

CRITICAL MAGNETIC FIELD AND ASYMPTOTIC BEHAVIOR OF SUPERCONDUCTING THIN FILMS*

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Abstract. In this paper, we discuss the vortex structure of the superconducting thin films placed in a magnetic field. The discussion is based on a system of simplified Ginzburg–Landau equations. We obtain the estimate for the lower critical magnetic field H_{c_1} , in the sense that it is the first critical value of h_{ex} , the applied field, for which the minimal energy among vortexless configurations is equal to the minimal energy among single-vortex configurations; moreover, it corresponds to the first phase transition in which vortices appear in the superconductor. We also discuss the location of these vortices and the asymptotic behavior of the local minimizers.

Key words. superconductivity, thin films, vortices, pinning, critical magnetic field

AMS subject classifications. 35J55, 35Q40

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1. Introduction. Consider a three-dimensional superconducting thin film that occupies the domain $\Omega_\delta = \Omega \times (-\delta a, \delta a)$, where Ω is a bounded smooth planar domain and $a \in C^\infty(\bar{\Omega})$ is a function measuring the variation in the film thickness. Assume that $a(x) \geq a_0 > 0$ for all $x \in \bar{\Omega}$; by taking integral averages along the vertical direction and setting δ going to zero, it was shown in [10] that the three-dimensional Ginzburg–Landau model of superconductivity [16, 26] defined on Ω_δ may be reduced to a two-dimensional one given by the minimization in $H^1(\Omega)$ of the functional

$$(1.1) \quad J_a(u) = \frac{1}{2} \int_{\Omega} a(x) \left[|\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right],$$

where $\mathbf{A}_0(x)$, the in-plane component of the magnetic potential, is determined by

$$(1.2) \quad \begin{cases} \operatorname{div}(a(x)\mathbf{A}_0) = 0, & \operatorname{curl}\mathbf{A}_0 = h_{ex} \text{ in } \Omega, \\ \mathbf{A}_0 \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, h_{ex} is the external magnetic field which is applied vertically to the (x_1, x_2) -plane, \mathbf{n} denotes the outward normal to $\partial\Omega$, u is the complex superconducting order parameter with $|u|^2$ representing the density of superconducting electrons ($|u| = 1$ corresponds to the superconducting state, $|u| = 0$ corresponds to the normal state), $\nabla_{\mathbf{A}_0} u = \nabla u - i\mathbf{A}_0 u$, and ε is proportional to the coherence length.

Let u be a critical point of the functional $J_a(u)$ in $H^1(\Omega)$ which satisfies the

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Euler–Lagrange (or simplified Ginzburg–Landau) equation

$$(1.3) \quad \begin{cases} -(\nabla - i\mathbf{A}_0) \cdot a(x)(\nabla u - i\mathbf{A}_0 u) = \frac{a(x)}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0 & \text{on } \partial\Omega. \end{cases}$$

The points where the zeros of u appear, with their topological degrees, are called the vortices of the map u . Understanding the vortex structures in the solutions and describing the vortices as h_{ex} varies is of great physical relevance and mathematical interests. Discussions on the vortex state in the thin film geometry have been given in [1, 16, 17, 19, 20, 26]; in particular, the variation in the film thickness is thought to provide an effective vortex pinning mechanism [10]. For works related to the mathematical analysis of the various pinning mechanisms, we refer to [2, 3, 4, 6, 10, 11, 12, 15].

In [7, 8], rigorous mathematical analysis of vortex solutions has been done for a similar problem with $a(x) = 1$, $\mathbf{A}_0 = 0$ and Dirichlet boundary condition $u = g: \Omega \rightarrow S^1$ of degree d . It was proved that, asymptotically, minimizers have d isolated vortices of degree one and their locations are determined by minimizing a renormalized energy. This result was extended to the case $a(x) \neq 1$, $\mathbf{A}_0 = 0$ with the same Dirichlet boundary conditions in [6] and [15] independently, and the vortices of the minimizers were shown to be located at the minimum of $a(x)$. Some results similar to those in [7] were obtained in [9] for the original Ginzburg–Landau functional $J(u, \mathbf{A})$,

$$J(u, \mathbf{A}) = \frac{1}{2} \int_{\Omega} \left[|\nabla_{\mathbf{A}} u|^2 + |\mathbf{curl} \mathbf{A} - h_{ex}|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right],$$

with $h_{ex} = 0$ and the gauge invariant Dirichlet conditions (a name given in [22]). This work was later extended in [14] to the case where a weight (thickness) appears in the functional $J(u, \mathbf{A})$; the corresponding renormalized energy was presented in [13]. Similar analysis based on the functional (1.1) was also presented in [18]. All the available results substantiate the pinning effect of the thickness variation; that is, the vortices turn to stay where the film is thin.

Recently, the minimizers of $J(u, \mathbf{A})$ with nonzero applied fields with natural boundary conditions were studied in [5, 18, 23, 24, 25, 21, 22]. In this case, there is no a priori bound on the number of the vortices for the minimizers in $H^1 \times H^1$. To overcome this difficulty, i.e., to have an a priori control on the numbers of the vortices, in [23, 24], the local minimizers of the functional

$$J(u, \mathbf{A}) = \frac{1}{2} \int_{\Omega} \left[|\nabla_{\mathbf{A}} u|^2 + |\mathbf{curl} \mathbf{A} - h_{ex}|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right]$$

in the set $\overline{D_M}$ were studied, where

$$D_M = \{ (u, \mathbf{A}) \in H^1(\Omega) \times H^1(\Omega) : F(u) < M |\ln \varepsilon| \}$$

and $F(u) = J_1(u) = J(u, 0)$ with $\mathbf{A}_0 = 0$. The minimizers were shown not to be on the boundary of D_M , hence the Ginzburg–Landau equations (the Euler–Lagrange equations for the functional J) are satisfied. Such analysis also provided estimates on the lower critical magnetic field H_{c_1} , the locations of the vortices, and the asymptotic behaviors of the minimizers. The lower critical field H_{c_1} may be defined as the value of h_{ex} for which the minimal energy among vortexless configurations is equal to the

minimal energy among single-vortex configurations. For $h_{ex} \leq H_{c_1}$, it was shown in [21] that the global minimizer (in $H^1 \times H^1$) of Ginzburg–Landau functional $J(u, \mathbf{A})$ is the vortexless solution found in [23]. For the case $H_{c_1} \ll h_{ex} \ll H_{c_2}$, in [22], it was shown that as $\varepsilon \rightarrow 0$ the energy minimizers have vortices whose density tends to be uniform and proportional to h_{ex} . For other discussions, we refer the reader to [2] and [25] and references therein.

In this paper, we study the minimizers of the functional (1.1) in the set

$$(1.4) \quad D_M^a = \{u \in H^1(\Omega) : F_a(u) < M |\ln \varepsilon| \},$$

where $F_a(u) = J_a(u)$ with $\mathbf{A}_0 = 0$. The main techniques of this paper come from [23, 24]. We also present the estimate on the lower critical magnetic field H_{c_1} and discuss the impact of the thickness function $a(x)$ and the given applied field $\mathbf{curl} \mathbf{A}_0$ on the vortices: their number and their locations. These new results have not been stated even in the physics literature. Our results also provide rigorous theoretical justification of the pinning mechanism due to the thickness variation based on the simplified Ginzburg–Landau model.

Let us introduce a few notation. By (1.2), there is a function $\xi \in H^2(\Omega)$ such that

$$a(x) \mathbf{A}_0(x) = \nabla^\perp \xi = (-\xi_{x_2}, \xi_{x_1}) \text{ in } \Omega.$$

Using the scaling $\xi = \xi_0 h_{ex}$, we have from (1.2) that

$$(1.5) \quad \begin{cases} -\operatorname{div}(\frac{1}{a(x)} \nabla \xi_0) = -1 & \text{in } \Omega, \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, we may easily see that $-C \leq \xi_0 < 0$ for some constant $C > 0$ and ξ_0 is a smooth function that depends only on Ω and $a = a(x)$. Let

$$(1.6) \quad \Lambda = \left\{ x \in \Omega, |\xi_0(x)/a(x)| = \max_{y \in \Omega} |\xi_0(y)/a(y)| \right\}.$$

To state our main results, the following assumption is made.

ASSUMPTION 1.1. *Assume that the constant M in (1.4) is chosen so that there is a positive integer $n \in \mathbb{N}$ such that*

$$(1.7) \quad \left[\frac{M}{\pi \max_{\Lambda} a(x)}, \frac{M}{\pi \min_{\Lambda} a(x)} \right] \subset (n, n + 1).$$

The above assumption on the existence of $n \in \mathbb{N}$ with the desired property (1.7) is needed in proving (see section 6) that the minimizer of $J_a(u)$ in $\overline{D_M^a}$ is in D_M^a (not on ∂D_M^a) and thus the minimizer is a solution of (1.3). Under the above assumption, we have the following theorem.

THEOREM 1.1. *There exists $k_a = \frac{1}{2 \max_{\Omega} |\xi_0(x)/a(x)|}$, $k_2^\varepsilon = O(1)$, $k_3^\varepsilon = o(1)$, and $\varepsilon_0 = \varepsilon_0(M) > 0$ such that*

$$(1.8) \quad H_{c_1} = k_a |\ln \varepsilon| + k_2^\varepsilon,$$

and, for $\varepsilon < \varepsilon_0$, the following holds:

(i) If $h_{ex} \leq H_{c1}$, there exists a solution u_ε of (1.3) which minimizes $J_a(u)$ in D_M^a , and it satisfies $1/2 \leq |u_\varepsilon| \leq 1$.

(ii) If $H_{c1} + k_3^\varepsilon \leq h_{ex} \leq H_{c1} + O(1)$, there exists a solution u_ε of (1.3) that minimizes $J_a(u)$ in D_M^a . The solution has a bounded positive number of vortices b_i^ε of degree one such that

$$(1.9) \quad \text{dist}(b_i^\varepsilon, \Lambda) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

and there exists a constant $\alpha > 0$ such that $\text{dist}(b_i^\varepsilon, b_j^\varepsilon) \geq \alpha$ for $i \neq j$.

REMARK 1.1. The main differences between our results and those in [23, 24] are as follows: first, A_0 is determined a priori, and it satisfies (1.2) and $\text{curl} A_0(x) = O(|\ln \varepsilon|)$ so that no London type equation is used; second, with a variable weight $a = a(x)$ in the functional, methods developed in [6] (see also [13]) and in [23] are needed to derive the energy lower bound.

REMARK 1.2. It follows from the proof of Theorem 1.1 that the number of the vortices, under our assumption, is bounded by

$$N = \min \left\{ \frac{M}{\pi \max_\Lambda a(x)}, \frac{\min_\Lambda a(x)}{\max_\Lambda a(x) - \min_\Lambda a(x)} \right\}.$$

REMARK 1.3. From (1.6) and (1.9), one may conclude that the distribution of the vortex locations are influenced both by the pinning effect due to thickness variation and the effect of the applied magnetic field. A similar phenomenon has also been explored in [2] with normal inclusion serving as pinning sites.

Let us discuss briefly Assumption 1.1 and the results of Theorem 1.1. The parameter M in (1.4) is chosen such that

$$\left[M/(\pi \max_\Lambda a(x)), M/(\pi \min_\Lambda a(x)) \right] \subset (n, n + 1)$$

for some positive integer n . For Λ defined by (1.6), it is easy to see that the above assumption can be equivalently replaced by

$$(1.10) \quad \max_\Lambda a(x) < 2 \min_\Lambda a(x).$$

Note that if Λ consists of only one point, or if $a(x)$ satisfies

$$\max_{\bar{\Omega}} a(x) < 2 \min_{\bar{\Omega}} a(x),$$

then (1.10) is automatically satisfied. With suitable choices of the domain Ω and the coefficient $a(x)$, it is indeed possible to make Λ a single point. A couple of simple examples are in order; let $\Omega = B(0, R_0)$ be a two-dimensional disc of radius R_0 and $r = |x|$ for $x \in \Omega$. Let $v(x) = \xi_0(x)/a(x)$; (1.5) for ξ_0 may be rewritten, in the polar coordinate system, as

$$(1.11) \quad \begin{cases} v''(r) + v'(r)/r + v'(r)(\ln a(r))' + v(r)\Delta \ln a(r) = 1 & \text{in } (0, R_0), \\ v'(0) = 0, \quad v(R_0) = 0, \end{cases}$$

where $v(r)$ and $a(r)$ represent the functions v and a in the polar coordinates.

Example 1. If $a(r) = e^{-\frac{r^2}{4}}$, then $\Lambda = \{0\}$.

Since $\Delta \ln a(r) = -1$, $(\ln a(r))' = -r/2$, we have

$$v''(r) + (1/r - r/2)v'(r) - v(r) = 1 \text{ in } (0, R_0).$$

One may verify that $v(r) = z(r)/z(R_0) - 1$ is nonpositive in $[0, R_0]$, where

$$z(r) = r^2 + r^3 + \sum_{n=0}^{+\infty} \left(\frac{1}{4^{n+1}(n+2)!} r^{2(n+2)} + \frac{1}{2^{n+1}(2n+5)!!} r^{2n+5} \right).$$

$v(r)$ is a solution of problem (1.11). Since $z(r)$ is strictly increasing in $[0, R_0]$, so is $v(r)$. We know $|v(r)|$ takes its maximum value only at 0, that is, $\Lambda = \{0\}$.

Example 2. If $a(r) = 2(1+r)$, then $\Lambda = \{0\}$.

In fact, let $\xi_0(r) = \frac{1}{3}r^3 + \frac{1}{2}r^2 - (\frac{1}{3}R_0^3 + \frac{1}{2}R_0^2)$. Then $\xi_0(r)$ is nonpositive in $[0, R_0]$, $\xi_0(R_0) = 0$, and ξ_0 is a solution of (1.11). $\xi_0 = \xi_0(r)$ is strictly increasing in $[0, R_0]$, so is $a = a(r)$. Therefore $|\xi_0(r)|/a(r)$ takes its maximum value only at $\{0\}$, so $\Lambda = \{0\}$.

For both of the above examples, depending on R_0 , the thickness function a may take on values of different magnitude at different locations in the domain $B(0, R_0)$. It is interesting to note that the coefficient $a(x)$ takes its minimum value at the boundary in Example 1 but at the origin in Example 2. Based on the analysis given in this paper, near H_{c_1} , the solution of (1.3) with a single vortex in Ω will have its vortex pinned near the origin in both cases for small enough ε even though the origin is the thickest position in Example 1. This illustrates that the vortex pinning phenomenon may be affected by the competition between the applied field and the thickness variation.

We now state the second main theorem.

THEOREM 1.2. *For a solution sequence $u_n = u_{\varepsilon_n}$ of (1.3) given by the part (ii) of Theorem 1.1, up to a subsequence, there exist d points $c_i \in \Lambda$ such that $u_n \rightarrow u_*$ weakly in $W^{1,p}$ ($p < 2$) and strongly in $H^1_{\text{loc}}(\Omega \setminus \cup_{i=1}^d \{c_i\})$, where u_* is a solution of*

$$(1.12) \quad \begin{cases} -\nabla \cdot (a(x)\nabla u_*) = a(x)u_*|\nabla u_*|^2 \text{ in } \Omega \setminus \cup_{i=1}^d \{c_i\}, \\ \frac{\partial u_*}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ |u_*| = 1 & \text{a.e. on } \Omega. \end{cases}$$

It is easy to see that the local minimizers in $\overline{D_M^a}$ may not be the solution of (1.3) (if it is on the boundary of $\overline{D_M^a}$). However, the vortex structure is only well defined for solutions that satisfy $|\nabla u| \leq C/\varepsilon$. For this reason, similar to [23], we introduce a regularization as follows.

Let $u_\varepsilon^\gamma \in H^1(\Omega, \mathbb{R}^2)$ be a minimizer of the following minimization problem:

$$(1.13) \quad \min_{v \in H^1(\Omega, \mathbb{R}^2)} \left\{ \int_\Omega a(x) \left[\frac{1}{2} |\nabla v|^2 + \frac{1}{4\varepsilon^2} (1 - |v|^2)^2 \right] + \int_\Omega \frac{|v - u_\varepsilon|^2}{2\varepsilon^{2\gamma}} \right\},$$

where $u_\varepsilon \in \overline{D_M^a}$. u_ε^γ is, in some sense, a regularization of u_ε in $\overline{D_M^a}$ and an a priori bound on the number of the vortices of u_ε^γ can be obtained. This in turn leads to a description of the vortices of u_ε . More careful examination of the minimizers u_ε of $J_a(u)$ in $\overline{D_M^a}$ shows that they are actually not on the boundary of $\overline{D_M^a}$, and hence they solve (1.3). For brevity, in the rest of the paper, unless explicitly stated to avoid ambiguity, the subscript ε is dropped from the notation u_ε and u_ε^γ ; i.e., u and u^γ are used instead.

This paper is organized as follows. In the next section we shall give some basic estimates for $J_a(u)$ and for the regularization u^γ . The main ideas are to define the vortices of u^γ and to expand the energy $J_a(u)$. Using the idea of [23] and the estimate in [6], we may then give the lower bound for the energy. In section 3, we shall provide estimates to the critical magnetic field. In section 4, the proof Theorem 1.1 is given, and in section 5 we shall prove the convergence of the sequence of the minimizers, i.e., Theorem 1.2.

In the following discussion, we always consider the case $h_{ex} \leq C|\ln \varepsilon|$ for some positive constant C and assume that the Abrikosov estimate $H_{c1} \leq C|\ln \varepsilon|$ holds.

2. Preliminaries. In this section we present technical estimates which can be proved by a slight modification of the results in [6, 23]. The detailed proofs are omitted. We begin by defining

$$(2.1) \quad J^0 = J_a(1) = \frac{1}{2} \int_{\Omega} \frac{1}{a(x)} |\nabla \xi|^2 \leq Ch_{ex}^2.$$

LEMMA 2.1. *For $u \in \overline{D_M^a}$ minimizing $J_a(u)$ in $\overline{D_M^a}$, we have*

$$(2.2) \quad J_a(u) \leq Ch_{ex}^2,$$

$$(2.3) \quad \int_{\Omega} a(x) |\nabla_{\mathbf{A}_0} u|^2 \leq Ch_{ex}^2,$$

$$(2.4) \quad \frac{1}{4\varepsilon^2} \int_{\Omega} a(x) (1 - |u|^2)^2 \leq Ch_{ex}^2.$$

Proof. Taking $v \equiv 1$ as a comparison function leads to the results. \square

For any \tilde{u} with $J_a(\tilde{u}) \leq Ch_{ex}^2$, let $\eta = |\tilde{u}|$. Since

$$a(x) |\nabla u - i\mathbf{A}_0 u|^2 = a(x) [|\nabla u|^2 + i\mathbf{A}_0 (u^* \nabla u - u \nabla u^*) + |\mathbf{A}_0|^2 |u|^2],$$

where u^* is the complex conjugate of u , we have the following lemma.

LEMMA 2.2. *For any \tilde{u} with $J_a(\tilde{u}) \leq Ch_{ex}^2$, we have*

$$J_a(\tilde{u}) = F_a(\tilde{u}) + \frac{1}{2} \int_{\Omega} \frac{1}{a(x)} |\nabla \xi|^2 + \int_{\Omega} (i\tilde{u}, \xi_{x_2} \tilde{u}_{x_1} - \xi_{x_1} \tilde{u}_{x_2}) + o(1).$$

LEMMA 2.3. *For $\tilde{u} \in \overline{D_M^a}$ such that $J_a(\tilde{u}) \leq Ch_{ex}^2$, there exists $u \in \overline{D_M^a}$ such that*

$$(2.5) \quad |u| \leq 1,$$

$$(2.6) \quad F_a(u) \leq F_a(\tilde{u}),$$

$$(2.7) \quad J_a(u) \leq J_a(\tilde{u}) + o(1).$$

If, in addition, $\tilde{u} \in \overline{D_M^a}$ is a minimizer of J_a in $\overline{D_M^a}$, then, as $\varepsilon \rightarrow 0$, there holds

$$(2.8) \quad F_a(u) = F_a(\tilde{u}) + o(1),$$

$$(2.9) \quad J_a(u) = J_a(\tilde{u}) + o(1).$$

LEMMA 2.4. *For $u \in \overline{D_M^a}$, we have $u^\gamma \in H^3(\Omega)$ (for any $0 < \gamma < 1$) which solves (1.13) and satisfies*

$$(2.10) \quad -\nabla \cdot (a(x) \nabla u^\gamma) = \frac{a(x)}{\varepsilon^2} u^\gamma (1 - |u^\gamma|^2) + \frac{u - u^\gamma}{\varepsilon^{2\gamma}},$$

$$(2.11) \quad F_a(u^\gamma) \leq F_a(u) \leq M |\ln \varepsilon|,$$

$$(2.12) \quad |u^\gamma| \leq 1, |\nabla u^\gamma| \leq \frac{C}{\varepsilon}.$$

This implies that $u^\gamma \in \overline{D_M^a}$. Taking u as a comparison function in (1.13) gives

$$\int_\Omega \frac{1}{2\varepsilon^{2\gamma}} |u - u^\gamma|^2 + F_a(u^\gamma) \leq F_a(u) \leq M |\ln \varepsilon|$$

so that $\|u - u^\gamma\|_{L^2(\Omega)} \leq C\varepsilon^\gamma |\ln \varepsilon|^{\frac{1}{2}}$. Since $|\nabla u^\gamma| \leq \frac{C}{\varepsilon}$, the vortices are well defined in the following sense.

LEMMA 2.5. *There exists $\lambda > 0$ and points a_i^ε ($i \in \mathcal{J}_1$) in Ω with $\text{Card}\mathcal{J}_1 \leq Ch_{ex}^2$ such that*

$$|u^\gamma| \geq \frac{1}{2} \quad \text{in } \Omega \setminus \cup_{i \in \mathcal{J}_1} B(a_i^\varepsilon, \lambda\varepsilon).$$

Proof. We know from [7] that there exists $\mu_0 > 0$ such that

$$\frac{1}{\varepsilon^2} \int_{B(a_i^\varepsilon, \lambda\varepsilon)} (1 - |u^\gamma|^2)^2 \geq \mu_0 \quad \forall i \in \mathcal{J}_1.$$

Using exactly the same arguments given in [7], this implies that $\text{Card}\mathcal{J}_1 \leq Ch_{ex}^2$ since $J_a(u) \leq Ch_{ex}^2$. \square

The balls $B(a_i^\varepsilon, \lambda\varepsilon)$ are called “bad” discs and a_i^ε together with its degree d_i^ε is called a vortex of “size” $\lambda\varepsilon$. We now pay attention to the minimizer u^γ . Although a weight is added to the functional on u^γ , i.e., (1.13), the proofs of the following four lemmas on the properties of u^γ can still be obtained directly from the corresponding ones in [23] and [24] by replacing the energy density with $e_\varepsilon(u) = \frac{1}{2}a(x)[|\nabla u|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2)^2]$. We omit the details.

LEMMA 2.6. *For any $0 < \gamma < \beta < 1$, u^γ has no vortex (i.e., $|u^\gamma| \geq 1/2$) in $\{x \in \Omega; \text{dist}(x, \partial\Omega) \leq \varepsilon^\beta\}$.*

LEMMA 2.7. *For small enough ε , $\text{Card}\mathcal{J}_1$ is uniformly bounded by a constant N which is independent of ε . Let $0 < \gamma < \beta < \mu < 1$ such that $\bar{\mu} = \mu^{N+1} > \beta$. For ε small enough, there exists a subset $\mathcal{J} \subset \mathcal{J}_1$ and a radius $\rho > 0$ with $\lambda\varepsilon \leq \varepsilon^\mu \leq \rho \leq \varepsilon^{\bar{\mu}} < \varepsilon^\beta$ such that*

$$\begin{aligned} &|u^\gamma| \geq 1/2 \quad \text{in } \Omega \setminus \cup_{i \in \mathcal{J}} B(a_i^\varepsilon, \rho), \\ &|u^\gamma| \geq 1 - 2|\ln \varepsilon|^{-2} \quad \text{on } \partial B(a_i^\varepsilon, \rho), \quad i \in \mathcal{J}, \\ &\int_{\partial B(a_i^\varepsilon, \rho)} e_\varepsilon(u^\gamma) \leq C(\beta, \mu)/\rho, \quad i \in \mathcal{J}, \\ &|a_i^\varepsilon - a_j^\varepsilon| \geq 8\rho, \quad i \neq j \in \mathcal{J}. \end{aligned}$$

Denote $d_i^\varepsilon = \text{deg}(u^\gamma, \partial B(a_i^\varepsilon, \rho))$. We have the following lemma.

LEMMA 2.8. *For small enough ε and $u \in \overline{D_M^a}$, $|d_i^\varepsilon| = O(1)$ for all $i \in \mathcal{J}$.*

Assume for the moment that $|\nabla u| \leq C/\varepsilon$ which is true if u is shown to be a solution of (1.3); then, in the sense of [7], the vortices of u are well defined and there exists the same uniform bound on the vortex number. One may also have bigger vortices of size ρ (where “bigger” means $\rho \geq \lambda\varepsilon$), $(b_i^\varepsilon, q_i^\varepsilon)$, such that u satisfies the same conclusions as in Lemma 2.7 for u^γ . As in [23], we may compare $(a_i^\varepsilon, d_i^\varepsilon)$ (the vortices of u^γ) with $(b_i^\varepsilon, q_i^\varepsilon)$ (the vortices of u) by the minimal connection between the vortices.

LEMMA 2.9. *For small ε , there holds $\text{dist}(a, b) \leq C\varepsilon^\gamma |\ln \varepsilon|$*

For the definition of $\text{dist}(a, b)$ and the proof of this lemma, we refer to [23]. The following lemma gives the splitting of the energy $J_a(u)$ as in [23].

LEMMA 2.10. For any \tilde{u} satisfying (2.2)–(2.4), let u be associated to \tilde{u} as in Lemma 2.3 and u^γ be associated to u by solving the minimization problem (1.13) with vortices (a_i, d_i) satisfying Lemma 2.7. Then we have

$$J_a(u) = F_a(u) + \frac{1}{2} \int_\Omega \frac{1}{a(x)} |\nabla \xi|^2 + 2\pi \sum_{i \in \mathcal{J}} d_i \xi(a_i), \quad \text{as } \varepsilon \rightarrow 0,$$

where $\xi = h_{ex} \xi_0$ and ξ_0 is the unique solution of problem (1.5).

Using this splitting, we have the following lemma.

LEMMA 2.11. The constant J^0 in (2.1) is asymptotically equal to the minimal energy among vortexless configurations; i.e., $\inf_{\{u: \mathcal{J}=\emptyset\}} J_a(u) = J^0 + o(1)$ as $\varepsilon \rightarrow 0$.

Let $e_\varepsilon(u) = \frac{1}{2} a(x) [|\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2]$ and $\Omega_\rho = \Omega \setminus \cup_{i \in \mathcal{J}} B(a_i, \rho)$, where $B(a_i, \rho)$'s are defined in Lemma 2.7. We have the following lemma.

LEMMA 2.12. Assume that $\mathcal{J} = \{1, 2, \dots, k\}$; then

$$\frac{1}{2} \int_{\Omega_\rho} a(x) |\nabla u^\gamma|^2 \geq \pi \sum_{i \in \mathcal{J}} a(a_i) d_i^2 |\ln \rho| + W((a_1, d_1), \dots, (a_k, d_k)) + O(1),$$

where

$$W((a_1, d_1), \dots, (a_k, d_k)) = -\pi \sum_{i \neq j \in \mathcal{J}} a(a_i) d_i d_j \ln |a_i - a_j| - \pi \sum_{i \in \mathcal{J}} d_i R_0(a_i)$$

and $R_0(x) = \Phi_0(x) - \sum_{i \in \mathcal{J}} a(a_i) d_i \ln |x - a_i|$ with $\Phi_0(x)$ solves

$$\begin{cases} -\operatorname{div}(\frac{1}{a(x)} \nabla \Phi_0) = 2\pi \sum_{i \in \mathcal{J}} d_i \delta_{a_i} & \text{in } \Omega, \\ \Phi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In the following lemma, we give a few more precise lower bounds on $F_a(u^\gamma)$.

LEMMA 2.13. For ε and ρ satisfying Lemma 2.7, we have

$$(2.13) \quad F_a(u^\gamma) \geq \pi \sum_{i \in \mathcal{J}} a(a_i) [d_i^2 |\ln \rho| + |d_i| |\ln(\rho/\varepsilon)| + W((a_1, d_1), \dots, (a_k, d_k)) + O(1),$$

$$(2.14) \quad F_a(u^\gamma) \geq \pi \sum_{i \in \mathcal{J}} a(a_i) |d_i| |\ln(\rho/\varepsilon)| + O(1).$$

3. Obtaining the critical magnetic field H_{c_1} . Using the splitting and the lower bound of $J_a(u)$, we now estimate the critical magnetic field H_{c_1} .

LEMMA 3.1. Let $h_{ex} = k_a |\ln \varepsilon| + o(|\ln \varepsilon|)$ and $\tilde{u} \in \overline{D_M^a}$ be a minimizer of $J_a(u)$ in $\overline{D_M^a}$ and $\{(a_i, d_i) : i \in \mathcal{J}\}$ be the vortices of u^γ . For ε small enough, if $\mathcal{J} \neq \emptyset$, say, $\mathcal{J} = \{1, \dots, k\}$, then

- (i) $d_i > 0$ for any $i \in \mathcal{J}$, and
- (ii) $\operatorname{dist}(a_i, \partial\Omega) \geq \alpha > 0$ for some positive constant α , and consequently
- (iii) $W((a_1, d_1), \dots, (a_k, d_k)) \geq C$ for some constant C .

Proof. We divide the proof into two steps.

Step 1. We first prove that, for ε small enough, $d_i > 0$ for $i \in \mathcal{J}$. Since $\tilde{u} \in \overline{D_M^a}$ is a minimizer of $J_a(u)$, it follows from Lemma 2.11 that $J_a(u) \leq J^0 + o(1)$, i.e.,

$$F_a(u) + J^0 + 2\pi h_{ex} \sum_{i \in \mathcal{J}} d_i \xi_0(a_i) + o(1) \leq J^0 + o(1).$$

Therefore

$$(3.1) \quad F_a(u) \leq -2\pi h_{ex} \sum_{i \in \mathcal{J}} d_i \xi_0(a_i) + o(1)$$

or equivalently (noting that $\xi_0 < 0$ in Ω)

$$\begin{aligned} F_a(u) &\leq 2\pi(k_a |\ln \varepsilon| + o(|\ln \varepsilon|)) \max_{d_i > 0} \left| \frac{\xi_0(x)}{a(x)} \right| \sum_{d_i > 0} a(a_i) d_i + o(1) \\ &\leq \pi |\ln \varepsilon| \sum_{d_i > 0} a(a_i) d_i + o(|\ln \varepsilon|). \end{aligned}$$

This inequality implies

$$(3.2) \quad F_a(u^\gamma) \leq F_a(u) \leq \pi |\ln \varepsilon| \sum_{d_i > 0} a(a_i) d_i + o(|\ln \varepsilon|).$$

Combining (3.2) with (2.14) in Lemma 2.13 we obtain

$$(3.3) \quad \pi(1 - \mu) \left(\sum_{i \in \mathcal{J}} a(a_i) |d_i| \right) |\ln \varepsilon| \leq \pi \left(\sum_{d_i > 0} a(a_i) d_i \right) |\ln \varepsilon| + o(|\ln \varepsilon|)$$

since $\varepsilon^\mu \leq \rho \leq \varepsilon^{\bar{\mu}}$. This implies

$$(3.4) \quad (1 - \mu) \sum_{d_i < 0} a(a_i) |d_i| \leq \mu \sum_{d_i > 0} a(a_i) d_i + o(1).$$

We estimate the first term on the right-hand side of (3.4). By (2.11), (2.14),

$$\mu \sum_{d_i > 0} a(a_i) d_i \leq \mu \sum_{i \in \mathcal{J}} a(a_i) |d_i| \leq M\mu / (\pi(1 - \mu)) + o(1).$$

Substituting this into (3.4), we get

$$\sum_{d_i < 0} a(a_i) |d_i| \leq M\mu / (\pi(1 - \mu)^2) + o(1).$$

This means $\{i \in \mathcal{J}; d_i < 0\} = \emptyset$ if one chooses μ small enough.

Step 2. We prove (ii) and (iii) in this step. It follows from Step 1 that

$$-\pi \sum_{i \neq j} a(a_i) d_i d_j \ln |a_i - a_j| \geq O(1)$$

and then

$$(3.5) \quad W((a_1, d_1), \dots, (a_k, d_k)) \geq -\pi \sum_{i \in \mathcal{J}} d_i R_0(a_i) + O(1).$$

For the proof of $\|R_0\|_{L^\infty(\Omega)} \leq C$, similar to [23] and [6], it suffices to prove that $\text{dist}(a_i, \partial\Omega)$ is uniformly bounded from below. Indeed, it can be shown as in [23] that

$$(3.6) \quad \|R_0(x)\|_{L^\infty(\Omega)} \leq C\beta |\ln \varepsilon| + O(1).$$

Therefore we deduce

$$(3.7) \quad W((a_1, d_1), \dots, (a_k, d_k)) \geq -C\beta |\ln \varepsilon|.$$

On the other hand, we know from (3.2) and (2.13) (in view of $d_i^2 \geq d_i > 0$) that

$$\begin{aligned} F_a(u^\gamma) &\leq F_a(u) \leq -2\pi h_{ex} \sum_{i \in \mathcal{J}} d_i \xi_0(a_i) + o(1), \\ F_a(u^\gamma) &\geq \pi \sum_{i \in \mathcal{J}} a(a_i) d_i |\ln \varepsilon| + W((a_1, d_1), \dots, (a_k, d_k)) + O(1) \\ &\geq \pi \sum_{i \in \mathcal{J}} a(a_i) d_i |\ln \varepsilon| - C\beta |\ln \varepsilon| + O(1). \end{aligned}$$

Putting these two inequalities together and using

$$h_{ex} = k_a |\ln \varepsilon| + o(|\ln \varepsilon|) = \frac{|\ln \varepsilon|}{2 \max_{\Omega} |\xi_0(x)/a(x)|} + o(|\ln \varepsilon|),$$

we get

$$(3.8) \quad 2\pi h_{ex} \sum_{i \in \mathcal{J}} a(a_i) d_i \left[\frac{\xi_0(a_i)}{a(a_i)} + \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| \right] \leq C\beta |\ln \varepsilon| + o(|\ln \varepsilon|).$$

Since $d_i \geq 1$ and $a(a_i) \geq \alpha_0 > 0$ for $i \in \mathcal{J}$, the above implies

$$(3.9) \quad \frac{\xi_0(a_i)}{a(a_i)} + \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| \leq C\beta \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| \quad \forall i \in \mathcal{J}.$$

Taking $\beta > 0$ such that $C\beta < 1/2$, we get

$$(3.10) \quad \frac{\xi_0(a_i)}{a(a_i)} \leq -\frac{1}{2} \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| < 0.$$

Since $\xi_0 = 0$ on $\partial\Omega$, we thus have $\text{dist}(a_i, \partial\Omega)$ being uniformly bounded from below. So, $\|R_0\|_{L^\infty(\Omega)} \leq C$. This implies $W \geq O(1)$ uniformly by (3.5). \square

Now, let $\overline{D_0} = \{u \in \overline{D_M^a}; \mathcal{J} = \emptyset\}$, and we have the following lemma.

LEMMA 3.2. *Suppose $\max_{\overline{\Omega}} a(x)\pi < M$. There are $k_2^\varepsilon = O(1)$, $k_3^\varepsilon = o(1)$, and $\varepsilon_0 > 0$ such that, for $h_{ex} = |\ln \varepsilon|/(2 \max_{\Omega} |\xi_0(x)/a(x)|) + t$, there holds*

(i) *if $t < k_2^\varepsilon$ and \tilde{u} is a minimizer of J_a in $\overline{D_M^a}$, then $\mathcal{J} = \emptyset$ and*

$$J_a(\tilde{u}) = \inf_{\overline{D_0}} J_a(u) = J^0 + o(1);$$

(ii) *if $t = k_2^\varepsilon$, there is $u \in \overline{D_M^a}$ with a simple vortex and $J_a(u) \leq \inf_{\overline{D_0}} J_a(v)$;*

(iii) *if $t \geq k_2^\varepsilon + k_3^\varepsilon$, there is $u \in \overline{D_M^a}$ with a simple vortex and $J_a(u) < \inf_{\overline{D_0}} J_a(v)$.*

Proof. Let J^0 be as in (2.1). We have

$$J_a(u) = F_a(u) + J^0 + 2\pi h_{ex} \sum_{i \in \mathcal{J}} d_i \xi_0(a_i) + o(1).$$

Clearly, $J^0 = \inf_{\overline{D_0}} J_a(v)$. If $\mathcal{J} \neq \emptyset$, then we consider two cases.

Case 1. $h_{ex} \leq (1 - \mu^*)k_a |\ln \varepsilon|$ for some $0 < \mu^* < 1$. Since $\rho \geq \varepsilon^\mu$ for some $\mu > 0$, it follows from Lemma 2.13 that

$$(3.11) \quad F_a(u) \geq F_a(u^\gamma) \geq (1 - \mu)\pi \sum_{i \in \mathcal{J}} a(a_i)|d_i| |\ln \varepsilon| + O(1),$$

and then

$$J_a(u) \geq J^0 + \pi(1 - \mu) \sum_{i \in \mathcal{J}} a(a_i)|d_i| |\ln \varepsilon| - 2\pi h_{ex} \sum_{i \in \mathcal{J}} a(a_i)|d_i| \left| \frac{\xi_0(a_i)}{a(a_i)} \right| + O(1).$$

Hence, $J_a(u) > \inf_{\overline{D_0}} J_a(v)$ as long as

$$2\pi h_{ex} \sum_{i \in \mathcal{J}} a(a_i)|d_i| \left| \frac{\xi_0(a_i)}{a(a_i)} \right| \leq (1 - \mu)\pi \sum_{i \in \mathcal{J}} a(a_i)|d_i| |\ln \varepsilon| + O(1)$$

which may be valid if we take $\mu < \mu^*$ since

$$h_{ex} \leq (1 - \mu) \frac{|\ln \varepsilon|}{2 \max_{\Omega} |\xi_0(x)/a(x)|}.$$

Case 2. $t < k_2^\varepsilon$ with $|t| = o(|\ln \varepsilon|)$. Then, by Lemma 3.1, we have

$$W((a_1, d_1), \dots, (a_k, d_k)) \geq C$$

for some constant C , thus, by Lemma 2.13, we get

$$F_a(u) \geq F_a(u^\gamma) \geq \pi \sum_{i \in \mathcal{J}} a(a_i)|d_i| |\ln \varepsilon| + O(1).$$

Then, similar to Case 1, we have $J_a(u) > \inf_{\overline{D_0}} J_a(v)$ as long as

$$h_{ex} \leq \frac{|\ln \varepsilon|}{2 \max_{\Omega} |\xi_0(x)/a(x)|} + O(1).$$

This verifies conclusion (i) in the lemma.

Next, let $k_a = 1/(2 \max_{\Omega} |\xi_0(x)/a(x)|)$. As in [23], set

$$Z^\varepsilon = \left\{ t \in \mathbb{R}; \text{ there exists } u \in \overline{D_M^a} \text{ with at least one vortex} \right. \\ \left. \text{and } J_a(u) < \inf_{\{\mathcal{J}=\emptyset\}} J_a \text{ for } h_{ex} = k_a |\ln \varepsilon| + t \right\}.$$

In the following, we prove $Z^\varepsilon \neq \emptyset$ which would allow us to define $k_2^\varepsilon = \inf Z^\varepsilon$ and to prove that there exists $k_3^\varepsilon = o(1)$ such that $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$.

Let $c \in \Omega$ such that $|\xi_0(c)/a(c)| = \max_{\Omega} |\xi_0(x)/a(x)|$. Consider the problem

$$(3.12) \quad \nu_\varepsilon(c) = \min_W \frac{1}{2} \int_{\Omega \setminus B(c, \varepsilon)} a(x) |\nabla u|^2,$$

where $W = \{ u \in H^1(\Omega \setminus B(c, \varepsilon), S^1), \deg(u, \partial B(c, \varepsilon)) = 1 \}$. Similar as before,

$$\nu_\varepsilon(c) = \pi a(c) |\ln \varepsilon| + O(1).$$

Let u be a minimizer of problem (3.12) which is well defined on $\Omega \setminus B(c, \varepsilon)$. Extending u to the whole domain Ω by defining it on $B(c, \varepsilon)$ as in [23] and denoting it by \bar{u} , we may get, as similarly done in [23],

$$F_a(\bar{u}, \Omega) = F_a(\bar{u}, B(c, \varepsilon)) + \frac{1}{2} \int_{\Omega \setminus B(c, \varepsilon)} a(x) |\nabla u|^2 \leq K + a(c)\pi |\ln \varepsilon|.$$

For $h_{ex} = \frac{1}{2 \max\{|\xi_0(x)/a(x)|\}} |\ln \varepsilon| + t = -\frac{1}{2\xi_0(c)/a(c)} |\ln \varepsilon| + t$, we have

$$\begin{aligned} J_a(\bar{u}) &\leq F_a(\bar{u}) + J^0 + 2\pi h_{ex} \xi_0(c) + o(1) \\ &= K - 2\pi |\xi_0(c)| t + J^0 + o(1). \end{aligned}$$

This implies $t \in Z^\varepsilon$ when $2\pi |\xi_0(c)| t \geq K + o(1)$. So, $Z^\varepsilon \neq \emptyset$ and $k_2^\varepsilon = \inf Z^\varepsilon \leq K/2\pi |\xi_0(c)| + o(1)$. On the other hand, $h_{ex} \leq k_a |\ln \varepsilon| + O(1)$; we thus know $k_2^\varepsilon \geq O(1)$ which gives $k_2^\varepsilon = O(1)$.

Finally, we prove that there exists $k_3^\varepsilon = o(1)$ such that $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$. In fact, let $t \in Z^\varepsilon$ and, for $h_{ex,1} = k_a |\ln \varepsilon| + t$, $J_a(u) < \inf_{\overline{D_0}} J_a(v)$ and u^γ has vortices (a_i, d_i) . Assume $t' > t$ and $h_{ex,2} = k_a |\ln \varepsilon| + t'$; we have

$$\begin{aligned} J_a(u) &= F_a(u) + J^0 + 2\pi h_{ex,2} \sum_{i=1}^k \xi_0(a_i) d_i + o(1) \\ &\leq J_a(u) + o(1) - (t' - t) \sum_{i=1}^k 2\pi |\xi_0(a_i)| d_i. \end{aligned}$$

Thus, if $t' - t \geq k_3^\varepsilon = o(1)$, then $J_a(u) < \inf_{\{\mathcal{J}=\emptyset\}} J_a$, i.e., $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$.

In summary, we have deduced that $H_{c_1} = k_a |\ln \varepsilon| + k_2^\varepsilon$ for the lower critical field. This completes the proof of the lemma. \square

4. Proof of Theorem 1.1. By Lemma 3.2 and the bounds in Lemma 2.3 and Lemma 2.11, we see that, for $h_{ex} \leq H_{c_1}$, the minimizer of J_a in $\overline{D_M^a}$ has no vortex and it is in the interior of $\overline{D_M^a}$, thus the first part of Theorem 1.1 follows.

To complete the proof of Theorem 1.1, we need the following lemmas.

LEMMA 4.1. *Let $h_{ex} = k_a |\ln \varepsilon| + o(|\ln \varepsilon|)$ and $\tilde{u} \in \overline{D_M^a}$ be a minimizer of $J_a(u)$ in $\overline{D_M^a}$. $\{(a_i, d_i)\}_{i=1}^k$ are the vortices of u^γ . Then $d_i = 1$ for any $i \in \mathcal{J} = \{1, \dots, k\}$.*

Proof. Using the lower bound on W proved in Lemma 3.1 and returning to Lemma 2.13, we have

$$\pi \sum_{i \in \mathcal{J}} a(a_i) d_i^2 |\ln \rho| + \pi \sum_{i \in \mathcal{J}} a(a_i) d_i \ln \frac{\rho}{\varepsilon} + O(1) \leq \pi \sum_{d_i > 0} a(a_i) d_i |\ln \varepsilon| + o(|\ln \varepsilon|).$$

This gives

$$\pi \sum_{i \in \mathcal{J}} a(a_i) (d_i^2 - d_i) |\ln \rho| \leq o(|\ln \varepsilon|).$$

Therefore we have from $\rho \geq \varepsilon^\mu$ that

$$\mu \sum_{i \in \mathcal{J}} a(a_i) (d_i^2 - d_i) |\ln \varepsilon| \leq \sum_{i \in \mathcal{J}} a(a_i) (d_i^2 - d_i) |\ln \rho| \leq o(|\ln \varepsilon|),$$

which implies

$$\mu \min_{\Omega} a(x) \sum_{i \in \mathcal{J}} (d_i^2 - d_i) \leq o(1).$$

This inequality is impossible if there is a $d_i > 1$ for small ε . So when $\varepsilon \leq \varepsilon_0$, we have $d_i = 1$ for all $i \in \mathcal{J}$. The lemma is proved. \square

We are now closer to a complete proof of Theorem 1.1. Consider

$$H_{c_1} + k_3^\varepsilon \leq h_{ex} \leq k_a |\ln \varepsilon| + O(1),$$

where

$$H_{c_1} = k_a |\ln \varepsilon| + k_2^\varepsilon, \quad k_a = 1/(2 \max_{\Omega} |\xi_0(x)/a(x)|),$$

$k_2^\varepsilon = O(1)$, and $k_3^\varepsilon = o(1)$. Let

$$\Lambda = \left\{ x, x \in \Omega : \left| \frac{\xi_0(x)}{a(x)} \right| = \max_{y \in \Omega} \left| \frac{\xi_0(y)}{a(y)} \right| \right\}.$$

The proof of Theorem 1.1 can be obtained by proving the following three lemmas.

LEMMA 4.2. *Let $u \in \overline{D_M^a}$ be a minimizer of $J_a(u)$, and let (a_i, d_i) ($d_i = 1$ for all $i \in \mathcal{J}$) be the vortices of u^γ . Then*

$$(4.1) \quad \text{dist}(a_i, \Lambda) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0 \quad \forall i \in \mathcal{J},$$

$$(4.2) \quad \text{dist}(a_i, a_j) \geq \alpha > 0 \quad \forall i \neq j \in \mathcal{J}.$$

The first result is also true under the assumption $h_{ex} \leq k_a |\ln \varepsilon| + o(|\ln \varepsilon|)$.

Proof. If $\mathcal{J} \neq \emptyset$, we have from Lemma 4.1 that $d_i = 1$ for all $i \in \mathcal{J}$. Now let $d = \sum_{i \in \mathcal{J}} d_i = \text{Card} \mathcal{J} = \text{deg}(u^\gamma, \partial\Omega)$. It follows from Step 1 in the proof of Lemma 3.1 and from Lemma 2.13 that

$$W(a_1, \dots, a_k) + \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1) \leq F_a(u^\gamma) \leq -2\pi h_{ex} \sum_{i \in \mathcal{J}} \xi_0(a_i) + o(|\ln \varepsilon|).$$

Then

$$2\pi h_{ex} \sum_{i \in \mathcal{J}} a(a_i) \left(\frac{\xi_0(a_i)}{a(a_i)} + \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| \right) \leq o(|\ln \varepsilon|).$$

Hence we have

$$\sum_{i \in \mathcal{J}} \left(\frac{\xi_0(a_i)}{a(a_i)} + \max_{\Omega} \left| \frac{\xi_0(x)}{a(x)} \right| \right) \leq \frac{o(|\ln \varepsilon|)}{h_{ex}} \rightarrow 0.$$

This implies the first conclusion

$$\text{dist}(a_i, \Lambda) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0 \quad \forall i \in \mathcal{J}.$$

Moreover, since $W(a_1, \dots, a_k) \geq O(1)$ and

$$W(a_1, \dots, a_k) + \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1) \leq -2\pi h_{ex} \pi \sum_{i \in \mathcal{J}} \xi_0(a_i) + O(1),$$

we have $W(a_1, \dots, a_k) \leq O(1)$ if $h_{ex} \leq k_a |\ln \varepsilon| + O(1)$. Therefore we conclude that $|a_i - a_j|$ remains bounded from below uniformly, since as in [7] we could prove that $W \rightarrow +\infty$ if $|a_i - a_j| \rightarrow 0$ for some $i \neq j$. Lemma 4.2 is proved. \square

LEMMA 4.3. *Let M, n satisfy Assumption 1.1, and let \tilde{u} be a minimizer of $J_a(u)$ in $\overline{D_M^a}$; then \tilde{u} satisfies (1.3) and $u = \tilde{u}$, where u is defined by \tilde{u} as in Lemma 2.3.*

Proof. It suffices to prove that \tilde{u} is not on the boundary of $\overline{D_M^a}$. Since we have proved that W is a bounded quantity and $\text{dist}(a_i, a_j) \geq \alpha > 0$, we get

$$\begin{aligned} \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1) &\leq -2\pi h_{ex} \sum_{i \in \mathcal{J}} \xi_0(a_i) + O(1) \\ &= 2\pi h_{ex} \sum_{i \in \mathcal{J}} a(a_i) |\xi_0(a_i)/a(a_i)| + O(1) \\ &\leq \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1). \end{aligned}$$

This inequality and Lemma 2.3 yield that

$$F_a(u) = F_a(\tilde{u}) + o(1) = \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1) \leq M |\ln \varepsilon| + O(1).$$

So, $\sum_{i \in \mathcal{J}} a(a_i) \leq M/\pi$. It follows from Lemma 4.2 that as $\varepsilon \rightarrow 0$, $a_i \rightarrow c_i \in \Lambda$. Then, for ε small enough, $d \leq M/(\pi m_d)$, where $m_d = (\sum_{i \in \mathcal{J}} a(c_i))/d$ and

$$\frac{M}{\pi m_d} \in \left[\frac{M}{\pi \max_{\Lambda} a(x)}, \frac{M}{\pi \min_{\Lambda} a(x)} \right] \subset (n, n + 1).$$

Thus, $M/(\pi m_d)$ is not an integer which implies $d < M/(\pi m_d)$. Hence $\pi \sum_{i \in \mathcal{J}} a(a_i) < M$ for $\varepsilon \leq \varepsilon_0$. Thus, there is a positive number $\eta > 0$ such that

$$F_a(\tilde{u}) \leq \pi \sum_{i \in \mathcal{J}} a(a_i) |\ln \varepsilon| + O(1) \leq (M - \eta) |\ln \varepsilon|,$$

which means that \tilde{u} is not on $\partial \overline{D_M^a}$. The lemma is proved. \square

REMARK 4.1. *It follows from the proof of Lemma 4.3 that $d < M/(\pi \max_{\Lambda} a(x))$. Otherwise, since we also have $d \min_{\Lambda} a(x) \leq \sum_{i=1}^d a(a_i) \leq M/\pi$, this implies that*

$$d \in \left[\frac{M}{\pi \max_{\Lambda} a(x)}, \frac{M}{\pi \min_{\Lambda} a(x)} \right] \subset (n, n + 1);$$

then d is not an integer. This leads to a contradiction.

Now we may continue the proof of Theorem 1.1 with the following lemma. Once u is a solution of (1.3), we may show that $|\nabla u| \leq C/\varepsilon$. Then u has bigger vortices of size $\rho: \{(b_i, q_i)\}_{i \in \mathcal{J}}$. The following lemma compares what we call the bigger vortices of u (i.e., the vortices of u^γ) with the real vortices of u . Its proof follows easily from the same arguments given in [23].

LEMMA 4.4. *For sufficiently small ε , we have*

(i) *if u is a solution of (1.3) such that $J_a(u) \leq Ch_{ex}^2$, then $|u| \leq 1$ and there exists a constant $C > 0$ such that $|\nabla u| \leq C/\varepsilon$;*

(ii) if u is a solution of (1.3) such that u^γ has no vortices (i.e., $|u^\gamma| \geq 1/2$) and $J_a(u) \leq J^0$, then u has no vortices on Ω ($|u| \geq 1/2$);

(iii) if u is a solution given in Theorem 1.1, then its vortices (of size ρ) satisfy the same conclusions as those of u^γ ;

(iv) if, in addition, $\{a_i\}_{i \in \mathcal{J}}$ are the vortices of u^γ of degree one, then the vortices $\{b_i\}_{i \in \mathcal{J}}$ are also of degree one and Lemma 2.7 (on u^γ) is satisfied by u .

The following lemma shows that the real vortices of u remain far from the boundary.

LEMMA 4.5. *If $u \in D_M^a$ is an energy minimizer satisfying (1.3), then, for any $0 < \beta < 1$, $|u| \geq 1/2$ on $\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon^\beta\}$. Moreover, u has no zero degree vortex.*

Finally, since $\text{dist}(a_i, \partial\Omega)$, $\text{dist}(b_i, \partial\Omega)$ remain bounded from below by a positive constant and $\text{dist}(a, b) \leq C\varepsilon^\gamma |\ln \varepsilon|$, we have for small ε that \mathbb{R}^2 is a hole of null multiplicity. This implies $\sum_i q_i = \sum_i q_i = \text{Card}\mathcal{J}'$, and the b_i 's tend to the a_i 's with the same multiplicities. However, $\inf_{i \neq j} |a_i - a_j| \leq C|\ln \varepsilon|^{-\frac{1}{2}}$; comparing with $\text{dist}(a, b) \leq C\varepsilon^\gamma |\ln \varepsilon|$, the b_i 's must be of multiplicity one, and $q_i = 1$ for all $i \in \mathcal{J}'$. Theorem 1.1 is proved.

5. Proof of Theorem 1.2. In this section we derive the convergence of u_ε and the limit equation of (1.3). The case $d = 0$ is again trivial to consider; we omit the details. Now, for $H_{c_1} + k_3^\varepsilon \leq h_{ex} \leq H_{c_1} + O(1)$, let us consider a sequence $\varepsilon_n \rightarrow 0$ and denote $u_n = u_{\varepsilon_n}$ an associated solution of (1.3) given by Theorem 1.1.

We also denote $\{b_i\}_{i \in \mathcal{J}}$ the real vortices of u_n (see Lemma 4.4) of size $\lambda\varepsilon$, and $\{b_i\}_{i \in \mathcal{J}' \subset \mathcal{J}}$ its vortices of size ρ , exactly as we did for u^γ . (Again, the superscript ε in the notation of b_i is removed.) This means

$$\begin{aligned} |u_n| &\geq 1/2 \quad \text{on } \Omega \setminus \cup_{i \in \mathcal{J}} B(b_i, \lambda\varepsilon) \quad \text{and on } \Omega \setminus \cup_{i \in \mathcal{J}'} B(b_i, \rho), \\ \cup_{i \in \mathcal{J}} B(b_i, \lambda\varepsilon) &\subset \cup_{i \in \mathcal{J}'} B(b_i, \rho). \end{aligned}$$

Extracting a subsequence if necessary, we may assume that $\text{Card}\mathcal{J}' \equiv d \geq 1$ and

$$b_i^{\varepsilon_n} \rightarrow c_i \in \Lambda \quad \text{for } i \in \mathcal{J}'.$$

Here, we put back the superscript ε_n to avoid ambiguity. First, we prove that

$$|\ln \varepsilon_n| |\nabla u_n| \rightarrow 0, \quad \text{strongly in } L^p(\Omega) \quad \forall p < 2.$$

In fact, we may rewrite (1.3) as

$$(5.1) \quad -\nabla \cdot (a(x)\nabla u) + 2ia(x)A_0 \cdot \nabla u = a(x)u \left[|A_0|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2) \right].$$

This equation is equivalent to the following system if we write locally $u = \rho e^{i\varphi}$:

$$(5.2) \quad \begin{cases} -\nabla \cdot (a(x)\nabla \rho) + a(x)\rho |\nabla \varphi|^2 - 2a\rho A_0 \cdot \nabla \varphi = a(x)\rho (|A_0|^2 + \frac{1-\rho^2}{\varepsilon^2}), \\ -\nabla \cdot (a(x)\rho^2 \nabla \varphi) + a(x)A_0 \cdot \nabla \rho^2 = 0. \end{cases}$$

Define

$$\bar{\rho} = \max\{\rho, 1 - |\ln \varepsilon|^{-4}\} \geq 1 - |\ln \varepsilon|^{-4}, \quad K = \{x \in \Omega, \rho \geq 1 - |\ln \varepsilon|^{-4}\};$$

then $\nabla\bar{\rho} = \nabla\rho$ on K , and $\nabla\bar{\rho} = 0$ on $\Omega \setminus K$. It follows from

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - \rho)^2 \leq \frac{1}{\varepsilon^2} \int_{\Omega} (1 - \rho^2)^2 \leq C |\ln \varepsilon|^2$$

that $\text{meas}(\Omega \setminus K) \leq C\varepsilon^2 |\ln \varepsilon|^{10}$.

Multiplying the first equation in (5.2) by $1 - \bar{\rho}$, and integrating over Ω , we get

$$2 \int_K a(x)\rho |\nabla\rho|^2 \leq C \int_{\Omega} a(x)(1 - \bar{\rho})\rho[|A_0|^2 + |A_0||\nabla u|] \leq C \|1 - \bar{\rho}\|_{L^\infty} |\ln \varepsilon|^2.$$

This inequality, together with the fact $0 \leq 1 - \bar{\rho} \leq |\ln \varepsilon|^{-4}$, yields

$$\int_K |\nabla\rho|^2 \leq C/|\ln \varepsilon|^2 \rightarrow 0.$$

On the other hand, we have for $p < 2$

$$\begin{aligned} \int_{\Omega \setminus K} |\nabla\rho|^p &\leq \left(\int_{\Omega \setminus K} |\nabla\rho|^2 \right)^{p/2} \text{meas}(\Omega \setminus K)^{1-p/2} \\ &\leq C\varepsilon^{2-p} |\ln \varepsilon|^{10-4p} \leq C/|\ln \varepsilon|^2 \rightarrow 0. \end{aligned}$$

Combining the above two estimates, we have

$$(5.3) \quad \int_{\Omega} |h_{ex} \nabla\rho|^p \leq \frac{C}{|\ln \varepsilon|^{2-p}} \rightarrow 0.$$

Now we rewrite the second equation of (5.2) as

$$-\nabla \cdot (a(x)\rho^2 \nabla\varphi) = f(x),$$

where $f(x) = -a(x)A_0 \cdot \nabla\rho^2$. Since $a(x)A_0 = h_{ex} \nabla^\perp \xi_0$ is a smooth function, we have

$$\int_{\Omega} |f(x)|^p \leq C/|\ln \varepsilon|^{2-p}.$$

Hence we have

$$(5.4) \quad \int_{\Omega} |h_{ex} \nabla\varphi|^p \leq C/|\ln \varepsilon|^{2-p} \rightarrow 0.$$

The estimates (5.3) and (5.4) yield

$$\int_{\Omega} |h_{ex} \nabla u|^p \rightarrow 0.$$

It follows that, for any smooth test function $\phi \in C_0^\infty(\Omega)$,

$$(5.5) \quad 2i \int_{\Omega} a \phi A_0 \cdot \nabla u = 2i \int_{\Omega} (\phi \nabla^\perp \xi_0) \cdot (h_{ex} \nabla u) \rightarrow 0$$

since we may take $(\phi \nabla^\perp \xi_0) \in L^q(\Omega)$ for any $q > 2$.

Let s be a positive integer. As in the proofs of Lemmas 2.12 and 2.13, by solving an auxiliary problem, we may get

$$\int_{B(c_i, \frac{1}{s}) \setminus B(b_i, \rho)} a(x) |\nabla u_n|^2 \geq \pi a(b_i) |\ln \rho| + O(1) \quad \forall i \in \mathcal{J}'$$

and

$$\int_{B(b_i, \rho)} e_\varepsilon(u_n) \geq \pi a(b_i) \ln \frac{\rho}{\varepsilon} + O(1) \quad \forall i \in \mathcal{J}'.$$

These two inequalities yield

$$\int_{B(c_i, \frac{1}{s})} e_\varepsilon(u_n) \geq \pi a(b_i) |\ln \varepsilon| + O(1) \quad \forall i \in \mathcal{J}'.$$

On the other hand, we have

$$F_a(u) \leq \pi \sum_{i=1}^d a(a_i) |\ln \varepsilon| + O(1) \leq \pi \sum_{i=1}^d a(b_i) |\ln \varepsilon| + O(1),$$

where we have used $\text{dist}(a, b) \leq C\varepsilon^\gamma |\ln \varepsilon|$ (see Lemma 2.9). We finally get

$$\int_{B(c_i, \frac{1}{s})} |\nabla u_n|^2 \leq C.$$

Extracting a subsequence if necessary, there exists u_* such that

$$(5.6) \quad u_n \rightharpoonup u_* \text{ weakly in } H^1(\Omega \setminus \cup_{i=1}^d B(c_i, 1/s)).$$

By standard diagonal extraction, we may find a subsequence such that this is true for any positive integer s . $|u_*| = 1$ follows from

$$\int_{\Omega} (1 - |u_n|^2)^2 \leq C\varepsilon^2 |\ln \varepsilon|^2.$$

Taking the cross product of (5.1) with u , we get

$$u_n \times [-\nabla \cdot (a(x)\nabla u_n) + 2ia(x)A_0 \cdot \nabla u_n] = 0.$$

Using (5.5) and (5.6), we have from the above that u_* solves

$$u_* \times (\nabla \cdot (a(x)\nabla u_*)) = 0 \text{ in } \mathcal{D}'(\Omega \setminus \cup\{c_i\}).$$

Now it is easy to get the limit (1.12) from the above inequality. Theorem 1.2 is proved.

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