55, 65N30, 65Y10, 35J20, 65K10

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1. Introduction. Domain decomposition is a realization of the *divide and conquer* strategy in which one attempts to treat a (computationally) intractable problem by replacing it by two or more simpler problems, each of which is presumably tractable. In particular, domain decomposition methodologies provide an effective means of introducing parallelism into problems which may not exhibit obvious, inherent parallelism.

The decomposition of problems based on a division of the computational domain into two or more subdomains is useful in two contexts. First, one may introduce (often artificial) subdomains and then define problems over the subdomains which, when solved repeatedly through an iterative procedure, yield practically the same solution as that of the original problem. Such algorithms have been studied extensively in the past decade; see, e.g., [1, 3, 6, 7, 8, 9, 15, 20, 21, 23, 32, 33]. On the other hand, a domain decomposition methodology can also be viewed as an integrator of subproblem solvers in a complicated system; see, e.g., [24, 38, 39]. Specifically, we have in mind problems where the subdivision into subdomains occurs naturally, resulting from changes in the mathematical models from one subdomain to another. In the simplest realization, we merely have different data in the model equations in the different subdomains. For example, we could have a wave propagation problem or a heat transfer problem wherein the media properties, e.g., the speed of sound or the diffusion coefficient, respectively, are piecewise continuous. In more complicated settings, we have completely different model equations in the different subdomains. For example, for aeroelastic problems, fluid dynamics equations in a flow

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domain are coupled with structural dynamic equations in an elastic body, e.g., an airplane, immersed in the fluid. It is perhaps for this last type of setting that the algorithms we discuss in this paper are most relevant and useful, especially when multidisciplinary simulations are merely an ingredient of a multidisciplinary optimization (MDO) problem. Specifically, one has available satisfactory subproblem solvers, e.g., a computational fluid dynamics (CFD) code for the fluid simulation and a linear elasticity code for the structural simulation. One then wants to use these existing codes in an efficient manner to solve a coupled fluid-structure interaction problem such as the response of an airfoil to wind gusts. By *efficient* we mean not only in terms of computational storage and CPU costs, but also in human programming effort and time.

Domain decomposition methods are characterized by two features which in fact serve to define specific methods. First, one must define the subdomain problems. The only aspect of their definition that is not obvious is the coupling conditions between solutions in the different subdomains. It is through these coupling conditions that one ensures that the solution of the subdomain problems are indeed solutions of the original problem. Here, we define the coupling conditions through an optimization-based strategy. (The idea of using optimization-based domain decompositions has been discussed in [15, 17, 22, 27, 28, 29]. The specific approach we follow here is related to that of [17, 27, 28, 29].) The second feature that characterizes domain decomposition methods is the iteration strategy used to update the data of the subdomain problems in terms of the solutions in other subdomains. Here, we examine a gradient method-based update strategy. Our discussion is based on a simple model problem; this is largely done for the sake of keeping the exposition simple. Most of what we say extends in an obvious manner, especially from an algorithmic viewpoint, to more complex and realistic problems.

The class of methods we study falls under the general framework of *generalized substructuring* or *nonoverlapping domain decomposition* methods. These methods have a long history in the mathematical and engineering literatures; see, e.g., [1, 30]. It is shown in [17] that the optimization-based approach employed here recovers many known substructuring methods and also defines some new methods. The methods discussed here may also be viewed as *interface relaxation methods*; indeed, some of the methods are related to those discussed in [38, 39].

The rest of the paper is organized as follows. We first present the model problem, then we present the more detailed theory using Neumann data as the control variable, followed by its finite element discretization. We then consider the Dirichlet control case. We also discuss how to modify the theory when an interior subdomain is present. Finally, we discuss the generalization to nonlinear problems such as the Navier–Stokes equations.

1.1. The model problem. To better clarify our presentation, let us consider a simple model problem, namely, the Poisson equation with a piecewise constant coefficient along with homogeneous Dirichlet boundary condition

$$(1.1) \quad -\operatorname{div}(a\nabla u) = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma,$$

where Ω is a bounded, simply connected domain in \mathbf{R}^n with Lipschitz boundary Γ . In the simplest case, Ω is partitioned into two simply connected nonoverlapping *subdomains* Ω_1 and Ω_2 , so that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and we assume that $a = a_i > 0$ on Ω_i for $i = 1, 2$. This is in fact an interface problem and the *interface* between the two subdomains is denoted by Γ_0 so that $\Gamma_0 = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Let $\Gamma_1 = \bar{\Omega}_1 \cap \Gamma$ and $\Gamma_2 = \bar{\Omega}_2 \cap \Gamma$. (See

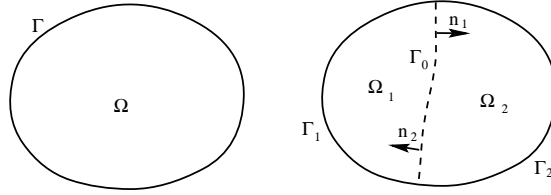


FIG. 1.1. The domain Ω divided into two nonoverlapping subdomains.

Figure 1.1 for a two-dimensional example.) Regularity conditions on the interface Γ_0 will be assumed later.

Let a pair of functions u_1, u_2 satisfy the given equations in the subdomains:

$$(1.2) \quad -a_i \Delta u_i = f \quad \text{in } \Omega_i \quad \text{and} \quad u_i = 0 \quad \text{on } \Gamma_i \quad \text{for } i = 1, 2.$$

In the standard setting, to piece u_1 and u_2 together to form the solution of (1.1), we need the *interface conditions*

$$(1.3) \quad u_1 = u_2 = \lambda \quad \text{on } \Gamma_0,$$

$$(1.4) \quad a_1 \frac{\partial u_1}{\partial n_1} = -a_2 \frac{\partial u_2}{\partial n_2} = g \quad \text{on } \Gamma_0.$$

Here, n_1 and n_2 are the unit outward normals of Ω_i on Γ_0 so that $n_1 = -n_2$.

For (λ, g) defined over suitable function spaces and for the solutions (u_1, u_2) of (1.2), an energy functional \mathcal{J} can now be defined such that its minimizer (u_1, u_2, λ, g) corresponds to those subdomain solutions that satisfy (1.3)–(1.4). Some equivalence property must be satisfied by \mathcal{J} . In light of the interface conditions (1.3)–(1.4), it is natural to have the following property.

EQUIVALENCE PROPERTY 1.1. *There exists a unique solution u to the problem (1.1), and*

$$\left\{ u|_{\Omega_1}, u|_{\Omega_2}, u|_{\Gamma_0}, a_1 \frac{\partial u}{\partial n_1} \Big|_{\Gamma_0} = -a_2 \frac{\partial u}{\partial n_2} \Big|_{\Gamma_0} \right\}$$

is the unique minimizer, over suitable function spaces of the functional $\mathcal{J}(u_1, u_2, \lambda, g)$ subject to (1.2).

The above equivalence property implies that solving the minimization problem *minimize $\mathcal{J}(u_1, u_2, \lambda, g)$ over u_1, u_2, λ , and g in suitable spaces, subject to (1.2), is formally equivalent to solving the original problem (1.1).*

There are more general forms of \mathcal{J} for various types of interface conditions; there are even many choices of the functional \mathcal{J} for (1.3)–(1.4). For example, in the case $a_1 = a_2$, given a constant $\delta > 0$, the functional

$$\mathcal{J}^\delta(u_1, u_2, \lambda, g) = \frac{1}{2} \sum_{i=1}^2 \left\{ \delta \|u_i - \lambda\|_{X_i}^2 + \left\| a_i \frac{\partial u_i}{\partial n_i} + (-1)^i g \right\|_{Y_i}^2 \right\}$$

with appropriate Hilbert spaces X_i and Y_i equipped with norms $\|\cdot\|_{X_i}$ and $\|\cdot\|_{Y_i}$ ($i = 1, 2$), respectively, was used in [17]. For the above functional, taking $\delta = 1$, $X_2 = L^2(\Gamma_0)$, and enforcing additional constraints that $\lambda = u_1$, $a_1 \partial u_1 / \partial n_1 = g$, and $a_2 \partial u_2 / \partial n_2 = -g$, we get

$$(1.5) \quad \mathcal{J}(u_1(g), u_2(g), g) = \|u_2 - u_1\|_{0, \Gamma_0}^2,$$

where $\|\cdot\|_X = \|\cdot\|_{0,\Gamma_0}$ is the standard L^2 norm on Γ_0 . This is the functional used in [28] for the case $a_1 = a_2$. There, an additional penalty term was incorporated to assure the bounds on the data; we show later that this is not necessary.

The above-defined functionals satisfy the equivalence principle given earlier, and we call the minimization problem the least squares approach [28]. There are many possible choices for the functional, depending on the choices of norms (or constraint spaces) for the variables. Let X be a function space for λ and Y be a function space for g . To illustrate the choices, we use $H^m(\Omega)$, $H_0^m(\Omega)$, and $H^s(\Gamma)$ to denote the standard Sobolev spaces and the trace spaces and also define

$$H_{0,i}^1(\Omega_i) = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \Gamma_i\}, \quad H_{00}^{1/2}(\Gamma_0) = \{v|_{\Gamma_0} : v \in H_0^1(\Omega)\}.$$

The connection of the optimization-based approach with some well-known methods studied in the literature has been made in [17]. For a simple alternating domain approach for solving the minimization problem, it was shown in [17] that the choice of $X = H_{00}^{1/2}(\Gamma_0)$, $Y = H^{-1/2}(\Gamma_0)$ leads to the well-known Dirichlet–Neumann alternating methods [9, 16, 35, 36, 37, 41], while the choice of $X = L^2(\Gamma_0)$, $Y = H^{-1/2}(\Gamma_0)$ or $X = H_{00}^{1/2}(\Gamma_0)$, $Y = L^2(\Gamma_0)$ leads to a well-known method analyzed in [34] that uses a Robin-type boundary condition for subdomain problems; see also [1, 14]. The choice $X = Y = L^2(\Gamma_0)$ used in [27, 28, 29] leads to a new algorithm.

The alternating domain approach may not be the most efficient minimization strategy in practice. In [27, 28, 29], gradient-type methods were considered, but only with the choice comparable to $X = Y = L^2(\Gamma_0)$ for the functional (1.5). We shall discuss various generalizations of the studies given in [27, 28, 29] and also discuss the case where no penalty term is added to the functional. The generalizations allow us to derive other methods, some of which have been examined by various authors [2, 15, 22, 31, 39], and to obtain some new features of those methods.

1.2. Features of the optimization-based framework. The optimization-based framework considered here offers the some advantageous features for integrating solvers for the subdomain problems in the multidisciplinary setting. First, we have the choice of solving specific types of boundary value problems in the subdomains. For instance, if for some technical reasons, the subdomain solvers can only deal with Neumann-type (*flux-like*) boundary conditions on the interface, then we have the appropriate choice of *control* variables and the spaces to be used. Different choices might be more suitable if Dirichlet-type boundary conditions on the interface are preferred by the subdomain solvers. Second, the integration of subdomain solvers can be completed with mismatched (or nonmatching) grids in the subdomains and different discretizations, and error estimates (usually of optimal order) can still be derived. Third, interior subdomains are allowed with slight modifications to take into account solvability conditions. Fourth, unlike most multidisciplinary simulations algorithms in use today, the algorithms discussed here can be implemented so that the individual disciplinary problems are solved in parallel. Of course, the generalizations given here have the promise of wide applications in cases with multisubdomains and for nonlinear equations (see [28, 29] and later discussions in section 6 for the case of the Navier–Stokes equations). For the sake of clarity, in this paper, we focus our discussion mostly on the model equations and the two nonoverlapping subdomain case. The ultimate goal is, naturally, to be able to treat both *small* and *large* numbers of subdomains, and this will be pursued in future research.

1.3. Notation. Before proceeding to more detailed discussion, we define some technical terms. With proper assumptions on the boundary, it is known by interpolation theory that $H_{00}^{1/2}(\Gamma_0) = [H_0^1(\Gamma_0), L^2(\Gamma_0)]_{1/2}$. For any $\theta \in H_{00}^{1/2}(\Gamma_0)$, the Steklov–Poincaré operators \mathcal{S}_i from $H_{00}^{1/2}(\Gamma_0)$ to $H^{-1/2}(\Gamma_0)$ are defined by

$$\mathcal{S}_i(\theta) = a_i \frac{\partial w_i}{\partial n_i} \quad \text{on } \Gamma_0,$$

where $\{w_i = \mathcal{R}_i(\theta), i = 1, 2\}$ are the harmonic extensions defined by

$$-\Delta w_i = 0 \quad \text{in } \Omega_i, \quad w_i = 0 \quad \text{on } \Gamma_i, \quad w_i = \theta \quad \text{on } \Gamma_0.$$

We let $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. We assume proper regularity of the boundary Γ and the interface Γ_0 [25], so that both $\mathcal{S}_1, \mathcal{S}_2$ (and consequently \mathcal{S}) and their inverses are bounded. The Steklov–Poincaré operators are self-adjoint positive definite operators, and they have been used frequently in the studies of the domain decomposition methods. Its discrete matrix analog is the Schur complement [9].

2. Domain decomposition algorithms via Neumann control. We now study gradient-type methods for iteratively solving the minimization problem. Let H be a Hilbert space equipped with an inner product $(\cdot, \cdot)_H$ and \mathcal{J} be a continuous functional on H which is Fréchet differentiable. Let \mathcal{J}' be the Fréchet derivative, i.e., $(\mathcal{J}'(v), w)_H = (d\mathcal{J}(v + \epsilon w)/d\epsilon)|_{\epsilon=0}$. The gradient-type iteration is given by choosing some appropriate sequence $\alpha_n > 0$ and setting

$$v_{n+1} = v_n - \alpha_n \mathcal{J}'(v_n).$$

To apply a gradient method, we must identify the independent variable which we wish to use since, by compatibility, for any $i = 1, 2$, the interface data λ_i and g_i cannot be chosen independently. We follow [28] by first considering g as the independent *control variable*. More precisely, letting X, Y be two Hilbert spaces, we have

$$\begin{aligned} \text{Min} \quad \mathcal{J}(g) &= \|u_1 - u_2\|_X^2 \\ \text{over } g \in Y \text{ and } u_1, u_2 \text{ that are subject to (1.2) and (1.4).} \end{aligned}$$

Note that, in the abstract setting, the minimizer g satisfies a linear Euler–Lagrange equation and the gradient-type iteration corresponds to the weighted Richardson method for the linear equation of g .

We consider here the choices $X = L^2(\Gamma_0)$ and $X = H_{00}^{1/2}(\Gamma_0)$, along with $Y = L^2(\Gamma_0)$ and $Y = H^{-1/2}(\Gamma_0)$. Notice that for the discrete case, we may use discrete mesh-dependent norms.

For convenience, we define operators \mathcal{P} and \mathcal{Q} by

$$(2.1) \quad (v, w)_X = (v, \mathcal{P}w)_{0, \Gamma_0} \quad \forall v, w \in X,$$

$$(2.2) \quad (v, w)_Y = (v, \mathcal{Q}w)_{0, \Gamma_0} \quad \forall v, w \in Y,$$

where $(\cdot, \cdot)_{0, \Gamma_0}$ denotes the standard inner product in $L^2(\Gamma_0)$. We can get the Fréchet derivative by

$$(\mathcal{J}'(g), \mathcal{Q}v)_{0, \Gamma_0} = (\mathcal{J}'(g), v)_Y = (u_1 - u_2, \mathcal{P}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})v)_{0, \Gamma_0} \quad \forall g, v \in Y.$$

Thus, viewing u_1 and u_2 as functions of g , we have on Γ_0 ,

$$\mathcal{J}'(g) = \mathcal{Q}^{-1}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})\mathcal{P}(u_1 - u_2).$$

Then, the gradient-type iteration is given by

$$(2.3) \quad g_{n+1} = g_n - \alpha_n \mathcal{J}'(g_n).$$

One easily obtains the following result.

THEOREM 2.1. *Given $\alpha_n > \beta > 0$, if the iteration (2.3) is convergent and*

$$u|_{\Omega_i} = \lim_{n \rightarrow \infty} u_{i,n} \quad \text{for } i = 1, 2,$$

then u is the solution of (1.1) on the whole domain Ω .

Let $e_n = g_n - \frac{\partial u}{\partial n_1}$; then

$$e_{n+1} = [I - \alpha_n \mathcal{Q}^{-1}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})\mathcal{P}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})] e_n := \mathcal{N}(\alpha_n)e_n .$$

As a result, we obtain the following convergence criterion.

THEOREM 2.2. *The algorithm (2.3) is convergent if and only if, for any e_0 ,*

$$(2.4) \quad \left\| \prod_{k=1}^n \mathcal{N}(\alpha_k)e_0 \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in some suitable norm and with some suitably chosen sequence $\{\alpha_n\}$.

2.1. Example 1. We first consider the case discussed in [28], $X = Y = L^2(\Gamma_0)$.

An additional penalty term is used there. In the limit when the penalty term goes to zero, the algorithm in [28] is a special case of (2.3) with the above choice of spaces, which corresponds to $\mathcal{P} = \mathcal{Q} = I$, the identity map. Thus the iteration is given by

$$g_{n+1} = g_n - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})(u_1^n - u_2^n) .$$

Implementation can be given by first solving for $u_i^n \in H_{i,0}^1(\Omega_i)$ ($i = 1, 2$) from

$$(2.5) \quad \begin{cases} -a_i \Delta u_i^n = f & \text{in } \Omega_i, \\ a_i \frac{\partial u_i^n}{\partial n_i} = (-1)^{i-1} g_n & \text{on } \Gamma_0. \end{cases}$$

Then, we solve for $\lambda_i^n \in H_{i,0}^1(\Omega_i)$ ($i = 1, 2$) from

$$\begin{cases} \Delta \lambda_i^n = 0 & \text{in } \Omega_i, \\ a_i \frac{\partial \lambda_i^n}{\partial n_i} = (-1)^i (u_1^n - u_2^n) & \text{on } \Gamma_i. \end{cases}$$

We then can update the iteration according to

$$(2.6) \quad g_{n+1} = g_n + \alpha_n (\lambda_1^n - \lambda_2^n) .$$

For convergence, we have the error equation $e_{n+1} = \mathcal{N}(\alpha_n)e_n$, where

$$\mathcal{N}(\alpha_n) = I - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2 .$$

To show convergence, let us rewrite the error equation as

$$[2I - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2](e_{n+1} - e_n) + (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2(e_{n+1} + e_n) = 0 .$$

Taking the inner product with $e_{n+1} - e_n$, we get

$$2\|(e_{n+1} - e_n)\|_{0,\Gamma_0}^2 - \alpha_n \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})(e_{n+1} - e_n)\|_{0,\Gamma_0}^2 \\ + \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})e_{n+1}\|_{0,\Gamma_0}^2 = \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})e_n\|_{0,\Gamma_0}^2.$$

Since for any $g \in L^2(\Gamma_0)$,

$$\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g\|_{0,\Gamma_0} \leq c\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g\|_{H^{1/2}} \leq c\|g\|_{H^{-1/2}} \leq c\|g\|_{0,\Gamma_0},$$

if we take

$$\theta > \sup_{g \in L^2(\Gamma_0)} \frac{\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g\|_{0,\Gamma_0}}{\|g\|_{0,\Gamma_0}},$$

and if for all n , we choose

$$(2.7) \quad 0 < \beta \leq \alpha_n \leq \frac{2}{\theta^2},$$

then we get as $n \rightarrow \infty$

$$e_{n+1} - e_n \rightarrow 0 \quad \text{in } L^2(\Gamma_0).$$

Consequently, the error equation gives

$$(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2 e_n \rightarrow 0 \quad \text{in } L^2(\Gamma_0).$$

On the other hand,

$$\|e_{n+1}\|_{0,\Gamma_0}^2 = \|e_n\|_{0,\Gamma_0}^2 - 2\alpha_n \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})e_n\|_{0,\Gamma_0}^2 + \alpha_n^2 \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2 e_n\|_{0,\Gamma_0}^2 \\ = \|e_n\|_{0,\Gamma_0}^2 - 2\alpha_n \|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})e_n\|_{0,\Gamma_0}^2 \left[1 - \frac{\alpha_n}{2} \left(\frac{\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^2 e_n\|_{0,\Gamma_0}}{\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})e_n\|_{0,\Gamma_0}} \right)^2 \right].$$

Under the condition on α_n , we see that $\|e_{n+1}\|_{0,\Gamma_0} \leq \|e_n\|_{0,\Gamma_0}$, i.e., e^n is uniformly bounded in $L^2(\Gamma_0)$. Therefore, from the above convergence, we get

$$e_n \rightarrow 0 \quad \text{weakly in } L^2(\Gamma_0) \quad \text{and thus } e_n \rightarrow 0 \quad \text{in } H^{-1/2}(\Gamma_0).$$

THEOREM 2.3. *The gradient-type algorithm (2.6) is convergent for any e^0 and for any sequence $\{\alpha_n\}$ satisfying (2.7).*

2.2. Example 2. We now consider $X = H_{00}^{1/2}(\Gamma_0)$ and $Y = H^{-1/2}(\Gamma_0)$. There are some choices for the norms. We consider, for example, that $\mathcal{P} = \mathcal{S}_2$ and $\mathcal{Q} = \mathcal{S}_1^{-1}$; i.e., the iteration is given by

$$g_{n+1} = g_n - \alpha_n \mathcal{S}_1 (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \mathcal{S}_2 (u_1^n - u_2^n) \quad \text{on } \Gamma_0.$$

Equivalently, we have

$$g_{n+1} = g_n - \alpha_n (\mathcal{S}_2 + \mathcal{S}_1) (u_1^n - u_2^n) \quad \text{on } \Gamma_0.$$

So, the implementation is to solve first (2.5), then solve

$$\begin{cases} \Delta \zeta_i = 0 & \text{in } \Omega_i, \\ \zeta_i = 0 & \text{on } \Gamma_i, \\ \zeta_i = u_1^n - u_2^n & \text{on } \Gamma_i, \end{cases}$$

and update by

$$(2.8) \quad g_{n+1} = g_n - \alpha_n \left(a_1 \frac{\partial \zeta_1}{\partial n_1} + a_2 \frac{\partial \zeta_2}{\partial n_2} \right).$$

As for convergence, one may derive a well-known geometric convergence result based on the error equation $e_{n+1} = \mathcal{N}(\alpha_n)e_n$, where

$$\mathcal{N}(\alpha_n) = I - \alpha_n(\mathcal{S}_1 + \mathcal{S}_2)(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}).$$

THEOREM 2.4. *There exist two constants α_* and α^* such that for any e_0 and for any sequence $\{\alpha_n\}$ satisfying*

$$(2.9) \quad 0 < \alpha_* \leq \alpha_n \leq \alpha^*,$$

the gradient-type algorithm (2.8) is geometrically convergent.

We omit the details here.

2.3. Example 3. We now consider $X = H_{00}^{1/2}(\Gamma_0)$ and $Y = L^2(\Gamma_0)$. There are some choices for the norms. We consider, for example, that $\mathcal{P} = (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})^{-1}$ and $\mathcal{Q} = I$; that is, the iteration is given by

$$(2.10) \quad g_{n+1} = g_n - \alpha_n(u_1^n - u_2^n) \quad \text{on } \Gamma_0,$$

where u_1^n and u_2^n satisfy (2.5).

This iteration has been studied in [15] with the use of the *Lagrange multipliers* formulation for saddle point-type problems and *Uzawa's* algorithm. For convergence, we have the error equation $e_{n+1} = \mathcal{N}(\alpha_n)e_n$, where

$$\mathcal{N}(\alpha_n) = I - \alpha_n(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}).$$

Using properties of the Steklov–Poincaré operators, we know that if

$$\tilde{\theta} > \sup_{g \in L^2(\Gamma_0)} \frac{(g, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0,\Gamma_0}}{\|g\|_{0,\Gamma_0}^2},$$

and, for all n , we choose

$$(2.11) \quad 0 < \beta \leq \alpha_n \leq \frac{2}{\tilde{\theta}},$$

then we have the following result.

THEOREM 2.5. *The gradient-type algorithm (2.10) is convergent for any e^0 and for any sequence $\{\alpha_n\}$ satisfying (2.11).*

3. Discrete implementation via Neumann control. Let us first define some bilinear forms; for $i = 1, 2$, $\forall u, v \in H_i^1(\Omega_i)$,

$$a_i(u, v) = a_i \int_{\Omega_i} \nabla u \cdot \nabla v \, d\Omega_i.$$

We also use the following notation for the inner products:

$$(u, v)_i = \int_{\Omega_i} uv \, d\Omega_i \quad \text{and} \quad (u, v)_{0,\Gamma_0} = \int_{\Gamma_0} uv \, d\Gamma.$$

Let V_1^h, V_2^h denote finite element spaces such that $V_i^h \subset H_{0,i}^1(\Omega_i)$ for $i = 1, 2$. We assume that the usual approximation properties hold, i.e., for $i = 1, 2$,

$$\inf_{v^h \in V_i^h} \|v - v^h\|_1 \rightarrow 0, \quad \text{as } h_i \rightarrow 0$$

for any $v \in H_{0,i}^1(\Omega_i)$, and there exist an integer k and a constant C such that

$$\inf_{v^h \in V_i^h} \|v - v^h\|_1 \leq Ch_i^m \|v\|_{m+1}$$

$\forall v \in H_{0,i}^1(\Omega_i) \cap H^{m+1}(\Omega_i), i = 1, 2$ and $0 \leq m \leq k$; see [11] and [19]. Here h_1, h_2 are the mesh parameters for the triangulation of the two subdomains, respectively.

For convenience, let us define some discrete Steklov–Poincaré operators. Let π_i^h be the projection operator from the trace space $H^{-1/2}(\Gamma_0)$ onto $V_i^h|_{\Gamma_0}$, that is,

$$(\pi_i^h(g), v^h)_{0,\Gamma_0} = (g, v^h)_{0,\Gamma_0} \quad \forall v^h \in V_i^h|_{\Gamma_0}.$$

Given the finite-dimensional space B^h on Γ_0 , for any $g^h \in B^h$, let us define the discrete harmonic extensions in V_i^h by

$$a_i(\mathcal{R}_i^h g^h, v_i^h) = 0 \quad \forall v_i^h \in V_i^h \cap H_0^1(\Omega_i),$$

$$(\mathcal{R}_i^h g^h)|_{\Gamma_0} = \pi_i^h g^h.$$

Then, we define $\mathcal{S}_i^h(g^h) \in V_i^h|_{\Gamma_0}$ by

$$(\mathcal{S}_i^h(g^h), v_i^h)_{0,\Gamma_0} = a_i(\mathcal{R}_i^h(g^h), v_i^h) \quad \forall v_i^h \in V_i^h.$$

If $B^h = V_i^h|_{\Gamma_0}$, then we see that \mathcal{S}_i^h is an operator defined from $V_i^h|_{\Gamma_0}$ to itself. Moreover, it is positive definite and its inverse is denoted by $\mathcal{S}_{i,h}^{-1}$. We also have the norm equivalence for small h , and $u^h \in V_i^h|_{\Gamma_0}$,

$$(\mathcal{S}_i^h(u^h), u^h)_{0,\Gamma_0}^{1/2} \sim \|\mathcal{R}_i^h u^h\|_{1,\Omega_i} \sim \|u^h\|_{1/2,\Gamma_0}.$$

It is also helpful to define some projection operators. Let $P^h \subset H^{1/2}(\Gamma_0)$ be a finite element space defined on the interface Γ_0 with mesh parameter h_0 with approximation properties to be given later. For any $v \in L^2(\Gamma_0)$, let us denote its L^2 projection in P^h by $\pi^h v$, that is,

$$(\pi^h v, w^h)_{0,\Gamma_0} = (v, w^h)_{0,\Gamma_0} \quad \forall w^h \in P^h.$$

In the case $P^h = V_i^h|_{\Gamma_0}$, we denote π^h by π_i^h for $i = 1, 2$. A few properties of the projection operators and the discrete Poincaré operators are included here for completeness. The proof requires that the finite element spaces satisfy the following type of inverse inequality [11]:

$$(3.1) \quad \|v^h\|_{s,\Gamma_0} \leq C_i h_i^{t-s} \|v^h\|_{t,\Gamma_0} \quad \forall v^h \in V_i^h|_{\Gamma_0}, \quad 0 \leq t \leq s \leq 1/2.$$

LEMMA 3.1. *For small $h_i, i = 1, 2$, there exist generic positive constants c_0, c_1 , and c_2 , independent of the mesh parameters h_1 and h_2 , such that*

$$(3.2) \quad \|\pi_i^h u\|_{0,\Gamma_0} \leq \|u\|_{0,\Gamma_0} \quad \forall u \in L^2(\Gamma_0), \quad i = 1, 2,$$

$$(3.3) \quad \|\pi_i^h u\|_{1/2,\Gamma_0} \leq c_0 \|u\|_{1/2,\Gamma_0} \quad \forall u \in H_{00}^{1/2}(\Gamma_0), \quad i = 1, 2,$$

$$\|\pi_i^h u\|_{-1/2,\Gamma_0} \leq c_0 \|u\|_{-1/2,\Gamma_0} \quad \forall u \in L^2(\Gamma_0), \quad i = 1, 2.$$

Proof. The inequality (3.2) is obvious. Let $\mathcal{R}_i u$ be an extension of u in $H^1_{i,0}(\Omega_i)$, such that $\|\mathcal{R}_i u\|_{1,\Omega_i} \leq c\|u\|_{1/2,\Gamma_0}$ for some constant $c > 0$. Define \hat{u}_i^h by

$$a_i(\hat{u}_i^h, v_i^h) = a_i(\mathcal{R}_i u, v_i^h) \quad \forall v_i^h \in V_i^h.$$

Using the H^2 regularity assumption for the Poisson equations in Ω_i with homogeneous Dirichlet condition on Γ_i and Neumann condition on Γ_0 [25], the standard finite element estimates, and the Aubin–Nitsche lemma [11], we have, when h_i is small,

$$\|\hat{u}_i^h - \mathcal{R}_i u\|_{s,\Omega_i} \leq ch_i^s \|\mathcal{R}_i u\|_{1,\Omega_i}, \quad s = 0, 1$$

for some generic constant $c > 0$, independent of h_i . By the trace theorem and an interpolation inequality between boundary and interior norms [25], we get

$$\|u - \hat{u}_i^h\|_{0,\Gamma_0} \leq c \left\{ h_i^{1/2} \|u - \hat{u}_i^h\|_{1,\Omega_i} + h_i^{-1/2} \|u - \hat{u}_i^h\|_{0,\Omega_i} \right\} \leq ch_i^{1/2} \|\mathcal{R}_i u\|_{1,\Omega_i}.$$

By the inverse inequality (3.1) and the best approximation property of $\pi_i^h u$ in $L^2(\Gamma_0)$,

$$\begin{aligned} \|\pi_i^h u\|_{1/2,\Gamma_0} &\leq \|\pi_i^h u - \hat{u}_i^h\|_{1/2,\Gamma_0} + \|u - \hat{u}_i^h\|_{1/2,\Gamma_0} \\ &\leq ch_i^{-1/2} \|\pi_i^h u - \hat{u}_i^h\|_{0,\Gamma_0} + \|u - \hat{u}_i^h\|_{1/2,\Gamma_0} \\ &\leq ch_i^{-1/2} \cdot \{2\|u - \hat{u}_i^h\|_{0,\Gamma_0}\} + \|u - \hat{u}_i^h\|_{1/2,\Gamma_0} \\ &\leq c\|\mathcal{R}_i u\|_{1,\Omega_i} \leq c_0\|u\|_{1/2,\Gamma_0} \end{aligned}$$

for some generic constant c_0 , independent of h_i . The last inequality in the lemma follows from the duality argument. \square

The above results are commonly referred to as “stability properties of the projection operators,” which were previously investigated in, e.g., [4, 5, 26].

LEMMA 3.2. *For small h_i , $i = 1, 2$, there exists a generic positive constant c , independent of the mesh parameters h_1, h_2 , such that for any $w_i^h \in V_i^h|_{\Gamma_0}$,*

$$\|\mathcal{S}_i^h w_i^h\|_{-1/2,\Gamma_0} \leq c\|w_i^h\|_{1/2,\Gamma_0}, \quad \|\mathcal{S}_{i,h}^{-1} w_i^h\|_{1/2,\Gamma_0} \leq c\|w_i^h\|_{-1/2,\Gamma_0}.$$

Proof. First, for any $v \in H_{00}^{1/2}(\Gamma_0)$, we have

$$\begin{aligned} (\mathcal{S}_i^h w_i^h, v)_{0,\Gamma_0} &= (\mathcal{S}_i^h w_i^h, \pi_i^h v)_{0,\Gamma_0} = a_i(\mathcal{R}_i^h w_i^h, \mathcal{R}_i^h \pi_i^h v)_{0,\Gamma_0} \\ &\leq c\|\mathcal{R}_i^h w_i^h\|_{1,\Omega_i} \|\mathcal{R}_i^h \pi_i^h v\|_{1,\Omega_i} \leq c\|w_i^h\|_{1/2,\Gamma_0} \|\pi_i^h v\|_{1/2,\Gamma_0} \end{aligned}$$

for some generic constant $c > 0$. Thus,

$$\begin{aligned} \|\mathcal{S}_i^h w_i^h\|_{-1/2,\Gamma_0} &= \sup_{v \in H_{00}^{1/2}(\Gamma_0)} \frac{(\mathcal{S}_i^h w_i^h, v)_{0,\Gamma_0}}{\|v\|_{1/2,\Gamma_0}} \\ &\leq c\|w_i^h\|_{1/2,\Gamma_0} \sup_{v \in H_{00}^{1/2}(\Gamma_0)} \frac{\|\pi_i^h v\|_{1/2,\Gamma_0}}{\|v\|_{1/2,\Gamma_0}} \leq c\|w_i^h\|_{1/2,\Gamma_0}. \end{aligned}$$

On the other hand, for $w_i^h \in V_i^h|_{\Gamma_0}$, let $u_i^h \in V_i^h$ be defined by

$$a_i(u_i^h, v_i^h) = (w_i^h, v_i^h)_{0,\Gamma_0} \quad \forall v_i^h \in V_i^h.$$

Let us also define $u_i \in H_{i,0}^1(\Omega_i)$ as the solution of

$$a_i(u_i, v_i) = (w_i^h, v_i)_{0,\Gamma_0} \quad \forall v_i \in H_{i,0}^1(\Omega_i).$$

Then, we have $\|u_i\|_{1,\Omega_i} \leq c\|w_i^h\|_{-1/2,\Gamma_0}$ for some constant c . By the standard finite element estimates, we also have

$$\|u_i^h - u_i\|_{1/2,\Gamma_0} \leq \|u_i^h - u_i\|_{1,\Omega_i} \leq c\|u_i\|_{1,\Omega_i} \leq c\|w_i^h\|_{-1/2,\Gamma_0}.$$

Thus, by the triangle inequality, we get

$$\|\mathcal{S}_{i,h}^{-1}w_i^h\|_{1/2,\Gamma_0} = \|u_i^h\|_{1/2,\Gamma_0} \leq c\|w_i^h\|_{-1/2,\Gamma_0}$$

for some constant c , independent of h_i . \square

3.1. Discrete example 1. The finite element discretization of (1.2)–(1.4) can be given by the following. Find $u_i^h \in V_i^h$ such that for $v_i^h \in V_i^h$,

$$(3.4) \quad a_i(u_i^h, v_i^h) = (f, v_i^h)_i - (-1)^i (g^h, v_i^h)_{0,\Gamma_0},$$

where $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ denote the standard inner products on $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively.

Then, the finite element approximation of the optimization problem is

$$\begin{aligned} & \text{Min} \quad \|\pi^h(u_1^h - u_2^h)\|_{1/2,\Gamma_0}^2 \\ & \text{over } (u_1^h, u_2^h, g^h) \in V_1^h \times V_2^h \times P^h, \text{ subject to (3.4) for } i = 1, 2. \end{aligned}$$

Before we discuss the convergence of the iterative scheme, we first consider the equations for the discrete minimizer and the discretization error.

The minimizer of the above optimization problem is given by the solution of the following optimality system. Find $(u_1^h, u_2^h, g^h) \in V_1^h \times V_2^h \times P^h$ such that

$$(3.5) \quad \begin{aligned} & a_1(u_1^h, v_1^h) + a_2(u_2^h, v_2^h) - (g^h, v_1^h)_{0,\Gamma_0} + (g^h, v_2^h)_{0,\Gamma_0} \\ & = (f, v_1^h)_1 + (f, v_2^h)_2 \quad \forall v_i^h \in V_i^h, \quad i = 1, 2, \end{aligned}$$

$$(3.6) \quad (u_1^h - u_2^h, w^h)_{0,\Gamma_0} = 0 \quad \forall w^h \in P^h.$$

We follow some techniques like those presented in [19, 26] to show the existence of the minimizer.

THEOREM 3.3. *Assume that $\min\{h_1, h_2\}$ is sufficiently small; comparing with h_0 , there exists a unique solution to the system (3.5)–(3.6). Moreover, letting u be the solution of the original equation (1.1), we have the following error estimate: there exists a constant $c > 0$ such that*

$$(3.7) \quad \begin{aligned} & \|u - u_1^h\|_{1,\Omega_1} + \|u - u_2^h\|_{1,\Omega_2} + \|a_1 \frac{\partial u}{\partial n_1} - g^h\|_{-1/2,\Gamma_0} \\ & \leq c \inf_{v_i^h \in V_i^h, w^h \in P^h} \left(\|u - v_1^h\|_{1,\Omega_1} + \|u - v_2^h\|_{1,\Omega_2} + \left\| a_1 \frac{\partial u}{\partial n_1} - w^h \right\|_{-1/2,\Gamma_0} \right). \end{aligned}$$

Proof. Since a_1, a_2 are bounded coercive symmetric bilinear forms on $H_{1,0}^1(\Omega_1)$ and $H_{0,2}^1(\Omega_2)$, respectively, in order to apply the general framework in [19], we just need to check the inf-sup condition

$$\inf_{0 \neq w^h \in P^h} \sup_{0 \neq (u_1^h, u_2^h) \in V_1^h \times V_2^h} \frac{(u_1^h - u_2^h, w^h)_{0,\Gamma_0}}{(\|u_1^h\|_1 + \|u_2^h\|_1)\|w^h\|_{-1/2}} \geq \beta$$

for some constant $\beta > 0$.

To verify this, for any $w^h \in P^h$, we consider for either $i = 1$ or 2

$$(3.8) \quad \begin{cases} \Delta u_i = 0 & \text{in } \Omega_i, \\ a_i \frac{\partial u_i}{\partial n_i} = (-1)^{i+1} w^h & \text{on } \Gamma_0, \quad u_i = 0 & \text{on } \Gamma_i. \end{cases}$$

Let U_i^h denote the set of all solutions for all functions $w^h \in P^h$. U_i^h forms a finite-dimensional subspace in $H_{i,0}^1(\Omega_i)$. For any small constant $\epsilon > 0$, since the unit ball of U_i^h is compact in $H_{i,0}^1(\Omega_i)$, by applying the standard approximation property of the finite element spaces, we can have, for sufficiently small h_i in comparison with h_0 , that there exists some constant C_i which depends only on the subdomain Ω_i , such that

$$\inf_{u_i^h \in V_i^h} \|u_i^h - u_i\|_{1/2, \Gamma_0} \leq c_i \inf_{u_i^h \in V_i^h} \|u_i^h - u_i\|_{1, \Omega_i} \leq c_i \epsilon \|u_i\|_{1, \Omega_i} \leq C_i \epsilon \|w^h\|_{-1/2, \Gamma_0}$$

uniformly for any $u_i \in U_i^h$ (or $w^h \in P^h$). On the other hand, using trace inequality and regularity estimates [13, 25], it is easy to check that

$$\|w^h\|_{-1/2, \Gamma_0} \leq \tilde{c}_i \|u_i\|_{1, \Omega_i} \leq \tilde{c}_i \|u_i\|_{1/2, \Gamma_0}$$

for some generic constant \tilde{c}_i ($i = 1$ or 2). Thus, there exists $u_i^h \in V_i^h$ such that

$$\begin{aligned} \|u_i^h\|_{1/2, \Gamma_0} &\geq \|u_i\|_{1/2, \Gamma_0} - \|u_i - u_i^h\|_{1/2, \Gamma_0} \geq \|u_i\|_{1/2, \Gamma_0} - C_i \epsilon \|w^h\|_{-1/2, \Gamma_0} \\ &\geq (1 - \tilde{c}_i C_i \epsilon) \|w^h\|_{-1/2, \Gamma_0}. \end{aligned}$$

If we choose a suitably small ϵ , i.e., sufficiently small h_i (either $i = 1$ or 2) in comparison with h_0 , and let $u_j^h = 0$ for $j \neq i$, then

$$\begin{aligned} (u_1^h - u_2^h, w^h)_{0, \Gamma_0} &= a_i (u_i^h, u_i^h) \geq c \|u_i^h\|_{1, \Omega_i}^2 \\ &\geq c (\|u_1^h\|_{1, \Omega_1} + \|u_2^h\|_{1, \Omega_2}) \|w^h\|_{-1/2, \Gamma_0}. \end{aligned}$$

The theorem then follows from the above inf-sup condition and the general framework presented in [19]. \square

Based on the above theorem, the order of the approximation can then be obtained from the best approximation properties and

$$\inf_{v^h \in P^h} \left\| a_1 \frac{\partial w}{\partial n_i} \Big|_{\Gamma_0} - v^h \right\|_{-1/2} \leq C h_0^{m-1} \inf_{w \in H^m(\Omega_i)} \|w\|_{m, \Omega_i}, \quad 1 \leq m \leq k, \quad i = 1, 2.$$

In the above theorem, the subdomain mesh parameters are assumed to be small compared with the mesh parameter on the interface. In special circumstances, such restriction is not always required. For example, if we let $P^h = V_1^h|_{\Gamma_0}$, then for any $w^h \in P^h$, we may take $u_2^h = 0$ and u_1^h to be an extension of $\mathcal{S}_{1,h}^{-1} w^h$ in V_1^h such that $\|u_1^h\|_{1, \Omega_1} \leq c \|\mathcal{S}_{1,h}^{-1} w^h\|_{1/2, \Gamma_0}$ for some constant c , independent of h . For example, u_i^h can be defined as a generalized interpolant of the harmonic extension of $\mathcal{S}_{1,h}^{-1} w^h$ in $H_{i,0}^1(\Omega_i)$ as defined in [40]. Moreover, assuming that the inverse inequality (3.1) holds, we get from the bounds on \mathcal{S}_1^h given in Lemma 3.2 that

$$\|w^h\|_{-1/2, \Gamma_0} \leq c \|\mathcal{S}_{1,h}^{-1} w^h\|_{1/2, \Gamma_0}$$

for some constant $c > 0$, independent of h_1 . Then,

$$\begin{aligned} (u_1^h - u_2^h, w^h)_{0, \Gamma_0} &= (\mathcal{S}_{1,h}^{-1}(w^h), w^h)_{0, \Gamma_0} \geq c \|\mathcal{S}_{1,h}^{-1}(w^h)\|_{1/2, \Gamma_0}^2 \\ &\geq c \|w^h\|_{-1/2, \Gamma_0} \|u^h\|_{1, \Omega_1} = c \|w^h\|_{-1/2, \Gamma_0} (\|u^h\|_{1, \Omega_1} + \|u^h\|_{1, \Omega_1}). \end{aligned}$$

So, we have

$$\inf_{0 \neq w^h \in P^h} \sup_{0 \neq (u_1^h, u_2^h) \in V_1^h \times V_2^h} \frac{(u_1^h - u_2^h, w^h)_{0, \Gamma_0}}{(\|u_1^h\|_1 + \|u_2^h\|_1) \|w^h\|_{-1/2}} \geq \beta$$

for some constant $\beta > 0$. Then, similar to the previous theorem, we get the following theorem.

THEOREM 3.4. *For $P^h = V_i^h|_{\Gamma_0}$, $i = 1$ or 2 , there exists a unique solution to the system (3.5)–(3.6). Moreover, the error estimate (3.7) remains valid.*

As for the order of approximation, since we used $V_1^h|_{\Gamma_0}$ for the approximation of both the interface values and their normal derivatives, the optimal order estimates hold if we assume that the normal derivatives are in $H_{00}^{1/2}(\Gamma_0)$. Notice that for the case where $V_i^h = V^h|_{\Omega_i}$ and $P^h = V^h|_{\Gamma_0}$, the solution is equivalent to the standard finite element approximation and the error analysis has been well documented (see, e.g., [10], [11]). On the other hand, one may view the optimality system given here as a method to incorporate two different finite element meshes on different subdomains together through a use of coarser grids. Such an idea has been used in [26] to treat inhomogeneous boundary conditions for the second-order elliptic problems as well as Stokes and Navier–Stokes equations. Those ideas have also been explored extensively in the context of domain decomposition methods via the use of *mortar elements* (see, e.g., [5] in the overlapping setting).

Next, we consider the finite element implementation of the gradient method which can be defined as follows. Given $g_h^n \in P^h$, seek $u_1^h \in V_1^h$, $u_2^h \in V_2^h$, $\lambda_1^h \in V_1^h$, and $\lambda_2^h \in V_2^h$ such that

$$a_i(u_{i,h}^n, v_i^h) = (f, v_i^h)_i - (-1)^i (g_h^n, v_i^h)_{0, \Gamma_0} \quad \forall v_i^h \in V_i^h,$$

$$a(\lambda_i^h, v_i^h) = (-1)^i (\pi^h(u_{2,h}^n - u_{1,h}^n), v_i^h)_{0, \Gamma_0} \quad \forall v_i^h \in V_i^h,$$

and seek $g_h^{n+1} \in P^h$ such that

$$(g_h^{n+1}, w^h)_{0, \Gamma_0} = (g_h^n, w^h)_{0, \Gamma_0} + \alpha_n (\lambda_1^h - \lambda_2^h, w^h) \quad \forall w^h \in P^h.$$

Its limit $(u_1^h, \lambda_1^h, u_2^h, \lambda_2^h, g^h) \in [V_1^h]^2 \times [V_2^h]^2 \times P^h$ satisfies the optimality system

$$\begin{cases} a_i(u_i^h, v_i^h) = (f, v_i^h)_i - (-1)^i (g_h^n, v_i^h)_{0, \Gamma_0} & \forall v_i^h \in V_i^h, \quad i = 1, 2, \\ a(\lambda_i^h, v_i^h) = (-1)^i (\pi^h(u_{2,h}^h - u_{1,h}^h), v_i^h)_{0, \Gamma_0} & \forall v_i^h \in V_i^h, \quad i = 1, 2, \\ (\lambda_1^h - \lambda_2^h, w^h) = 0 & \forall w^h \in P^h. \end{cases}$$

For $P^h = V_1^h|_{\Gamma_0}$, there exists a unique solution given by $(u_1^h, 0, u_2^h, 0, g^h)$, where (u_1^h, u_2^h, g^h) satisfies (3.5)–(3.6). The error equation is then given by

$$e_{n+1}^h = e_n^h - \alpha_n (\mathcal{S}_{1,h}^{-1} + \pi_1^h \pi_2^h \mathcal{S}_{2,h}^{-1} \pi_2^h \pi_1^h)^2 e_n^h.$$

Let $\sigma(\cdot)$ denote the spectral radius; then the iteration is convergent, provided that

$$0 < \theta_* \leq \alpha_n \leq \theta^* < 2/\sigma(\pi_1^h \pi_2^h \mathcal{S}_{2,h}^{-1} \pi_2^h \pi_1^h + \mathcal{S}_{1,h}^{-1})^2.$$

The convergence rate depends on the mesh parameter.

3.2. Discrete example 2. In this case, the finite element approximation of the optimization problem is

$$\begin{aligned} & \text{Min} \quad (\pi^h(u_1^h - u_2^h), \mathcal{S}_1^h \pi_1^h \pi^h(u_1^h - u_2^h))_{0,\Gamma} \\ & \text{over } (u_1^h, u_2^h, g^h) \in V_1^h \times V_2^h \times P^h \text{ subject to (3.4) for } i = 1, 2. \end{aligned}$$

By viewing P^h as a subspace of $H^{-1/2}(\Gamma_0)$, the finite element implementation of the gradient method can be given by the following. Given $g_h^n \in P^h$, solve for $u_{1,h}^n \in V_1^h$, $u_{2,h}^n \in V_2^h$ by

$$(3.9) \quad a_i(u_i^h, v_i^h) = (f, v_i^h)_i - (-1)^i (g_h^n, v_1^h)_{0,\Gamma_0} \quad \forall v_i^h \in V_i^h.$$

Then update by

$$(3.10) \quad \begin{aligned} (g_h^{n+1}, w^h)_{0,\Gamma_0} &= (g_h^n, w^h)_{0,\Gamma_0} \\ &+ \alpha_n ((\mathcal{S}_1^h \pi_1^h + \mathcal{S}_2^h \pi_2^h) \pi^h(u_2^h - u_1^h), w^h) \quad \forall w^h \in P^h. \end{aligned}$$

Note that if we let $\lambda_i^h \in V_i^h$,

$$\begin{aligned} a_i(\lambda_i^h, v_i^h) &= 0 \quad \forall v_i^h \in V_i^h \cap H_0^1(\Omega_i), \\ \lambda_i^h|_{\Gamma_0} &= \pi_i^h \pi^h(u_2^h - u_1^h), \end{aligned}$$

and $\theta_i^h \in V_i^h|_{\Gamma_0}$ such that

$$(\theta_i^h, v_i^h)_{0,\Gamma_0} = a_i(\lambda_i^h, v_i^h) \quad \forall v_i^h \in V_i^h,$$

then $\mathcal{S}_i^h \pi_i^h \pi^h(u_2^h - u_1^h) = \theta_i^h$ and (3.10) is the same as

$$(g_h^{n+1}, w^h)_{0,\Gamma_0} = (g_h^n, w^h)_{0,\Gamma_0} + \alpha_n (\theta_1^h + \theta_2^h, w^h) \quad \forall w^h \in P^h.$$

Any limit $(u_1^h, u_2^h, g^h, \lambda_1^h, \lambda_2^h, \theta^h) \in [V_1^h \times V_2^h \times P^h]^2$ satisfies the optimality system

$$\begin{aligned} & a_1(u_1^h, v_1^h) + a_2(u_2^h, v_2^h) - (g^h, v_1^h - v_2^h)_{0,\Gamma_0} \\ & = (f, v_1^h)_1 + (f, v_1^h)_2 \quad \forall v_i^h \in V_i^h, \quad i = 1, 2, \\ & a_1(\lambda_1^h, w_1^h) + a_2(\lambda_2^h, w_2^h) + (\theta^h, w_1^h - w_2^h)_{0,\Gamma_0} \\ & = 0 \quad \forall w_i^h \in V_i^h, \quad i = 1, 2, \\ & (\lambda_1^h, z_1^h)_{0,\Gamma_0} - (\pi^h(u_1^h - u_2^h), z_1^h)_{0,\Gamma_0} = 0 \quad \forall z_1^h \in V_1^h|_{\Gamma_0}, \\ & (\lambda_2^h, z_2^h)_{0,\Gamma_0} - (\pi^h(u_1^h - u_2^h), z_2^h)_{0,\Gamma_0} = 0 \quad \forall z_2^h \in V_2^h|_{\Gamma_0}. \end{aligned}$$

It is not difficult to check that for $P^h = V_1^h|_{\Gamma_0}$ (or $P^h = V_2^h|_{\Gamma_0}$), the solution to the above system is unique and is given by $(u_1^h, u_2^h, g^h, 0, 0, 0)$, where (u_1^h, u_2^h, g^h) satisfies the weak form (3.5)–(3.6). In this case, the error equation is given by

$$e_{n+1}^h = e_n^h - \alpha_n (\mathcal{S}_1^h + \pi_1^h \mathcal{S}_2^h \pi_2^h) (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) e_n^h := \mathcal{N}^h(\alpha_n) e_n^h.$$

Let $\mathcal{S}^h = \mathcal{S}_1^h + \pi_1^h \pi_2^h \mathcal{S}_2^h \pi_2^h \pi_1^h$. Note that \mathcal{S}^h is positive definite and let \mathcal{S}^{-h} denote its inverse. Thus we have the following lemma.

LEMMA 3.5. *For small h_i , $i = 1, 2$, there exist generic positive constants c_1 and c_2 , independent of the mesh parameters h_1 and h_2 , such that for any $w_1^h \in V_1^h|_{\Gamma_0}$,*

$$(3.11) \quad c_1 (w_1^h, \mathcal{S}^{-h} w_1^h)_{0,\Gamma_0} \leq (w_1^h, (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) \pi_2^h w_1^h)_{0,\Gamma_0}$$

and

$$(3.12) \quad \begin{aligned} & \left((\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) w_1^h, \mathcal{S}^h (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) w_1^h \right)_{0,\Gamma_0} \\ & \leq c_2 (w_1^h, \mathcal{S}^{-h} w_1^h)_{0,\Gamma_0}. \end{aligned}$$

Proof. The inequalities (3.11)–(3.12) follow from Lemma 3.1 and the bounds on the discrete Steklov–Poincaré operators and their inverses given in Lemma 3.2. \square

From the above lemma, we get

$$\begin{aligned} (e_{n+1}^h, \mathcal{S}^{-h} e_{n+1}^h)_{0,\Gamma_0} &= (\mathcal{N}_n^h e_n^h, \mathcal{S}^{-h} \mathcal{N}_n^h e_n^h)_{0,\Gamma_0} \\ &= (e_n^h, \mathcal{S}^{-h} e_n^h)_{0,\Gamma_0} - 2\alpha_n (e_n^h, (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) e_n^h)_{0,\Gamma_0} \\ &\quad + \alpha_n^2 ((\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) e_n^h, \mathcal{S}^h (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) e_n^h)_{0,\Gamma_0} \\ &\leq (e_n^h, \mathcal{S}^{-h} e_n^h)_{0,\Gamma_0} - 2c_1 \alpha_n (e_n^h, \mathcal{S}^{-h} e_n^h)_{0,\Gamma_0} + \alpha_n^2 c_2 (e_n^h, \mathcal{S}^{-h} e_n^h)_{0,\Gamma_0} \\ &= (1 - 2c_1 \alpha_n + c_2 \alpha_n^2) (e_n^h, \mathcal{S}^{-h} e_n^h)_{0,\Gamma_0}. \end{aligned}$$

If we take any two positive constants σ_* and σ^* such that

$$0 < \sigma_* < \sigma^* < 2c_1/c_2,$$

then for

$$(3.13) \quad \sigma_* \leq \alpha_n \leq \sigma^*$$

we have the mesh-independent geometric convergence of the iteration.

THEOREM 3.6. *The discrete gradient-type algorithm (3.9)–(3.10) is convergent for any e_h^0 and for any sequence $\{\alpha_n\}$ satisfying (3.13).*

3.3. Discrete example 3. Using the same optimization as before but viewing P^h as a subspace of $L^2(\Gamma_0)$, we get the finite element implementation of the gradient method as follows. Given $g_h^n \in P^h$, seek $u_1^h \in V_1^h$, $u_2^h \in V_2^h$, such that

$$a_i(u_i^h, v_i^h) = (f, v_i^h)_i - (-1)^i (g_h^n, v_i^h)_{0,\Gamma_0} \quad \forall v_i^h \in V_i^h.$$

Then update by

$$(g_h^{n+1}, w^h)_{0,\Gamma_0} = (g_h^n, w^h)_{0,\Gamma_0} - \alpha_n (u_2^h - u_1^h, w^h) \quad \forall w^h \in P^h.$$

The limit (u_1^h, u_2^h, g^h) satisfies the weak form (3.5)–(3.6). If we take $P^h = V_i^h|_{\Gamma_0}$, then we have a unique solution like before. The error equation is given by

$$e_{n+1}^h = e_n^h - \alpha_n (\mathcal{S}_{1,h}^{-1} + \pi_1^h \mathcal{S}_{2,h}^{-1} \pi_2^h) e_n^h.$$

It is convergent, provided that

$$0 < \gamma_* \leq \alpha_n \leq \gamma^* < 2/\sigma(\pi_1^h \pi_2^h \mathcal{S}_{2,h}^{-1} \pi_2^h \pi_1^h + \mathcal{S}_{1,h}^{-1}),$$

where, as before, $\sigma(\cdot)$ denotes the spectral radius. The convergence rate will depend on the mesh parameter.

3.4. Implementation issues and other forms of gradient-based iterations. Comparing the different examples, we see that though a mesh-independent rate can be obtained for the algorithm in section 3.2, its implementation requires both Dirichlet- and Neumann-type solvers with respect to the interface data. On the other hand, the algorithm in section 3.3 can be implemented with only one Neumann-type solver with respect to the interface data and such solvers need to be performed twice for the algorithm in section 3.3. Similarly, it will be shown in the next section that, via the Dirichlet control, various algorithms can be implemented, including, in particular, algorithms that only require Dirichlet solvers with respect to the interface data. Though those algorithms may not be the most efficient iterations, the freedom given by them in choosing specified types of solvers can be an advantage in some applications as discussed in the introduction.

Besides implementing the simple gradient-type iterations, one may also apply conjugate gradient-type methods (with or without preconditioning/bistabilization) to solve the optimization problem. An example can be found in [27]. A least squares formulation is employed there. The convergence analysis of the conjugate gradient type methods, in the finite-dimensional settings, again rests upon the spectral properties of the gradient operators \mathcal{J}' . Using standard techniques and the properties of the discrete Steklov–Poincaré operators, a mesh-independent convergence rate can be proved for the iteration presented in section 3.2.

4. Domain decomposition algorithms via Dirichlet control. Now, we switch to consider λ as the independent *control variable*. Again, let X, Y be two Hilbert spaces; we consider

$$\begin{aligned} \text{Min } \mathcal{J}(g) &= \left\| a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} \right\|_Y^2 \\ \text{over } \lambda \in Y, u_1, u_2, & \text{ subject to (1.2) and (1.3).} \end{aligned}$$

Different choices of X and Y can again be considered as before. Notice that

$$(\mathcal{J}'(\lambda), \mathcal{P}w)_{0, \Gamma_0} = (\mathcal{J}'(\lambda), w)_X = \left(a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2}, \mathcal{Q}(\mathcal{S}_1 + \mathcal{S}_2)w \right)_{0, \Gamma_0} \quad \forall \lambda, w \in X,$$

or

$$\mathcal{J}'(\lambda) = \mathcal{P}^{-1}(\mathcal{S}_1 + \mathcal{S}_2)\mathcal{Q} \left(a_1 \frac{\partial u_1^n}{\partial n_1} + a_2 \frac{\partial u_2^n}{\partial n_2} \right) \quad \text{on } \Gamma_0,$$

where u_i^n satisfies the equations

$$(4.1) \quad \begin{cases} -a_i \Delta u_i^n = f & \text{in } \Omega_i, \\ u_i^n = 0 & \text{on } \Gamma_i, \\ u_i^n = \lambda^n & \text{on } \Gamma_0. \end{cases}$$

Thus, the gradient-type iteration is given by

$$(4.2) \quad \lambda_{n+1} = \lambda_n - \alpha_n \mathcal{J}'(\lambda_n).$$

Again, we have the following result.

THEOREM 4.1. *Given $\alpha_n > \theta > 0$, if the iteration (4.2) is convergent and*

$$u|_{\Omega_i} = \lim_{n \rightarrow \infty} u_i^n \quad \text{for } i = 1, 2,$$

then u is the solution of (1.1) on the whole domain Ω .

Let $e_n = \lambda_n - u|_{\Gamma_0}$; then

$$e_{n+1} = [I - \alpha_n \mathcal{P}^{-1}(\mathcal{S}_1 + \mathcal{S}_2)\mathcal{Q}(\mathcal{S}_1 + \mathcal{S}_2)] e_n := \mathcal{D}(\alpha_n)e_n .$$

As a result, we get the following convergence criterion.

THEOREM 4.2. *The algorithm (4.2) is convergent if and only if for any e^0 ,*

$$\left\| \prod_{k=1}^n \mathcal{D}(\alpha_k)e^0 \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in some suitable norm and with some suitably chosen sequence $\{\alpha_n\}$.

In comparison with the Neumann control case, we now consider some examples which lead to algorithms that, in particular, always require Dirichlet-type subdomain solvers with respect to the interface data.

4.1. Example 1. We first consider $X = H_{00}^{1/2}(\Gamma_0)$, $Y = H^{-1/2}(\Gamma_0)$ with $\mathcal{P} = \mathcal{S}$, $\mathcal{Q} = \mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}$. Thus the iteration is

$$(4.3) \quad \lambda_{n+1} = \lambda_n - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \left(a_1 \frac{\partial u_1^n}{\partial n_1} + a_2 \frac{\partial u_2^n}{\partial n_2} \right),$$

which has been studied in [2]. It can be implemented as follows: given λ^n , first solve for $u_1^n, u_2^n \in H_{i,0}^1(\Omega_i)$ from (4.1); then, solve for $\zeta_1^n, \zeta_2^n \in H_{i,0}^1(\Omega_i)$ from

$$\begin{cases} \Delta \zeta_i^n = 0 & \text{in } \Omega_i, \\ a_i \frac{\partial \zeta_i^n}{\partial n_i} = (-1)^i (a_1 \frac{\partial u_1^n}{\partial n_1} + a_2 \frac{\partial u_2^n}{\partial n_2}) & \text{on } \Gamma_0; \end{cases}$$

then we can update by

$$\lambda_{n+1} = \lambda_n + \alpha_n (\zeta_1^n - \zeta_2^n).$$

The error equation is $e_{n+1} = \mathcal{D}(\alpha_n)e_n$, where

$$\mathcal{S}(\alpha_n) = (I - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})(\mathcal{S}_1 + \mathcal{S}_2))e_n.$$

Let $f_n = \mathcal{S}e_n$; thus we have

$$f_{n+1} = (I - \alpha_n (\mathcal{S}_1 + \mathcal{S}_2)(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}))f_n.$$

The above iteration is then equivalent to the case discussed before in section 2.2. Thus, similar to Theorem 2.4, we have Theorem 4.3.

THEOREM 4.3. *There exist two constants α_* and α^* such that for any e_0 and for any sequence $\{\alpha_n\}$ satisfying (2.9), the gradient-type algorithm (4.3) is geometrically convergent.*

As for the discretization, we first have the weak form for the feasible solution: for $i = 1, 2$, $g_i^h \in V_i^h|_{\Gamma_0}$,

$$(4.4) \quad a_i(u_i^h, v_i^h) = (f, v_i^h)_i + (g_i^h, v_i^h)_{0,\Gamma_0} \quad \forall v_i^h \in V_i^h,$$

$$(4.5) \quad (u_i^h - \lambda^h, w_i^h)_{0,\Gamma_0} = 0 \quad \forall w_i^h \in V_i^h|_{\Gamma_0} .$$

The finite element approximation of the optimization problem is

$$\begin{aligned} & \text{Min} \quad \left(\pi_2^h(g_1^h + g_2^h), \mathcal{S}_{2,h}^{-1} \pi_2^h(g_1^h + g_2^h) \right)_{0,\Gamma} \\ & \text{over } (u_1^h, u_2^h, \lambda^h) \in V_1^h \times V_2^h \times P^h, \text{ subject to (4.4)–(4.5) for } i = 1, 2. \end{aligned}$$

Taking P^h as a subspace of $H^{-1/2}(\Gamma_0)$, the finite element implementation of the gradient method can be defined by the following. Given $\lambda_h^n \in P^h$, seek $u_1^h \in V_1^h$, $u_2^h \in V_2^h$, such that

$$\begin{aligned} a_i(u_i^h, v_i^h) - (g_{i,h}^{n+1}, v_i^h) &= (f, v_i^h)_i \quad \forall v_i^h \in V_i^h, \\ (u_i^h, w_i^h) &= (\lambda_h^n, w_i^h) \quad \forall w_i^h \in V_i^h \mid_{\Gamma_0}, \end{aligned}$$

and

$$a_i(\zeta_i^h, v_i^h) = (-1)^i (\pi^h(g_{1,h}^{n+1} + g_{2,h}^{n+1}), v_i^h) \quad \forall v_i^h \in V_i^h.$$

Then we update by

$$(\lambda_h^{n+1}, w^h)_{0,\Gamma_0} = (\lambda_h^n, w^h)_{0,\Gamma_0} - \alpha_n (\zeta_1^h - \zeta_2^h, w^h) \quad \forall w^h \in P^h.$$

We again first consider equations for the discrete minimizer and the error. Find $(u_1^h, g_1^h, \zeta_1^h, u_2^h, g_2^h, \zeta_2^h, \lambda^h) \in V_1^h \times V_1^h \mid_{\Gamma_0} \times V_1^h \times V_2^h \times V_2^h \mid_{\Gamma_0} \times V_2^h \times P^h$ such that

$$\begin{aligned} a_1(u_1^h, v_1^h) + a_2(u_2^h, v_2^h) - (g_1^h, v_1^h)_{0,\Gamma_0} - (g_2^h, v_2^h)_{0,\Gamma_0} \\ &= (f, v_1^h)_1 + (f, v_2^h)_2 \quad \forall v_i^h \in V_i^h, \quad i = 1, 2, \\ (u_1^h - \lambda^h, w_1^h)_{0,\Gamma_0} &= 0 \quad \forall w_1^h \in V_1^h \mid_{\Gamma_0}, \\ (u_2^h - \lambda^h, w_2^h)_{0,\Gamma_0} &= 0 \quad \forall w_2^h \in V_2^h \mid_{\Gamma_0}, \\ a_i(\zeta_i^h, v_i^h) &= (-1)^i (\pi^h(g_1^h + g_2^h), v_i^h) \quad \forall v_i^h \in V_i^h, \\ (\zeta_1^h - \zeta_2^h, w^h)_{0,\Gamma_0} &= 0 \quad \forall w^h \in P^h. \end{aligned}$$

For $P^h = V_2^h \mid_{\Gamma_0}$, the solution again is given by $(u_1^h, g^h, 0, u_2^h, -\pi_2^h g^h, 0, u_2^h \mid_{\Gamma_0})$, where (u_1^h, u_2^h, g^h) satisfies the weak form (3.5)–(3.6).

The error equation is given by

$$e_h^{n+1} = e_h^n - \alpha_n \pi_2^h (\mathcal{S}_{1,h}^{-1} \pi_1^h + \mathcal{S}_{2,h}^{-1}) \pi_2^h (\mathcal{S}_1^h \pi_1^h + \mathcal{S}_2^h) e_h^n.$$

The convergence property is the same as that for the algorithm (3.9)–(3.10). Thus, we may get as before that for α_n satisfying the condition (3.13), the iteration is geometrically convergent with a rate independent of the mesh parameters.

4.2. Example 2. We now consider $X = H_{00}^{1/2}(\Gamma_0)$, $Y = L^2(\Gamma_0)$ with $\mathcal{P} = \mathcal{S}_1$, $\mathcal{Q} = I$. Thus the iteration is

$$\lambda_{n+1} = \lambda_n - \alpha_n \left(a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} \right).$$

Its implementation is straightforward. The error equation is given by

$$e_{n+1} = (I - \alpha_n \mathcal{S}) e_n.$$

Since \mathcal{S} is not bounded from $H_{00}^{1/2}(\Gamma_0)$ into itself, no constant choice of α_n yields a globally convergent iteration. In some special geometry where the spectral information of \mathcal{S} can be more explicitly stated, it may be possible to choose appropriate sequences of $\{\alpha_n\}$ to get the convergence.

In the discrete case, using notation as before, we may consider minimizing

$$\|\pi^h(g_1^h + g_2^h)\|_{0,\Gamma_0}.$$

Viewing P^h as a subspace of $H_{00}^{1/2}(\Gamma_0)$, the implementation of the gradient iteration is defined by the following. Given $\lambda_h^n \in P^h$, first solve

$$\begin{aligned} a_i(u_{i,h}^n, v_i^h) &= (f, v_1^h)_1 \quad \forall v_i^h \in V_i^h \cap H_0^1(\Omega_i), \quad i = 1, 2, \\ (u_{1,h}^n - \lambda_h^n, w_1^h)_{0,\Gamma_0} &= 0 \quad \forall w_1^h \in V_1^h|_{\Gamma_0}, \\ (u_{2,h}^n - \lambda_h^n, w_2^h)_{0,\Gamma_0} &= 0 \quad \forall w_2^h \in V_2^h|_{\Gamma_0}. \end{aligned}$$

Then,

$$\begin{aligned} (g_{i,h}^n, v_i^h)_{0,\Gamma_0} &= a_i(u_{i,h}^n, v_i^h) - (f, v_1^h)_1 \quad \forall v_i^h \in V_i^h, \quad i = 1, 2, \\ (\lambda_h^{n+1}, w^h) &= (\lambda_h^n, w^h) - \alpha_n(\pi^h(g_{1,h}^n + g_{2,h}^n), w^h) \quad \forall w^h \in P^h. \end{aligned}$$

Similar to before, for $P^h = V_2^h|_{\Gamma_0}$, we get the limit of $(u_{1,h}^n, g_{1,h}^n, u_{2,h}^n, g_{2,h}^n, \lambda_h^n)$ to be $(u_1^h, g^h, u_2^h, -\pi_2^h g^h, u_2^h|_{\Gamma_0})$, where (u_1^h, u_2^h, g^h) satisfied the weak form (3.5)–(3.6).

In the discrete case, the error equation is given by

$$e_{n+1}^h = (I - \alpha_n(\pi_2^h \mathcal{S}_1^h \pi_2^h + \mathcal{S}_2^h))e_n^h.$$

It can be convergent, provided that

$$0 < \theta_* \leq \alpha_n \leq \theta^* < 2/\sigma(\pi_2^h \mathcal{S}_1^h \pi_2^h + \mathcal{S}_2^h).$$

Again, $\sigma(\cdot)$ denotes the spectral radius. The convergence rate will depend on the mesh parameter.

The algorithm given above is, to some extent, similar to that given in [39], where the normal derivatives on the interface are calculated explicitly via differences, resulting in limiting solutions that differ from the standard Galerkin finite element solution. Due to the mesh-dependent convergence rate, a preconditioner is also proposed there. Similar algorithms and conjugate gradient variants were also studied much earlier, in [15].

4.3. Example 3. We now consider $X = Y = L^2(\Gamma_0)$ with $\mathcal{P} = \mathcal{Q} = I$; thus, the iteration is given by

$$\lambda_{n+1} = \lambda_n - \alpha_n(\mathcal{S}_1 + \mathcal{S}_2) \left(a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} \right).$$

It can be implemented as follows: First solve for $u_1^n, u_2^n \in H_{i,0}^1(\Omega_i)$ from (4.1), then solve for $\lambda_1^n, \lambda_2^n \in H_{i,0}^1(\Omega_i)$ from

$$\begin{cases} \Delta \zeta_i^n = 0 & \text{in } \Omega_i, \\ \zeta_i^n = \left(a_1 \frac{\partial u_1^n}{\partial n_1} + a_2 \frac{\partial u_2^n}{\partial n_2} \right) & \text{on } \Gamma_i, \end{cases}$$

and then make the update

$$\lambda_{n+1} = \lambda_n - \alpha_n \left(a_1 \frac{\partial \zeta_1^n}{\partial n_1} + a_2 \frac{\partial \zeta_2^n}{\partial n_2} \right).$$

The discretization can be given just like those presented for the previous examples. Similar to before, we only expect global convergence for the discrete case and the convergence rate may depend on the mesh parameter.

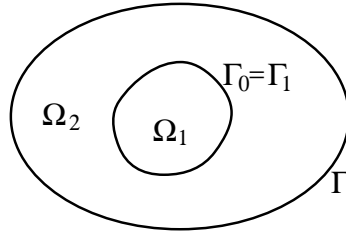


FIG. 5.1. Two nonoverlapping subdomains, including one in the interior.

4.4. Additional comment on the regularity of solution. Before we conclude this section, let us note that for the Dirichlet control with only $X = L^2(\Gamma_0)$, the derivation of the algorithm in the continuous level (PDEs) is only *formal* due to the lack of sufficient regularity estimates. Nevertheless, its discrete version can still be rigorously validated if the finite element spaces for the Dirichlet interface data are in fact subspaces of $H_{00}^{1/2}(\Gamma_0)$.

5. Neumann control with interior subdomains. We now consider a partition of the domain Ω into two subdomains $\Omega_i, i = 1, 2$, such that $\partial\Omega_1 \cap \Gamma = \emptyset$, i.e., such that Ω_1 is an interior subdomain. See Figure 5.1. In this case, $\Gamma_0 = \partial\Omega_1$. This case is of interest in many applications, such as the simulation of three-dimensional superconductors in the external applied magnetic field [18], as well as the simulation of the motion of elastic structure in the fluids.

What distinguishes the situation here from what we discussed previously is the fact that the interface data λ and g lie in spaces different from those used earlier. We discuss the necessary modifications here, which will also be helpful for the study of the decomposition with a large number of subdomains including interior ones.

To this end, we define the following spaces:

$$H_a^{1/2}(\Gamma_0) = \left\{ v \in H_a^{1/2}(\Gamma_0), \int_{\Gamma_0} v \, d\Gamma = 0 \right\}, \quad H_a^{1/2,*}(\Gamma_0) = \text{Dual of } H_a^{1/2}(\Gamma_0).$$

Since $H_a^{1/2,*}(\Gamma_0) = H^{-1/2}(\Gamma_0)/R$, we equate $H_a^{1/2,*}(\Gamma_0)$ with

$$H_a^{-1/2}(\Gamma_0) = \{v \in H^{-1/2}(\Gamma_0), (v, 1)_{0,\Gamma_0} = 0 \}.$$

We consider, for any $g_2 \in H^{-1/2}(\Gamma_0)$,

$$(5.1) \quad \begin{cases} -\Delta w_2 = 0 & \text{in } \Omega_2, \\ a_2 \frac{\partial w_2}{\partial n_2} = g_2 & \text{on } \Gamma_0, \\ w_2|_{\Gamma} = 0. \end{cases}$$

The solution w_2 is unique. We now define the Steklov–Poincaré operator \mathcal{S}_2 by

$$\mathcal{S}_2(w_2|_{\Gamma_0}) = g_2 \quad \text{and} \quad \mathcal{S}_2^{-1}(g_2) = w_2|_{\Gamma_0} \quad \text{on } \Gamma_0,$$

where w_2 and g_2 satisfy (5.1). Moreover,

$$\|w_2\|_{1/2,\Gamma_0} \sim \|w_2\|_{1,\Omega_1} \sim \|g_2\|_{-1/2,\Gamma_0}.$$

Now, for $w_1 \in H^{1/2}(\Gamma_0)$, let $\mathcal{R}w_1$ be its harmonic extension in Ω_1 and define

$$\mathcal{S}_1(w_1) = a_1 \frac{\partial}{\partial n_1} \mathcal{R}w_1 \quad \text{on } \Gamma_0.$$

We have $\mathcal{S}_1(w_1) \in H_a^{-1/2}(\Gamma_0)$. Meanwhile, for any $g_1 \in H_a^{-1/2}(\Gamma_0)$, we consider

$$(5.2) \quad \begin{cases} -\Delta w_1 = 0 & \text{in } \Omega_1, \\ a_1 \frac{\partial w_1}{\partial n_1} = g_1 & \text{on } \Gamma_0, \\ (1, w_1)_{0, \Gamma_0} = (1, \mathcal{S}_2^{-1} g_1)_{0, \Gamma_0}. \end{cases}$$

The solution w_1 is unique. Moreover,

$$\|w_1\|_{1/2, \Gamma_0} \sim \|w_1\|_{1, \Omega_1} \sim \|g_1\|_{-1/2, \Gamma_0}.$$

We then define the inverse of \mathcal{S}_1 by

$$\mathcal{S}_1^{-1}(g_1) = w_1 \quad \text{on } \Gamma_0,$$

where w_1 and g_1 satisfy (5.2). We have $(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})(g_1) \in H_a^{1/2}(\Gamma_0)$ for any $g_1 \in H_a^{-1/2}(\Gamma_0)$. Moreover, from the elliptic regularity, we get that \mathcal{S}_i^{-1} ($i = 1, 2$) are bounded operators from $H_a^{-1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$ to $H^{1/2}(\Gamma_0)$, respectively, and

$$\|\mathcal{S}_i^{-1}(g_i)\|_{1/2, \Gamma_0} \leq c \|g_i\|_{-1/2, \Gamma_0}, \quad i = 1, 2 \quad \forall g_i \in H_a^{-1/2}(\Gamma_0).$$

Meanwhile, \mathcal{S}_1 and \mathcal{S}_2 are bounded operators from $H^{1/2}(\Gamma_0)$ to $H_a^{-1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$, respectively, so that

$$\|\mathcal{S}_i(w_i)\|_{-1/2, \Gamma_0} \leq c \|w_i\|_{1/2, \Gamma_0}, \quad i = 1, 2 \quad \forall w_i \in H^{1/2}(\Gamma_0).$$

Let Λ denote the projection of $H^{-1/2}(\Gamma_0)$ to $H_a^{-1/2}(\Gamma_0)$, i.e.,

$$\Lambda(g) = g - \frac{1}{|\Gamma_0|} (1, g)_{0, \Gamma_0} \quad \forall g \in H^{-1/2}(\Gamma_0).$$

Notice that

$$(5.3) \quad \|\Lambda(g)\|_{-1/2, \Gamma_0} \leq \bar{c}_0 \|g\|_{0, \Gamma_0} \quad \forall g \in H^{-1/2}(\Gamma_0)$$

for some positive constant \bar{c}_0 . We also let $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$.

Let X and Y be subspaces of $H_a^{1/2}(\Gamma_0)$ and $H_a^{-1/2}(\Gamma_0)$, respectively. For the case of Neumann control, we have the following equations for $i = 1, 2$:

$$\begin{cases} -a_i \Delta u_i = f & \text{in } \Omega_i, \\ u_2 = 0 & \text{on } \Gamma, \\ a_i \frac{\partial u_i}{\partial n_i} = (-1)^i g & \text{on } \Gamma_0, \\ (1, u_1)_{0, \Gamma_0} = (1, u_2)_{0, \Gamma_0}. \end{cases}$$

Then we minimize $\mathcal{J}'(g) = \frac{1}{2} \|u_1 - u_2\|_X^2$ subject to the above equations.

Using notation similar to that given before, we have

$$(\mathcal{Q}\mathcal{J}'(g), v)_{0, \Gamma_0} = (\mathcal{P}(u_1 - u_2), (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})v)_{0, \Gamma_0} \quad \forall v \in Y.$$

So, the general gradient iteration is given by

$$g_{n+1} = g_n - \alpha_n \Lambda \mathcal{Q}^{-1} (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \Lambda \mathcal{P} (u_1^n - u_2^n)$$

and the error equations are given by

$$e_{n+1} = [I - \alpha_n \Lambda \mathcal{Q}^{-1} (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \Lambda \mathcal{P} (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})] e_n := \mathcal{N}(\alpha_n) e_n.$$

Similarly, we may derive iterative schemes for the case of using Dirichlet control. Details are omitted here.

5.1. Example 1. Here, $\mathcal{P} = \mathcal{Q} = I$, and the iteration is given by

$$(5.4) \quad g_{n+1} = g_n - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) (u_1^n - u_2^n).$$

Implementation is done by solving first

$$\begin{cases} -a_i \Delta u_i^n = f & \text{in } \Omega_i, \\ u_2^n = 0 & \text{on } \Gamma, \\ a_i \frac{\partial u_i}{\partial n_i} = (-1)^i g^n & \text{on } \Gamma_0, \\ (1, u_1)_{0, \Gamma_0} = (1, u_2)_{0, \Gamma_0} \end{cases}$$

for $i = 1, 2$, then

$$\begin{cases} \Delta \lambda_i = 0 & \text{in } \Omega_i, \\ \lambda_2 = 0 & \text{on } \Gamma, \\ a_i \frac{\partial \lambda_i}{\partial n_i} = (-1)^i (u_1^n - u_2^n) & \text{on } \Gamma_0, \\ (1, \lambda_1)_{0, \Gamma_0} = (1, \lambda_2)_{0, \Gamma_0}, \end{cases}$$

and then updating by

$$g_{n+1} = g_n + \alpha_n (\lambda_1 - \lambda_2).$$

Note that in the above implementation, we may solve for u_1 and u_2 independently and also λ_1 and λ_2 independently and match the conditions

$$(1, u_1)_{0, \Gamma_0} = (1, u_2)_{0, \Gamma_0} \quad \text{and} \quad (1, \lambda_1)_{0, \Gamma_0} = (1, \lambda_2)_{0, \Gamma_0} = 0$$

by adding appropriate constants to u_1 and λ_1 .

The error equation is given by

$$e_{n+1} = e_n + \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e^n.$$

As discussed before, let $L_a^2(\Gamma_0) = L^2(\Gamma_0) \cap H_a^{-1/2}(\Gamma_0)$; if we take

$$\hat{\theta} > \sup_{g \in L_a^2(\Gamma_0)} \frac{\|(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g\|_{0, \Gamma_0}}{\|g\|_{0, \Gamma_0}},$$

and if for all n we choose

$$(5.5) \quad 0 < \beta \leq \alpha_n \leq 2/\hat{\theta}^2,$$

then we obtain the following result.

THEOREM 5.1. *The gradient-type algorithm (5.4) is convergent for any e_0 and for any sequence $\{\alpha_n\}$ satisfying (5.5).*

5.2. Example 2. We now consider $X = H_{00}^{1/2}(\Gamma_0)$ and $Y = H^{-1/2}(\Gamma_0)$. For example, we let $\mathcal{P} = \mathcal{S}_1$ and $\mathcal{Q} = \Lambda\mathcal{S}_2^{-1}$; that is, the iteration is given by

$$(5.6) \quad g_{n+1} = g_n - \alpha_n \Lambda \mathcal{S}_2 (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \mathcal{S}_1 (u_1^n - u_2^n) \quad \text{on } \Gamma_0.$$

So, the implementation is to solve first

$$\begin{cases} -a_i \Delta u_i^n = f & \text{in } \Omega_i, \\ u_2^n = 0 & \text{on } \Gamma_i, \\ a_i \frac{\partial u_i}{\partial n_i} = (-1)^i g^n & \text{on } \Gamma_0, \\ (1, u_1)_{0, \Gamma_0} = (1, u_2)_{0, \Gamma_0}, \end{cases}$$

followed by

$$\begin{cases} \Delta \zeta_i = 0 & \text{in } \Omega_i, \\ \zeta_2 = 0 & \text{on } \Gamma_i, \\ \zeta_i = u_1^n - u_2^n & \text{on } \Gamma_0, \end{cases}$$

and then the iteration is updated by

$$g_{n+1} = g_n - \alpha_n \left(a_1 \frac{\partial \zeta_1}{\partial n_1} + a_2 \frac{\partial \zeta_2}{\partial n_2} - \frac{a_2}{|\Gamma_0|} \int_{\Gamma_0} \frac{\partial \zeta_2}{\partial n_2} d\Gamma_0 \right).$$

The error equation is

$$e_{n+1} = e_n - \alpha_n \Lambda (\mathcal{S}_1 + \mathcal{S}_2) (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e^n.$$

By properties of the Steklov–Poincaré operators, for any $g \in H_a^{-1/2}(\Gamma_0)$, we have

$$(g, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0, \Gamma_0} \sim ((\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g, \mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0, \Gamma_0}.$$

Thus,

$$\begin{aligned} & (\mathcal{N}_n e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \mathcal{N}_n e_n)_{0, \Gamma_0} = (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \quad - 2\alpha_n ((\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n, \mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \quad + \alpha_n^2 ((\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) \Lambda \mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n, \Lambda \mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \leq (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} - 2\bar{c}_1 \alpha_n (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \quad + \alpha_n^2 \bar{c}_2 \|\Lambda \mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n\|_{-1/2, \Gamma_0} \\ & \leq (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} - 2\bar{c}_1 \alpha_n (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \quad + \alpha_n^2 \bar{c}_2 \bar{c}_0 \|\mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n\|_{-1/2, \Gamma_0} \\ & \leq (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} - 2\bar{c}_1 \alpha_n (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{0, \Gamma_0} \\ & \quad + \alpha_n^2 \bar{c}_2 \bar{c}_0 \bar{c}_3 (e_n, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e_n)_{-1/2, \Gamma_0} \\ & = (1 - 2\bar{c}_1 \alpha_n + \bar{c}_0 \bar{c}_2 \bar{c}_3 \alpha_n^2) (e_n, \mathcal{S}^{-1} e_n)_{0, \Gamma_0}, \end{aligned}$$

where \bar{c}_0 is given in (5.3) and the positive constants \bar{c}_1 , \bar{c}_2 , and \bar{c}_3 are defined as

$$\bar{c}_1 = \inf_{g \in H^{-1/2}(\Gamma_0)} \frac{(g, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0, \Gamma_0}}{(g, \mathcal{S}^{-1}g)_{0, \Gamma_0}},$$

$$\bar{c}_2 = \sup_{g \in H_a^{-1/2}(\Gamma_0)} \frac{((\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g, g)_{0, \Gamma_0}}{\|g\|_{-1/2, \Gamma_0}},$$

and

$$\bar{c}_3 = \sup_{g \in H^{-1/2}(\Gamma_0)} \frac{\|\mathcal{S}(\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g\|_{-1/2, \Gamma_0}^2}{(g, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0, \Gamma_0}}.$$

So, if we take any two positive constants α_* and α^* such that

$$(5.7) \quad 0 < \alpha_* < \alpha_n \alpha^* < 2\bar{c}_1 / (\bar{c}_0 \bar{c}_2 \bar{c}_3),$$

then we have the geometric convergence of the iteration.

THEOREM 5.2. *The gradient-type algorithm (5.6) is convergent for any e_0 and for any sequence $\{\alpha_n\}$ satisfying (5.7).*

5.3. Example 3. We now consider $X = L^2(\Gamma_0)$ and $Y = H^{-1/2}(\Gamma_0)$. If we consider, for example, that $\mathcal{P} = I$ and $\mathcal{Q} = \mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}$, i.e., that the iteration is given by

$$(5.8) \quad g_{n+1} = g_n - \alpha_n \Lambda(u_1^n - u_2^n) \quad \text{on } \Gamma_0,$$

then the implementation is to solve first

$$\begin{cases} -a_i \Delta u_i^n = f & \text{in } \Omega_i, \\ u_2^n = 0 & \text{on } \Gamma, \\ a_i \frac{\partial u_i}{\partial n_i} = (-1)^i g^n & \text{on } \Gamma_0, \\ (1, u_1)_{0, \Gamma_0} = (1, u_2)_{0, \Gamma_0}, \end{cases}$$

and then

$$g_{n+1} = g_n - \alpha_n (u_1^n - u_2^n).$$

The error equation is

$$e_{n+1} = e_n - \alpha_n (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1}) e^n.$$

If we take

$$\bar{\theta} > \sup_{g \in L_a^2(\Gamma_0)} \frac{(g, (\mathcal{S}_1^{-1} + \mathcal{S}_2^{-1})g)_{0, \Gamma_0}}{\|g\|_{0, \Gamma_0}^2},$$

and if for all n , we choose

$$(5.9) \quad \beta \leq \alpha_n \leq 2/\bar{\theta}$$

for any positive constant β , then we obtain the following result.

THEOREM 5.3. *The gradient-type algorithm (5.8) is convergent for any e_0 and for any sequence $\{\alpha_n\}$ satisfying (5.9).*

6. The case of nonlinear constraints. We now give a brief discussion about the extension of the framework to nonlinear problems. We consider a generic nonlinear problem given by

$$(6.1) \quad \begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega, \\ \mathcal{B}(u) = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

and also the associated subdomain problems

$$(6.2) \quad \begin{cases} \mathcal{L}^i(u_i) = 0 & \text{in } \Omega_i, \\ \mathcal{B}^i(u_i) = 0 & \text{on } \Gamma \cap \partial\Omega_i, \\ \mathcal{C}^i(u_i) = g & \text{on } \Gamma_0 \cap \partial\Omega_i. \end{cases}$$

We consider the case of two nonoverlapping subdomains, as in section 1, with a *control* variable g defined on the interface Γ_0 , and we seek g to minimize the functional

$$\mathcal{J}(g) = \|\mathcal{A}^1(u_1) - \mathcal{A}^2(u_2)\|_X,$$

where X is a suitable Hilbert space and $\mathcal{A}^i, \mathcal{A}^j$ are suitable operators. To calculate the Fréchet derivative of the functional \mathcal{J} with respect to \tilde{g} , we differentiate the equations and boundary conditions given in (6.2) to obtain

$$\begin{cases} \mathcal{L}_u^i(u_i(g)) \mu_i = 0 & \text{in } \Omega_i, \\ \mathcal{B}_u^i(u_i(g)) \mu_i = 0 & \text{on } \Gamma_i, \\ \mathcal{C}_u^i(u_i(g)) \mu_i = \tilde{g} & \text{on } \Gamma_0, \end{cases}$$

where $\mu_i = \partial_g u_i(g) \tilde{g}$. It is convenient to denote the map $\tilde{g} \rightarrow \mu_i$ by

$$\mu_i = \mathcal{S}_i(g) \tilde{g}, \quad i = 1, 2.$$

Notice that \mathcal{S}_i may be viewed as generalizations of the Steklov–Poincaré operators defined in earlier sections.

Let \mathcal{P}, \mathcal{Q} be defined as in (2.1)–(2.2); then, the gradient iteration can be given by

$$(6.3) \quad g_{n+1} = g_n - \alpha_n \mathcal{Q}^{-1} \mathcal{D}_n^* \mathcal{P} [\mathcal{A}^1(u_1^n) - \mathcal{A}^2(u_2^n)],$$

with \mathcal{D}_n^* denoting the adjoint operator of $\partial_u \mathcal{A}_1(u_1^n) \mathcal{S}_1(g_n) - \partial_u \mathcal{A}_2(u_2^n) \mathcal{S}_2(g_n)$, $u_i^n = u_i(g_n)$, and $\partial_u \mathcal{A}_i$ denoting the derivative of \mathcal{A}_i for $i = 1, 2$. One may try to pick a suitably chosen parameter α_n so that the iteration is convergent to the solution of the minimizer of \mathcal{J} , i.e., the solution of the original nonlinear equation (6.1).

More generally, one may consider some modified gradient-type algorithms to accelerate the convergence or simplify the computation; for example, instead of using a parameter α_n as in (6.3), we may define the iteration by

$$g_{n+1} = g_n - \Lambda_n \mathcal{Q}^{-1} \mathcal{D}_n^* \mathcal{P} [\mathcal{A}^1(u_1^n) - \mathcal{A}^2(u_2^n)],$$

where Λ_n is a suitably defined operator.

We illustrate the possible generalizations with the Navier–Stokes equations of incompressible, viscous flow, again in the simple two-subdomain setting of section 1. Let \mathbf{u} denote the velocity vector, p the pressure, \mathbf{f} a given body force, and ν the constant kinematic viscosity; then the problem to be solved is

$$(6.4) \quad \begin{cases} -\nu \nabla \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma. \end{cases}$$

We may use either $\mathbf{u}|_{\Gamma_0}$ or the boundary flux $-p\mathbf{n} + \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \cdot \mathbf{n} - \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})\mathbf{u}$ as the control. For simplicity, let us only consider the case of using the Neumann (flux)

control; that is, instead of (6.4), we solve the pair of problems

$$(6.5) \quad \begin{cases} -\nu \nabla \cdot (\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T) + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla p_i = \mathbf{f} & \text{in } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \Omega_i, \\ \mathbf{u}_i = 0 & \text{on } \Gamma_i, \\ p_i \mathbf{n}_i - \nu (\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T) \cdot \mathbf{n}_i + \frac{1}{2} (\mathbf{u}_i \cdot \mathbf{n}_i) \mathbf{u}_i = (-1)^i \mathbf{g} & \text{on } \Gamma_0 \end{cases}$$

for $i = 1, 2$, where \mathbf{n}_1 and \mathbf{n}_2 denote the unit outer normal vectors to Ω_1 and Ω_2 , respectively. We choose $\mathbf{g} \in \mathbf{Y}$ so that

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{X}}^2$$

is minimized with respect to some appropriate norms in the space \mathbf{X} , subject to the constraint (6.5). Again, the choices of spaces \mathbf{X} and \mathbf{Y} can be similar to (the vectorized version of) $H_{00}^{1/2}(\Gamma_0)$ or $L^2(\Gamma_0)$ and $L^2(\Gamma_0)$ or $H^{-1/2}(\Gamma_0)$, respectively. We now give a couple of examples that illustrate the various implementations of the gradient algorithms or the modified gradient algorithm.

6.1. Example 1. Letting $\mathbf{X} = [H_{00}^{1/2}(\Gamma_0)]^3$, $\mathbf{Y} = [H^{-1/2}(\Gamma_0)]^3$, we have the following:

- Step 1. Given $\mathbf{g} = \mathbf{g}_n$, find \mathbf{u}_i^n , $i = 1, 2$, from (6.5).
- Step 2. Solve the problems

$$\begin{cases} \Delta \xi_i = 0 & \text{in } \Omega_i, \\ \xi_i = 0 & \text{on } \Gamma_i, \\ \xi_i = \mathbf{u}_1^n - \mathbf{u}_2^n & \text{on } \Gamma_0. \end{cases}$$

Step 3. For $\tilde{g}_n = \frac{\partial \xi_1}{\partial \mathbf{n}_1} + \frac{\partial \xi_2}{\partial \mathbf{n}_2}$, solve for (μ_i, ϕ_i) from the Stokes-type problems:

$$(6.6) \quad \begin{cases} -\nu \nabla \cdot (\nabla \mu_i + (\nabla \mu_i)^T) + \frac{1}{2} (\mu_i \cdot (\nabla \mathbf{u}_i^n)^T - \mathbf{u}_i^n \cdot (\nabla \mu_i)^T) - \mathbf{u}_i^n \cdot \nabla \mu_i + \nabla \phi_i = \mathbf{f} & \text{in } \Omega_i, \\ \nabla \cdot \mu_i = 0 & \text{in } \Omega_i, \\ \mathbf{u}_i = 0 & \text{on } \Gamma_i, \\ \phi_i \mathbf{n}_i + \nu (\nabla \mu_i - (\nabla \mu_i)^T) \cdot \mathbf{n}_i + \frac{1}{2} (\mathbf{u}_i^n \cdot \mathbf{n}_i) \mu_i = (-1)^i \tilde{g}_n & \text{on } \Gamma_0. \end{cases}$$

Step 4. Solve the problems

$$\begin{cases} \Delta \eta_i = 0 & \text{in } \Omega_i, \\ \eta_i = 0 & \text{on } \Gamma_i, \\ \eta_i = \mu_1 - \mu_2 & \text{on } \Gamma_0. \end{cases}$$

Step 5. Update $\mathbf{g}_{n+1} = \mathbf{g}_n - \alpha_n \left(\frac{\partial \eta_1}{\partial \mathbf{n}_1} + \frac{\partial \eta_2}{\partial \mathbf{n}_2} \right)$.

Using the convergence theory on the gradient algorithm (see, e.g., [12]), one may verify that, under a small data assumption which guarantees the uniqueness of the solutions for (6.4), one can find suitably chosen parameters α_n to get the convergence of the above iteration. Notice that the majority of the computational time is spent on solving the nonlinear equations in the first step.

6.2. Example 2. One may use the modified gradient algorithm to simplify or precondition the iteration. For example, by choosing $\mathbf{X} = [H_{00}^{1/2}(\Gamma_0)]^3$ and $\mathbf{Y} = [L^2(\Gamma_0)]^3$, one simplified algorithm is given by the following iteration.

- Step 1. Given $\mathbf{g} = \mathbf{g}_n$, find \mathbf{u}_i^n , $i = 1, 2$, from (6.5).

Step 2. Solve the problems

$$\begin{cases} \Delta \eta_i = 0 & \text{in } \Omega_i, \\ \eta_i = 0 & \text{on } \Gamma_i, \\ \eta_i = \mathbf{u}_1 - \mathbf{u}_2 & \text{on } \Gamma_0. \end{cases}$$

Step 3. Update $\mathbf{g}_{n+1} = \mathbf{g}_n - \alpha_n(\eta_1^n - \eta_2^n)$.

This implies that the operator Λ_n is defined by

$$\Lambda_n^{-1} = \mathcal{Q}^{-1} [\mathcal{S}_1(g_n) - \mathcal{S}_2(g_n)]^* \mathcal{P},$$

where the operators $\mathcal{S}_i(g_n)$ are defined by $\mathcal{S}_i(g_n)\tilde{g} = \mu_i|_{\Gamma_0}$, with μ_i solving the linear Stokes-type problem given by (6.6). The implementation can be further simplified by removing the second step and letting $\eta_i^n = \mathbf{u}_i^n$; then the iteration will involve only the solution of nonlinear subdomain problems. The convergence of the above iterations as well as more systematic (both theoretical and computational) studies of using general modified gradient as well as preconditioned conjugate gradient algorithms for the construction of optimization-based domain decomposition methods for nonlinear problems will be discussed in future works.

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