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On Ginzburg–Landau Vortices of Superconducting Thin Films

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Abstract In this paper, we discuss the vortex structure of the superconducting thin films placed in a magnetic field. We show that the global minimizer of the functional modelling the superconducting thin films has a bounded number of vortices when the applied magnetic field $h_{ex} < H_{c_1} + K \log|\log \varepsilon|$ where H_{c_1} is the lower critical field of the film obtained by Ding and Du in *SIAM J. Math. Anal.*, 2002. The locations of the vortices are also given.

Keywords Superconducting thin films, Global minimizer, Vortices, Pinning mechanism
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1 Introduction

Consider a three-dimensional superconducting thin film that is symmetric with respect to the (x_1, x_2)-plane. Thus the superconducting domain is given by $\Omega_\delta = \Omega \times (-\delta a(x), \delta a(x))$, where $a(x)$ is a given smooth function in $\overline{\Omega}$ and $a(x) \geq a_0 > 0$ for all $x \in \overline{\Omega}$ and Ω is a bounded smooth domain in $\mathbb{R}^2$. Making integral averaging along the direction of x_3-axis and sending δ to zero, Chapman and Du [1] modeled this three-dimensional Ginzburg–Landau model of superconductivity as a two-dimensional one by minimizing the following functional in $H^1(\Omega, \mathbb{R}^2)$

$$J_a(u) = \frac{1}{2}\int_\Omega a(x)\left[|\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right], \tag{1.1}$$

where $\mathbf{A}_0(x)$ is a given smooth magnetic potential vector, $\mathbf{A}_0(x) = (A_0^1(x), A_0^2(x))$ (or denoted as a 1-form $\mathbf{A}_0(x) = A_0^1(x)dx_1 + A_0^2(x)dx_2$), which satisfies

$$\operatorname{div}(a(x)\mathbf{A}_0) = 0, \quad \mathbf{curl}\mathbf{A}_0 = h_{ex}, \quad \text{in } \Omega, \tag{1.2}$$

and

$$\mathbf{A}_0 \cdot \mathbf{n} = \mathbf{0}, \quad \text{on } \partial\boldsymbol{\Omega}, \tag{1.3}$$

where h_{ex} is the applied field which is applied vertically to the (x_1, x_2)-plane, and $\mathbf{n}$ denotes the outward normal to $\partial\Omega$, $\nabla_{\mathbf{A}_0} u = \nabla u - i\mathbf{A}_0 u$, u is the complex superconducting order parameter such that $|u|^2$ represents the number density of superconducting electrons ($|u| = 1$ corresponds to the superconducting state, $|u| = 0$ corresponds to the normal state), $\kappa = \frac{1}{\varepsilon}$ is a material parameter which determines the type of superconducting material; $\kappa < \frac{1}{\sqrt{2}}$ describes a type-I superconductor, and $\kappa > \frac{1}{\sqrt{2}}$ describes a type-II superconductor.

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Let u be a critical point of the functional J_a. u satisfies the Euler–Lagrange equation which is also called the simplified Ginzburg–Landau equation

$$\begin{cases} -(\nabla - i\mathbf{A}_0)\cdot a(x)(\nabla u - i\mathbf{A}_0 u) = \dfrac{a(x)}{\varepsilon^2}u(1-|u|^2), & \text{in } \Omega \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In [2], by finding the local minimizers in

$$D_M^a = \left\{ u \in H^1(\Omega, \mathbb{R}^2) : \ F_a(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 + \frac{a(x)}{4\varepsilon^2}(1-|u|^2)^2 < M|\ln\varepsilon| \right\}, \tag{1.5}$$

we found the lower critical magnetic field (where "critical" means the first phase transition in which vortex appears in the superconductor)

$$H_{c_1} = k_a|\ln\varepsilon| + O(1), \tag{1.6}$$

where $k_a = \frac{1}{2\max_\Omega |\xi_0(x)/a(x)|}$ with ξ_0 the solution of the following problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{a(x)}\nabla\xi_0\right) = -1, & \text{in } \ \Omega, \\ \xi_0 = 0, & \text{on } \ \partial\Omega. \end{cases} \tag{1.7}$$

Defining

$$\Lambda = \left\{ x \in \Omega, |\xi_0(x)/a(x)| = \max_{y\in\Omega} |\xi_0(y)/a(y)| \right\} \tag{1.8}$$

and assuming

$$\max_\Lambda a(x) < 2\min_\Lambda a(x), \tag{1.9}$$

we proved (in [2]) the following theorem:

Theorem 1.1 *There exist* $k_2^\varepsilon = O_\varepsilon(1)$ *and* $k_3^\varepsilon = o_\varepsilon(1)$, *such that*

$$H_{c_1} = k_a|\ln\varepsilon| + k_2^\varepsilon \tag{1.10}$$

and $\varepsilon_0 = \varepsilon_0(M) > 0$ *such that for* $\varepsilon < \varepsilon_0$, *the following holds*:

(i) *if* $h_{ex} \le H_{c_1}$, *a solution of* (*G.L*) *that is minimizing* $J_a(u)$ *in* D_M^a *exists, and satisfies* $\frac{1}{2} \le |u| \le 1$;

(ii) *if* $H_{c_1} + k_3^\varepsilon \le h_{ex} \le H_{c_1} + O_\varepsilon(1)$, *a solution of* (*G. L*) *that is minimizing* $J_a(u)$ *in* D_M^a *exists, it has a bounded positive number of vortices* a_i^ε *of degree one, such that*

$$\mathrm{dist}(a_i^\varepsilon, \Lambda) \to 0, \ \ \text{as } \varepsilon \to 0. \tag{1.11}$$

However, the assumption (1.9) seems unnatural for general cases of superconducting thin films.

Our aim of this paper is to rule out the assumption (1.9) and to prove that for any global minimizers, there is a bounded number of vortices only under the assumption that the applied field is near the first critical magnetic field H_{c_1}.

Our main results are the following two theorems:

Theorem A *Let* $\ln a(x)$ *be a real analytic function. For any* $K > 0$, *there exist* $\varepsilon_0 > 0$ *and* $M > 0$ *such that for* $h_{ex} \le H_{c_1} + K\log|\log\varepsilon|$, $\varepsilon < \varepsilon_0$, *the global minimizer of* (1.1) *is gauge equivalent to an element in* D_M^a.

Theorem B *Let* $\ln a(x)$ *be a real analytic function. For any* $K > 0$, *there exist* $\varepsilon_0 > 0$, $C > 0$ *and* $\alpha > 0$ *such that for* $h_{ex} \le H_{c_1} + K\log|\log\varepsilon|$, $\varepsilon < \varepsilon_0$, *the vortices* (a_i, d_i) *of the global minimizer of* (1.1) *have the following properties*:

(1) $d_i \ge 0, \forall i$;

(2) $\sum d_i \le C$;

(3) $\mathrm{dist}(a_i, \Lambda) \le C|\log\varepsilon|^{-\alpha}$.

The main idea in the proof is to give much more careful analysis on the lower bounds of the energy. Compared with [10], because of the freezing of the variable $\mathbf{A}$, we no longer have the London equation which is crucial in the proof of [3]. We deal with this difficulty as in [4] by introducing a function h (see Section 2). The ideas in [3] are also used.

2 Lower Bound of the Energy

First of all, we give the definition of vortices, this definition is somewhat different from that in [5], see [3, 6–10] or [2, 4]. Modifying the proof of [3, Proposition 1] (see also [11], [12]), we can prove

Proposition 2.1 *For any $K > 0$, there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $\varepsilon < \varepsilon_0$, $h_{ex} < K|\log\varepsilon|$ and any u satisfying $J_a(u) < K|\log\varepsilon|^2$, there exist a finite family of disjoint balls $B_i = B(a_i, r_i)$, $i \in I$ such that*

$$(1)\quad \left\{x : |u(x)| < 1 - \frac{1}{|\log\varepsilon|^2}\right\} \subset \cup_i B_i, \tag{2.1}$$

$$(2)\quad \sum_{i\in I} r_i < \frac{1}{|\log\varepsilon|^{10}}, \tag{2.2}$$

$$(3)\quad \frac{1}{2}\int_{B_i} a(x)\rho^2|\nabla\varphi - A_0|^2 \geq \pi a(a_i)|d_i|(|\log\varepsilon| - C\log|\log\varepsilon|) - o(1), \tag{2.3}$$

where a_i is the center of the vortices and r_i is the radius, d_i is the winding number of $\frac{u}{|u|}$ on the boundary of B_i.

Remark Here and in the following, $o(1)$ means $\lim_{\varepsilon\to 0} o(1) = 0$.

This proposition gives the definition of the vortex. In this definition, the vortex is slightly different from that in [5], in this definition the radius of vortex is larger than that one in [5].

Proposition 2.2 (Lower bound for the energy of critical points) *For any $K > 0$, there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $\varepsilon < \varepsilon_0$, $h_{ex} < K|\log\varepsilon|$, if u is a critical point of $J_a(u)$ satisfying $J_a(u) < K|\log\varepsilon|^2$, and (a_i, d_i) are the vortices defined in Proposition* 2.1*, then there holds*:

$$\begin{aligned} J_a(u) \geq{}& h_{ex}^2 J_0 + \pi\sum_i a(a_i)|d_i|(|\log\varepsilon| - C\log|\log\varepsilon|) \\ &+\frac{1}{2}\int_{\tilde\Omega} \frac{1}{a(x)}|\nabla(h - h_{ex}\xi_0)|^2 + 2\pi h_{ex}\sum_i d_i\xi_0(a_i) + o(1). \end{aligned} \tag{2.4}$$

where $J_0 = J_a(1)/h_{ex}^2 = \frac{1}{2}\int_\Omega \frac{|\nabla\xi_0|^2}{a(x)}$, h is so chosen such that

$$-\frac{1}{a(x)}\nabla^\perp h = (iu, \nabla_{A_0}u), \tag{2.5}$$

$\xi_0(x)$ is the unique solution of (1.7).

Such a function in (2.5) can be found by the unique solution of the following problem:

$$-\mathrm{div}\left(\frac{1}{a(x)}\nabla h\right) = \mathbf{curl}(iu, \nabla u) - \mathbf{curl}(A_0|u|^2) \tag{2.6}$$

with $h = h_{ex}$ on the boundary.

In order to prove this proposition, we introduce some lemmas. Denote $\tilde\Omega = \Omega \setminus \cup_i B_i$. As in [4], it is not difficult to prove the following lemmas:

Lemma 2.1 *For u as in Proposition* 2.2*, then one may find $\tilde u$ such that*

(1) *$|\tilde u| = 1$ on $\tilde\Omega$, $\tilde u$ and $\frac{u}{|u|}$ have the same degree on ∂B_i, for $i \in I$;*

(2) *Denote $\tilde u = \tilde\rho e^{i\tilde\varphi}$, $u = \rho e^{i\varphi}$. Then $\|\rho(\nabla\varphi - A_0) - \tilde\rho(\nabla\tilde\varphi - A_0)\|_{L^2(\Omega)} = o(1)$;*

(3) *$J_a(\tilde u) \leq J_a(u) + o(1)$.*

Lemma 2.2 *For any $p < 2$ we have*

$$\left\| - \mathrm{div}\left(\frac{1}{a(x)}\nabla h\right) + h_{ex} - 2\pi\sum_i d_i\delta_{a_i}\right\|_{W^{-1,p}(\Omega)} \leq o(1). \tag{2.7}$$

Proof First of all, it follows from the definition that

$$\begin{aligned}-\mathrm{div}\Big(\frac{1}{a(x)}\nabla h\Big) &= \mathbf{curl}(iu,\nabla_{A_0}u) = \mathbf{curl}(iu,\nabla u) - \mathbf{curl}(A_0|u|^2)\\ &= \mathbf{curl}(iu,\nabla u) - \mathbf{curl}A_0 + \mathbf{curl}(A_0(1-|u|^2))\\ &= \mathbf{curl}(iu,\nabla u) - h_{ex} + \mathbf{curl}(A_0(1-|u|^2)),\end{aligned} \tag{2.8}$$

since $h_{ex} = \mathbf{curl}A_0$.

By Lemma 2.1, we may assume $|u| = 1$ on $\tilde{\Omega}$. Let q be such that

$$\frac{1}{q} + \frac{1}{p} = 1, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{2}. \tag{2.9}$$

For any $\xi \in W_0^{1,q}(\Omega)$, we have (noting that $\sum_i r_i \le \frac{1}{|\log\varepsilon|^{10}}$)

$$\begin{aligned}\Big|\int_{\cup_i B_i} \nabla^\perp\xi\cdot(iu,\nabla u)\Big| &\le \|\nabla u\|_{L^2(\Omega)}\|\nabla\xi\|_{L^q(\Omega)}(|\cup_i B_i|)^{\frac{1}{q'}} \le Ch_{ex}\|\nabla\xi\|_{L^q(\Omega)}\Big(\sum_i r_i\Big)^{\frac{1}{q'}}\\ &\le o(1)\|\nabla\xi\|_{L^q(\Omega)}.\end{aligned} \tag{2.10}$$

Integrating by parts, we have

$$\int_\Omega \xi\mathbf{curl}(iu,\nabla u) = -\int_\Omega \nabla^\perp\xi(iu,\nabla u). \tag{2.11}$$

Hence, it follows from above inequality that

$$\Big\|\int_\Omega \xi\mathbf{curl}(iu,\nabla u) + \int_{\tilde{\Omega}} \nabla^\perp\xi(iu,\nabla u)\Big\|_{W^{-1,p}(\Omega)} \le o(1). \tag{2.12}$$

On the other hand, since $|u| = 1$ on $\tilde{\Omega}$, then $\mathbf{curl}(iu,\nabla u) \equiv 0$ on $\tilde{\Omega}$. It follows from Stokes Theorem that

$$\int_{\tilde{\Omega}} \nabla^\perp\xi\cdot(iu,\nabla u) = \int_{\partial\tilde{\Omega}} \xi\Big(iu,\frac{\partial u}{\partial\tau}\Big) = \sum_i \int_{\partial B_i\cap\Omega} \xi\Big(iu,\frac{\partial u}{\partial\tau}\Big) \tag{2.13}$$

and

$$\Big|\sum_i \int_{\partial B_i\cap\Omega} (\xi-\xi(a_i))\Big(iu,\frac{\partial u}{\partial\tau}\Big)\Big| = \Big|\sum_i \int_{B_i} \nabla^\perp\xi\cdot(iu,\nabla u) + (\xi-\xi(a_i))\mathbf{curl}(iu,\nabla u)\Big|. \tag{2.14}$$

Noting that $q > 2$, $W_0^{1,q}(\Omega)$ embeds in some $C^{0,\gamma}$ with $0 < \gamma < 1$, and $|\mathbf{curl}(iu,\nabla u)| \le |\nabla u|^2$, then

$$\begin{aligned}\sum_i \int_{B_i} |(\xi-\xi(a_i))\mathbf{curl}(iu,\nabla u)| &\le (\min_i r_i)^\gamma\|\xi\|_{C^{0,\gamma}}\sum_i\int_{B_i}|\nabla u|^2\\ \le (\min_i r_i)^\gamma\|\xi\|_{W_0^{1,q}}\|\nabla u\|^2_{L^2(\Omega)} &\le Ch_{ex}^2(\min_i r_i)^\gamma\|\xi\|_{W_0^{1,q}} \le \frac{C}{|\log\varepsilon|^8}\|\xi\|_{W_0^{1,q}(\Omega)}.\end{aligned} \tag{2.15}$$

Therefore we have

$$\Big|\int_{\tilde{\Omega}} \nabla^\perp\xi\cdot(iu,\nabla u) - 2\pi\sum_i d_i\xi(a_i)\Big| \le o(1)\|\xi\|_{W_0^{1,q}(\Omega)}. \tag{2.16}$$

In summary, we have proved

$$\Big\|\mathbf{curl}(iu,\nabla u) - 2\pi\sum_i d_i\xi(a_i)\Big\|_{W_0^{-1,p}(\Omega)} \le o(1). \tag{2.17}$$

Similarly we have

$$\Big|\int_\Omega \xi\mathbf{curl}((1-|u|^2)A_0)\Big| = \Big|\int_\Omega \nabla^\perp\xi((1-|u|^2)A_0)\Big| \le o(1)\|\nabla\xi\|_{L^2(\Omega)}. \tag{2.18}$$

It follows from $\|A_0\|_{L^\infty} \le Ch_{ex}$ that

$$\|\mathbf{curl}((1-|u|^2)A_0)\|_{W^{-1,p}(\Omega)} \le o(1). \tag{2.19}$$

But, by (2.8), we have

$$-\mathrm{div}\Big(\frac{1}{a(x)}\nabla h\Big) + h_{ex} = \mathbf{curl}(iu,\nabla u) - \mathbf{curl}((1-|u|^2)A_0) \tag{2.20}$$

We may end the proof of this lemma by multiplying (2.20) by ξ.

Now we are in a position to prove Proposition 2.2.

Proof Since

$$-\frac{1}{a(x)}\nabla^{\perp}h=(iu,\nabla_{A_0}u)=\rho^2(\nabla\varphi-A_0),\tag{2.21}$$

then

$$\frac{1}{a^2(x)}|\nabla h|^2\le\rho^4|\nabla\varphi-A_0|^2+|\nabla\rho|^2\le|\nabla u-iA_0u|^2.\tag{2.22}$$

Hence

$$J_a(u)=\frac{1}{2}\int_{\Omega}a(x)|\nabla u-iA_0u|^2+\frac{1}{2\varepsilon^2}a(x)(1-|u|^2)^2\ge J_a(u,\cup_iB_i)+\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla h|^2.\tag{2.23}$$

However, it follows from Proposition 2.1 that

$$J_a(u,\cup_iB_i)\ge\pi\sum_i a(a_i)|d_i|(|\log\varepsilon|-C\log|\log\varepsilon|)-o(1).\tag{2.24}$$

Let $f=h-h_{ex}\xi_0$. Then

$$\begin{aligned}\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla h|^2&=\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla f+h_{ex}\nabla\xi_0|^2=\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}[|\nabla f|^2+h_{ex}^2|\nabla\xi_0|^2+2h_{ex}\nabla f\cdot\nabla\xi_0]\\&=\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla f|^2+\frac{1}{2}\int_{\tilde\Omega}\frac{h_{ex}^2}{a(x)}|\nabla\xi_0|^2+\int_{\tilde\Omega}\frac{h_{ex}}{a(x)}\nabla f\cdot\nabla\xi_0\\&\ge h_{ex}^2J_0+\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla(h-h_{ex}h_0)|^2+\int_{\tilde\Omega}\frac{h_{ex}}{a(x)}\nabla f\cdot\nabla\xi_0\end{aligned}\tag{2.25}$$

Substituting (2.24) and (2.25) in (2.23) we get

$$\begin{aligned}J_a(u)&\ge h_{ex}^2J_0+\pi\sum_i a(a_i)|d_i|(|\log\varepsilon|-C\log|\log\varepsilon|)\\&\quad+\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla(h-h_{ex}h_0)|^2+\int_{\Omega}\frac{h_{ex}}{a(x)}\nabla f\cdot\nabla\xi_0+o(1).\end{aligned}\tag{2.26}$$

For the fourth term on the right hand side of (2.26), we have from Lemma 2.2 that

$$\begin{aligned}\int_{\Omega}\frac{1}{a(x)}\nabla f\cdot\nabla\xi_0&=-\int_{\Omega}\xi_0\mathrm{div}\Big(\frac{1}{a(x)}\nabla f\Big)=\int_{\Omega}\Big[-\mathrm{div}\Big(\frac{1}{a(x)}\nabla h\Big)+h_{ex}\Big]\xi_0\\&=2\pi\sum_i d_i\xi_0(a_i)+o(1).\end{aligned}\tag{2.27}$$

We finally get

$$\begin{aligned}J_a(u)&\ge h_{ex}^2J_0+\pi\sum_i a(a_i)|d_i|(|\log\varepsilon|-C\log|\log\varepsilon|)\\&\quad+\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla(h-h_{ex}\xi_0)|^2+2\pi h_{ex}\sum_i d_i\xi_0(a_i)+o(1).\end{aligned}\tag{2.28}$$

Proposition 2.2 is proved.

Lemma 2.3 *Let u be a minimizer of $J_a(u)$. Then*

$$\pi\sum_i a(a_i)|d_i|(|\log\varepsilon|-C\log|\log\varepsilon|)+\frac{1}{2}\int_{\tilde\Omega}\frac{1}{a(x)}|\nabla(h-h_{ex}\xi_0)|^2+2\pi h_{ex}\sum_i d_i\xi_0(a_i)\le o(1).\tag{2.29}$$

Proof Since u is a minimizer, we have $J_a(u)\le J_a(1)=h_{ex}^2J_0$. This combined with the lower bound in Proposition 2.2 yields the conclusion.

3 Proof of Theorem B

Now we are in a position to prove Theorem B.

Firstly, we prove that the vortices are close to the set Λ. This is a consequence of the following lemma which implies (3) of Theorem B (see [3]).

Lemma 3.1 *Assume that* $\ln a(x)$ *is an analytic function. Then the set* Λ*, the critical points of* $\frac{\xi_0(x)}{a(x)}$ *is a finite set of points* $\{p_1, p_2, \dots, p_n\}$*. In particular, the set* Λ *where* $\frac{\xi_0(x)}{a(x)}$ *attains its minumum is finite and there exist* $\delta > 0$, $N > 0$ *such that for every* $p \in \Omega$

$$\frac{\xi_0(p)}{a(p)} \geq \min_{\Omega} \frac{\xi_0(x)}{a(x)} + \delta \mathrm{dist}(p, \Lambda)^N. \tag{3.1}$$

Proof According to the proof of Lemma 4 of [3], it suffices to prove that (1.7) admits a solution ξ_0 such that $\frac{\xi_0(x)}{a(x)}$ is analytic. Let $\xi = \frac{\xi_0(x)}{a(x)}$. It is not difficult to see that ξ_0 is the solution of (1.7) if and only if $\xi(x)$ is the solution of the following equation:

$$-\Delta \xi - \nabla \xi \cdot \nabla \ln a(x) - \xi \Delta \ln a(x) = 0, \text{ in } \Omega \tag{3.2}$$

with $\xi = 0$ on $\partial\Omega$. Under the assumption of Lemma 3.1, Equation (3.2) with the zero Dirichlet boundary condition admits an analytic solution. In other words, under the assumption of Lemma 3.1, (1.7) admits a solution ξ_0 so that $\frac{\xi_0(x)}{a(x)}$ is analytic. Now we may use the proof of Lemma 4 of [3] to verify our claim.

Applying the maximum principle to problem (1.7), we know $\xi_0(x) < 0$ on Ω. So, denoting $m = \min_\Omega(\frac{\xi_0(x)}{a(x)})$, then it follows from $|\xi_0(x)| = -\xi_0(x)$ that $m = \min_\Omega(\frac{\xi_0(x)}{a(x)}) = -\max_\Omega \frac{|\xi_0(x)|}{a(x)} < 0$. We rewrite (2.29) as

$$\begin{aligned}\pi \sum_i a(a_i)|d_i|(|\log\varepsilon| - C\log|\log\varepsilon|) + \frac{1}{2}\int_{\tilde{\Omega}} \frac{1}{a(x)}|\nabla(h - h_{ex}\xi_0)|^2 \\ + 2\pi h_{ex} \sum_i d_i a(a_i) \frac{\xi_0(a_i)}{a(a_i)} \leq o(1).\end{aligned} \tag{3.3}$$

Denote $I_- = \{i \in I : d_i < 0\}$, $I_+ = I \setminus I_-$, $D = \sum_i a(a_i)|d_i|$, $D_+ = \sum_{d_i > 0} a(a_i)|d_i|$, $D_- = \sum_{d_i<0} a(a_i)|d_i|$. Then $D = D_+ + D_-$. If $d_i < 0$, then $d_i \frac{\xi_0(a_i)}{a(a_i)} > 0$. If $d_i > 0$, then $d_i \frac{\xi_0(a_i)}{a(a_i)} \geq -|d_i| \max_\Omega \frac{|\xi_0(x)|}{a(x)} = m|d_i|$. We get from (3.3)

$$\pi D(|\log\varepsilon| - C\log|\log\varepsilon|) + 2\pi m D_+ h_{ex} \leq o(1). \tag{3.4}$$

In the following, we prove the uniform boundedness of D.

Let C_t be a circle completely included in $\tilde{\Omega}$ with radius t and circling a ball B_t. On $\tilde{\Omega}$, $|u| = 1$, we may write $u = e^{i\varphi}$, then

$$-\frac{1}{a(x)}\nabla^\perp h = \nabla\varphi - A_0, \text{ in } \tilde{\Omega}. \tag{3.5}$$

We have

$$\int_{C_t} \frac{1}{a(x)}\frac{\partial \xi_0}{\partial \nu} = -\int_{B_t} dx, \tag{3.6}$$

$$\begin{aligned}\int_{C_t} \frac{1}{a(x)}\frac{\partial h}{\partial \nu} &= \int_{C_t} \frac{\partial\varphi}{\partial\tau} - A_0 \cdot \tau = \int_{C_t} \frac{\partial\varphi}{\partial\tau} - \int_{B_t} \mathbf{curl} A_0 = \int_{C_t} \frac{\partial\varphi}{\partial\tau} - h_{ex}\int_{B_t} dx \\ &= \int_{C_t} \frac{\partial\varphi}{\partial\tau} + h_{ex}\int_{C_t} \frac{1}{a(x)}\frac{\partial\xi_0}{\partial\nu}.\end{aligned} \tag{3.7}$$

Therefore,

$$\int_{C_t} \frac{1}{a(x)}\frac{\partial f}{\partial \nu} = \int_{C_t} \frac{\partial\varphi}{\partial\tau} = 2\pi d_t, \tag{3.8}$$

where d_t is the winding number of u on C_t. Let $a_0 = \min_\Omega a(x)$. We get by Hölder inequality

$$a_0 \frac{\pi}{2t} d_t^2 \leq \frac{1}{2}\int_{C_t} \frac{1}{a(x)}\left(\frac{\partial f}{\partial\nu}\right)^2. \tag{3.9}$$

It follows from Lemma 3.1 that $\Lambda = \{p_1, p_2, \dots, p_n\}$, a finite set and for some $0 < \alpha < 1$ there holds

$$\frac{|p_i - p_j|}{2} > \alpha, \quad \alpha < \mathrm{dist}(p_i, \partial\Omega). \tag{3.10}$$

Let $C_{i,t}$ be a circling centered at p_i with radius t and circling a ball $B_{i,t}$. Denote

$$E = \{0 < t < \alpha : \forall\, 1 \le i \le n, C_{i,t} \subset \tilde{\Omega}\}.$$

It follows from the above discussions that

$$a_0 \frac{\pi}{2t} d_{i,t}^2 \le \frac{1}{2} \int_{C_{i,t}} \frac{1}{a(x)} |\nabla f|^2. \tag{3.11}$$

Hence

$$a_0 \frac{\pi}{2t} \sum_i d_{i,t}^2 \le \frac{1}{2} \int_{\cup_i C_{i,t}} \frac{1}{a(x)} |\nabla f|^2. \tag{3.12}$$

Since a_i is close to Λ, we have from Lemma 3.1 that for $t \ge |\log\varepsilon|^{-\frac{1}{2N}}$,

$$\sum_i a(a_i)|d_{i,t}| = D(1 + o(1)) + o(1), \tag{3.13}$$

hence

$$\frac{1}{2} \int_{\cup_i C_{i,t}} \frac{1}{a(x)} |\nabla f|^2 \ge \frac{C}{t} D^2 + o(1). \tag{3.14}$$

On the other hand, $\sum_i r_i \le |\log\varepsilon|^{-10}$, then

$$\int_{\{t\in E, t \ge |\log\varepsilon|^{-\frac{1}{2N}}\}} \frac{dt}{t} \ge \int_{|\log\varepsilon|^{-\frac{1}{2N}} + |\log\varepsilon|^{-10}}^{\alpha} \frac{dt}{t}. \tag{3.15}$$

Then

$$\frac{1}{2} \int_{\tilde{\Omega}} \frac{1}{a(x)} |\nabla f|^2 \ge C D^2 \log|\log\varepsilon| + o(1). \tag{3.16}$$

We get

$$-CD \log|\log\varepsilon| + CD^2 \log|\log\varepsilon| \le o(1). \tag{3.17}$$

This implies that D is bounded uniformly in ε.

Finally, we prove that all the vortices have positive winding numbers. In fact, it follows from the fact $H_{c_1} = -\frac{|\log\varepsilon|}{2m}$, $h_{ex} < H_{c_1} + C\log|\log\varepsilon|$ and (3.4) that

$$\pi D(|\log\varepsilon| - C\log|\log\varepsilon|) - \pi D_+(|\log\varepsilon| + C\log|\log\varepsilon|) \le o(1). \tag{3.18}$$

Therefore,

$$D_- \le C D_+ \frac{\log|\log\varepsilon|}{|\log\varepsilon|} + o(1), \tag{3.19}$$

and then it follows from the bounedeness of D and $a(x) \ge \alpha > 0$ for some constant $\alpha > 0$ and any $x \in \overline{\Omega}$ that

$$D_- = 0 \tag{3.20}$$

for $\varepsilon < \varepsilon_0$, since D_- must be a nonnegative integer. In other words, $d_i \ge 0$ for all i and $\varepsilon < \varepsilon_0$. Theorem B is proved.

4 Proof of Theorem A

Now we give the proof of Theorem A.

Proof of Theorem A Since we have

$$J_a(u) = F_a(u) + \frac{1}{2} \int_\Omega a(x)|u|^2|A_0|^2 - 2(a(x)A_0 \cdot \nabla u, iu) \tag{4.1}$$

and

$$J_a(u) \le J_0 h_{ex}^2 = \frac{1}{2} \int_\Omega a(x)|A_0|^2, \tag{4.2}$$

then

$$F_a(u) \le \frac{1}{2} \int_\Omega a(x)(1 - |u|^2)|A_0|^2 + 2(a(x)A_0 \cdot \nabla u, iu). \tag{4.3}$$

However,

$$\frac{1}{2}\int_\Omega a(x)(1-|u|^2)|A_0|^2 = o(1); \tag{4.4}$$

we only need to prove that

$$\left|\int_\Omega (a(x)A_0\cdot\nabla u, iu)\right| \le C|\log\varepsilon|. \tag{4.5}$$

This inequality is true. In fact, we may assume that $|u| = 1$ on $\tilde{\Omega}$. Since $a(x)A_0 = \nabla^\perp\xi$ in Ω where $\xi = 0$ on $\partial\Omega$ and $\|\xi\|_{H^3(\Omega)} \le Ch_{ex}$, then

$$\left|\int_{\cup_i B_i} (a(x)A_0\cdot\nabla u, iu)\right| \le C|\cup_i B_i|^{1/4}\|A_0\|_{L^4(\Omega)}\|u\|_{H^1(\Omega)} = o(1); \tag{4.6}$$

while on $\tilde{\Omega}$, we denote $u = e^{i\varphi}$ to get

$$\int_{\tilde{\Omega}} (a(x)A_0\cdot\nabla u, iu) = \int_{\tilde{\Omega}} \nabla^\perp\xi\cdot\nabla\varphi = \sum_i\int_{\partial B_i}\xi\frac{\partial\varphi}{\partial\tau} = 2\pi\sum_i d_i\xi(a_i) + o(1). \tag{4.7}$$

In this inequality, $\|\xi\|_{L^\infty(\Omega)} \le Ch_{ex}$ and $\sum_i |d_i|$ is uniformly bounded, so we have

$$\left|\int_\Omega (a(x)A_0\cdot\nabla u, iu)\right| \le Ch_{ex} \le C|\log\varepsilon|. \tag{4.8}$$

This implies that u belongs to D^a_M for any $M > C$. Theorem A is proved.

Final Remark After this paper was submitted, there have been published some other related papers on the analysis of the behaviors of the minimizers to some Ginzburg–Landau models for the applied fields near the lower critical fields. We refer to the references [13]–[16]. The authors are grateful to the referee to introduce to them these references.

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