COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **1**, Number **3**, September **2002**

Website: http://AIMsciences.org

pp. **327–340**

THE GLOBAL MINIMIZERS AND VORTEX SOLUTIONS TO A GINZBURG-LANDAU MODEL OF SUPERCONDUCTING FILMS

Shijin Ding

Department of Mathematics
South China Normal University
Guangzhou, Guangdong 510631, China.

Qiang Du

Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Abstract. In this paper, we discuss the global minimizers of a free energy for the superconducting thin films placed in a magnetic field h_{ex} below the lower critical field H_{c_1} or between H_{c_1} and the upper critical field H_{c_2}. For h_{ex} is near but smaller than H_{c_1}, we prove that the global minimizer having no vortex is unique. For $H_{c_1} << h_{ex} << H_{c_2}$, we prove that the density of the vortices of the global minimizer is proportional to the applied field.

1. **Introduction.** For a sufficiently thin superconducting film, it was shown in [3] that the three-dimensional Ginzburg-Landau model of superconductivity [5, 12] may be reduced to a two-dimensional one given by the minimization of the functional :

$$J_a(u) = \frac{1}{2}\int_\Omega a(x)\left[|\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right], \tag{1}$$

in $H^1(\Omega, \mathbb{R}^2)$ where Ω is a bounded smooth domain in $\mathbb{R}^2$ representing the planar cross-section, $a \in C^\infty(\overline{\Omega})$ is a given function measuring the variation in the thickness, and $\mathbf{A}_0(x)$, the in-plane component of the magnetic potential, is determined from the vertical component h_{ex} of the applied magnetic field:

$$\mathrm{div}\,(a(x)\mathbf{A}_0) = 0,\ \mathbf{curl}\mathbf{A}_0 = h_{ex},\ \text{in } \Omega, \mathbf{A}_0\cdot\mathbf{n} = \mathbf{0},\ \text{on } \partial\mathbf{\Omega}. \tag{2}$$

Here, we assume that $a(x) \geq \alpha_0 > 0$ for all $x \in \overline{\Omega}$, $\mathbf{n}$ denotes the outward normal to $\partial\Omega$, $\nabla_{\mathbf{A}_0} u = \nabla u - i\mathbf{A}_0 u$, u is the complex order parameter, and ε measures the coherence length. Let u be a critical point of the functional J_a, the points where the zeros of u appear, with their topological degrees, are called the vortices of u. It is important to understand the vortex structures in the solutions u from both mathematical and in physical point of view [1, 5, 6, 9, 12].

The lower critical field H_{c_1} is defined as the value of h_{ex} for which the energy of the Meissner solution becomes equal to the energy of the solution with a single-vortex [12]. The upper critical field H_{c_2} is the field at which the densely packed

1991 *Mathematics Subject Classification.* 37J55,35Q40.
Key words and phrases. Superconductivity, thin films, vortices, pinning.

vortex solutions disappear into the normal state and it is estimated as $O(\varepsilon^{-2})$. The *conjectured* behavior of the minimizers of the free energy functional are as follows:

1: When $h_{ex} = 0$, $u \equiv 1$ is a minimizer;

2: If $h_{ex} < H_{c_1}$, $|u|$ is about 1 everywhere, this is, u is the Meissner solution;

3: When $h_{ex} = O(H_{c_1})$, a few vortices of degree one appear, and their positions are determined by the geometry of Ω and the thickness variation ;

4: If $h_{ex} >> H_{c_1}$, the density of the vortices of the minimizers are expected to be uniform and proportional to h_{ex}. They repel one another through the Colombian interaction;

5: If $h_{ex} \to H_{c_2} = O(\varepsilon^{-2})$, the density of vortices is such that they are separated by a distance shorter than ε. Then $|u|$ is about 0 everywhere.

Partial, but rigorous, verifications of the above statements have been carried out in [4]. For easy reference, let us state some previously proved results. First, let $\xi = h_{ex}\xi_0 \in H^2(\Omega, R)$ satisfy

$$a(x)\mathbf{A}_0(x) = \nabla^{\perp}\xi = (-\xi_{x_2}, \xi_{x_1}), \text{ in } \Omega,$$

and ξ_0 solves

$$\begin{cases} -\mathrm{div}(\frac{1}{a(x)}\nabla\xi_0) = -1, & \text{in } \Omega, \\ \xi_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

We may easily see that $-C \leq \xi_0 < 0$ for some constant $C > 0$ only depending on Ω and $a = a(x)$. Now, define

$$\Lambda = \left\{ x \in \Omega, |\xi_0(x)/a(x)| = \max_{y\in\Omega} |\xi_0(y)/a(y)| \right\},$$

and

$$D_M^a = \left\{ u \in H^1(\Omega, \mathrm{I\!R}^2) : \ F_a(u) < M|\ln\varepsilon| \ \right\} \tag{3}$$

were discussed. In [4], the minimizers of the free energy functional (1) for the thin film in the set D_M^a was studied under the following assumption:

Assumption 1.1. *Assume that the constant M in (3) is so chosen that there is a positive integer $n \in \mathrm{I\!N}$ such that*

$$\left[\frac{M}{\pi \max_\Lambda a(x)}, \frac{M}{\pi \min_\Lambda a(x)}\right] \subset (n, n+1). \tag{4}$$

The above assumption on the existence of $n \in \mathrm{I\!N}$ with the desired property (4) is needed in proving (see [4] section 6) that the minimizer of $J_a(u)$ in $\overline{D_M^a}$ is actually in D_M^a (not on ∂D_M^a), thus the minimizer is a solution of the Ginzburg-Landau equation (5), the Euler-Lagrange equations of the functional $J_a(u)$:

$$-(\nabla - i\mathbf{A}_0)\cdot a(x)(\nabla u - i\mathbf{A}_0 u) = \frac{a(x)}{\varepsilon^2}u(1-|u|^2), \text{ in } \Omega\,, \ \ \partial_{\mathbf{n}}u = 0, \text{ on } \partial\Omega \tag{5}$$

The assumption can be equivalently replaced by $\max_\Lambda a(x) < 2\min_\Lambda a(x)$. More detailed discussion on the assumption 1.1 has been given in the section 8 of [4]. The following theorems have been proved in [4]:

Theorem A. *Let $k_a = 1/(2\max_\Omega |\xi_0(x)/a(x)|)$, there exist $k_2^\varepsilon = O(1)$, $k_3^\varepsilon = o(1)$, and $\varepsilon_0 = \varepsilon_0(M) > 0$ such that*

$$H_{c_1} = k_a|\ln\varepsilon| + k_2^\varepsilon \tag{6}$$

for $\varepsilon < \varepsilon_0$. Moreover, the following holds

(i) *if $h_{ex} \leq H_{c_1}$, there exists a minimizer of $J_a(u)$ in D_M^a which satisfies* (5) *and*

$1/2 \le |u_\varepsilon| \le 1$;
(ii) *if* $H_{c_1} + k_3^\varepsilon \le h_{ex} \le H_{c_1} + O(1)$, *there exists a solution* u_ε *of* (5) *that minimizes* $J_a(u)$ *in* D_M^a. *The solution has a bounded positive number of vortices* b_i^ε *of degree one, such that*

$$\mathrm{dist}(b_i^\varepsilon, \Lambda) \to 0, \ \text{as } \varepsilon \to 0,$$

and there exists a constant $\alpha > 0$ *such that* $\mathrm{dist}(b_i^\varepsilon, b_j^\varepsilon) \ge \alpha$ *for* $i \neq j$.

Theorem B. *Considering a sequence* $u_n = u_{\varepsilon_n}$ *of solutions of (5) given by Theorem A, then up to subsequence, there exist d points* $c_i \in \Lambda$ *such that* $u_n \to u_*$ *weakly in* $W^{1,p}$ $(p < 2)$ *and strongly in* $H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^d \{c_i\})$, *where* u_* *is a solution of*

$$\begin{cases} -\nabla \cdot (a(x)\nabla u_*) = a(x) u_* |\nabla u_*|^2, & \text{in } \Omega \setminus \cup_{i=1}^d \{c_i\} \\ \frac{\partial u_*}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega \\ |u_*| = 1, & \text{a.e. on } \Omega\ . \end{cases} \tag{7}$$

In this paper, we consider the cases

$$h_{ex} < H'_{c_1} := H'_{c_1} = H_{c_1} + O(|\ln|\ln \varepsilon||)$$

and

$$H_{c_1} << h_{ex} << H_{c_2} := c_0/\varepsilon^2\ .$$

In both cases we let $u \in H^1(\Omega)$ be a global minimizer of problem

$$\nu(\varepsilon) = \inf_{v \in H^1(\Omega)} J_a(v)\ . \tag{8}$$

The main results of this paper are stated as follows:

Theorem 1.1. *Let* $h_{ex} < H'_{c_1}$, *there exists, for small* ε, *a globally minimizing solution of (5) satisfying* $|u(x)| \ge \frac{3}{4}$ *on* Ω *which coincides with the solution found in the Theorem A (i).*

Theorem 1.2. *Let* $H_{c_1} << h_{ex} << H_{c_2} = c_0/\varepsilon^2$ *and* u_ε *be a minimizer of problem (8). Then*

$$J_a(u_\varepsilon) = \frac{1}{2} A_\Omega h_{ex} \ln \frac{1}{\varepsilon\sqrt{h_{ex}}}(1 + o(1)), \ \text{as } \varepsilon \to 0,$$

where $A_\Omega = \int_\Omega a(x)$. *Moreover, the upper bound*

$$J_a(u_\varepsilon) \le \frac{1}{2} A_\Omega h_{ex} \ln \frac{1}{\varepsilon\sqrt{h_{ex}}}(1 + o(1)), \ \text{as } \varepsilon \to 0,$$

is also true under weaker assumption $h_{ex} \le c_0/\varepsilon^2$ *for some* $c_0 > 0$.

Theorem 1.3. *Let* h_{ex} *be as in Theorem 1.2,* u_ε *be a minimizer of problem (8). Then there is* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$, *there exists a family of disjoint balls* $\{B_\varepsilon^i\}_{1 \le i \le k_{(\varepsilon)}}$ *with radii each less than* $\sqrt{h_{ex}}$ *and the sum less than* $|\Omega|\sqrt{h_{ex}}$ *so that*

$$|u_\varepsilon| \ge \frac{1}{2}, \quad \text{on } \partial B_\varepsilon^i.$$

If a_i^ε *is the center of* B_ε^i *and* d_i^ε *is the degree of* u_ε *on* ∂B_ε^i, *then*

$$\mu_\varepsilon = \frac{2\pi}{h_{ex}} \sum_{i=1}^{k_\varepsilon} d_i \delta_{a_i} \to a(x)dx\ .$$

The proof of the first theorem relies on a uniqueness result which is interesting by itself. We thus state it in a lemma and prove it here.

LEMMA 1.1. *The stable vortex-less solution of (5) is unique for sufficiently small ϵ and*

$$h_{ex} \le C\varepsilon^{-\alpha}$$

where $C > 0$ and $1 > \alpha > 0$ are positive constants.

Proof. Let $u_1, u_2 \in H^1(\Omega, \mathbb{R}^2)$ be two vortex-less stable minimizers. Similar to the results given in [7], we may assume that, without loss of generality, u_1 and u_2 satisfy, for $1 > \alpha > 0$, that $J_a(u_1) \le J_a(u_2)$ and

$$\int_\Omega |\nabla u_j|^2 \le C\ , \quad j = 1, 2\ .$$

Denote $\eta_j = |u_j|$, $j = 1, 2$. For vortex-less solutions, we have $\eta_j \ge 3/4$ on Ω, thus we may write

$$u_j = \eta_j e^{i\phi_j}, \quad j = 1, 2.$$

$(u_j, \mathbf{A}_0)$ is Gauge equivalent to $(\eta_j, \mathbf{A}_0')$ where $\mathbf{A}_0' = \mathbf{A}_0 - d\phi_j$. And, we have

$$J_a(u_j) = \frac{1}{2}\int_\Omega a(x)[|\nabla \eta_j|^2 + \eta_j^2|\mathbf{A}_0'|^2 + \frac{1}{2\varepsilon^2}(1 - \eta_j^2)^2].$$

It is easy to see that the first two terms of J_a are strictly convex in η_j, moreover, $F(x) = (1 - x^2)^2$ is also strictly convex for $x \ge 3/4$, so we have

$$J_a((1 - t)\eta_1 + t\eta_2) < J_a(\eta_2), \quad \forall t \in [0, 1],$$

which contradicts with the stability of η_2. Gauge invariance thus implies $u_1 = u_2$. The uniqueness is obtained. □

To prove the main results, we essentially follow the ideas used in [11] and combine them with the analysis given in [4]. The rest of the paper is organized as follows: in the section 2, the proof of theorem 1.1 is provided. The global minimizer for $H_{c_1} << h_{ex} << H_{c_2} = c_0/\varepsilon^2$ is considered in the section 3 along with the proofs of both theorems 1.2 and 1.3.

2. **Proof of Theorem 1.1.**

Let us first recall a proposition in [10].

PROPOSITION 2.1. (Proposition III.2 of [10]) *Let $u : \Omega \to \mathbb{R}^2$ be such that $|\nabla u| \le \frac{C}{\varepsilon}$ and that $F(u) \le C|\ln \varepsilon|^2$. Then for any $\alpha > 0$ there exist disjoint balls $\{B_i\}_{i\in I}$ of radii $\{r_i\}$ such that, for small ε,*

1). *$\{|u| < \frac{3}{4}\} \subset \cup_{i\in I} B_i$;*
2). *$Card(I) \le C|\ln \varepsilon|^2$;*
3). *$r_i \le C|\ln \varepsilon|^{-\alpha}$;*
4). *If $\overline{B_i} \subset \Omega$ and define $d_i = \deg(u, \partial B_i)$, then*

$$F(u, B_i \cap \Omega) \ge \pi|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)).$$

where $F(u, V) = \frac{1}{2}\int_V [|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2]$.

Using this proposition, we may prove the following

LEMMA 2.1. *Let $u : \Omega \to \mathbb{R}^2$ be such that $|\nabla u| \le \frac{C}{\varepsilon}$ and that $F_a(u) \le C|\ln \varepsilon|^2$. Then for any $\alpha > 0$ there exist disjoint balls $\{B_i\}_{i\in I}$ of radii $\{r_i\}$ centered at $\{a_i\}$ such that, for small ε,*

1). *$\{|u| < \frac{3}{4}\} \subset \cup_{i\in I} B_i$;*
2). *$Card(I) \le C|\ln \varepsilon|^2$;*
3). *$r_i \le C|\ln \varepsilon|^{-\alpha}$;*

4). *If* $\overline{B_i} \subset \Omega$ *and define* $d_i = \deg(u, \partial B_i)$*, then*

$$F_a(u, B_i \cap \Omega) \geq \pi a(a_i)|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)) - C|\ln|^{1-\alpha}. \tag{9}$$

where $F_a(u, V) = \frac{1}{2}\int_V a(x)[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2]$.

Proof. Conclusions 1)-3) can be proved exactly as in [10]. Now we prove the equation (9). Since $r_i \leq C|\ln \varepsilon|^{-\alpha}$, we have

$$|a(x) - a(a_i)| \leq Cr_i \leq C|\ln \varepsilon|^{-\alpha}, \ \forall x \in B_i(a_i).$$

It follows from Proposition 2.1 that

$$\begin{aligned}
F_a(u, B_i \cap \Omega) &= \frac{1}{2}\int_{B_i(a_i)} a(x)\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right] \\
&= \frac{1}{2}\int_{B_i(a_i)} a(a_i)\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right] \\
&= \frac{1}{2}\int_{B_i(a_i)} (a(x) - a(a_i))\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right] \\
&\geq a(a_i)F(u, B_i \cap \Omega) - \frac{C}{|\ln|^{\alpha}}F(u, B_i \cap \Omega) \\
&\geq \pi a(a_i)|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)) - C|\ln \varepsilon|^{1-\alpha}\ .
\end{aligned}$$

The Lemma is proved. □

As in [4], we have

LEMMA 2.2. *Let* u *be a solution of (5). There holds*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}. \tag{10}$$

If u *is in addition a minimizer of the energy, then*

$$J_a(u) \leq J^0 \leq Ch_{ex}^2, \tag{11}$$

$$\|\nabla u\|_{L^2(\Omega)} \leq Ch_{ex}. \tag{12}$$

Proof. (10) and (11) have been proved in [4]. We only need to prove (12). It follows from (11) that

$$\int_\Omega a(x)|\nabla_{\mathbf{A}_0} u|^2 \leq Ch_{ex}^2\ .$$

Since $|u| \leq 1$ and $|\mathbf{A}_0| \leq Ch_{ex}$, we get

$$\int_\Omega a(x)|\nabla u|^2 \leq Ch_{ex}^2\ .$$

Therefore we have (12). The Lemma is proved. □

Then, we may define the vortices of u with their degrees, by defining balls $\{B_i\}_{i\in I}$ of radius $\{r_i\}$ centered at $\{a_i\}$ satisfying Lemma 2.1 such that $|u| \geq 3/4$ on $\Omega \setminus \cup_{i\in I} B_i$ and $d_i =$deg$(u/|u|, \partial B_i)$ with

$$r_i \leq C|\ln \varepsilon|^{-\alpha}$$

where α is to be chosen below. Now, we prove

LEMMA 2.3. *If $\alpha > 5$, then*

$$J_a(u) = F_a + J^0 + 2\pi h_{ex} \sum_{i \in I} d_i \xi_0(a_i) + o(1) \tag{13}$$

where I and F_a satisfy Lemma 2.1,

$$J^0 = J_a(1) = \frac{1}{2} h_{ex}^2 \int_\Omega \frac{|\nabla \xi_0|^2}{a(x)}.$$

Proof. Since

$$\mathbf{A}_0 = h_{ex}(-\frac{\xi_{0x_2}}{a(x)}, \frac{\xi_{0x_1}}{a(x)}),$$

we have

$$\begin{aligned} a(x)|\nabla u - i\mathbf{A}_0 u|^2 &= a(x)|\nabla u|^2 + h_{ex}^2 |u|^2 \frac{|\nabla \xi_0|^2}{a(x)} + i h_{ex}(u^* \nabla^\perp \xi_0 \nabla u - u \nabla^\perp \xi_0 \nabla u^*) \\ &= a(x)|\nabla u|^2 + h_{ex}^2 |u|^2 \frac{|\nabla \xi_0|^2}{a(x)} + 2(\nabla u, -i h_{ex} \nabla^\perp \xi_0 u). \end{aligned}$$

However, we have from [4]

$$h_{ex}^2 \int_\Omega |u|^2 \frac{|\nabla \xi_0|^2}{a(x)} = h_{ex}^2 \int_\Omega \frac{|\nabla \xi_0|^2}{a(x)} + o(1)$$

and similar to [10], for $\alpha > 5$, that

$$\int_\Omega (\nabla u, -i h_{ex} \nabla^\perp \xi_0 u) = 2\pi h_{ex} \sum_{i \in I} d_i \xi_0(a_i) + o(1).$$

The conclusion of Lemma 2.3 follows easily from the above two relations. □

Since the balls in the set $\{B_i\}_{i \in I}$ are disjoint, we have

$$F_a(u, \Omega) \geq \sum_{i \in I} F_a(u, B_i) \ .$$

Using the lower bound on $F_a(u, B_i)$ in Lemma 2.2, we have

LEMMA 2.4. *If $\alpha > 5$, then*

$$\begin{aligned} J_a(u) \quad \geq \quad & \sum_{i \in I} \pi a(a_i)|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)) \\ & - \frac{C}{|\ln|^{\alpha - 3}} + J^0 + 2\pi h_{ex} \sum_{i \in I} d_i \xi_0(a_i) + o(1). \end{aligned} \tag{14}$$

Proof. It follows from

$$F_a(u, B_i) \geq \pi a(a_i)|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)) - C|\ln \varepsilon|^{1-\alpha}$$

that

$$\sum_{i \in I} F_a(u, B_i) \geq \pi \sum_{i \in I} a(a_i)|d_i|(|\ln \varepsilon| - O(|\ln|\ln \varepsilon||)) - C \,\mathrm{Card}(I)\, |\ln \varepsilon|^{1-\alpha}.$$

But

$$\mathrm{Card}(I) \leq C h_{ex}^2 \leq C|\ln \varepsilon|^2$$

we get (14) by Lemma 2.3. This Lemma is proved. □

Now we are in the position to prove the existence of the global minimizer having no vortices in Theorem 1.1. In fact, it follows from the minimality that

$$J_a(u) \leq J^0. \tag{15}$$

On the other hand, we have from Lemma 2.3 and 2.4 that

$$\begin{aligned} J_a(u) &= F_a + J^0 + 2\pi h_{ex} \sum_{i\in I} d_i \xi_0(a_i) + o(1) \\ &\geq J^0 + 2\pi h_{ex} \sum_{i\in I} d_i \xi_0(a_i) \\ &\quad + \pi \sum_{i\in I} a(a_i)|d_i|(|\ln\varepsilon| + O(|\ln|\ln\varepsilon||)) + o(1). \end{aligned} \tag{16}$$

Combining (15) with (16) , we have, since ξ_0 is negative, that

$$\begin{aligned} &\pi \sum_{i\in I} a(a_i)|d_i|(|\ln\varepsilon| + O(|\ln|\ln\varepsilon||)) \\ \leq\ & 2\pi h_{ex} \sum_{i\in I} |d_i||\xi_0(a_i)| + o(1) \\ \leq\ & 2\pi h_{ex} \max_{\Omega} \left|\frac{\xi_0(x)}{a(x)}\right| \sum_{i\in I} a(a_i)|d_i|. \end{aligned}$$

If $\sum_{i\in I} |d_i| \neq 0$, we get

$$\begin{aligned} h_{ex} &\geq \frac{1}{2\max\limits_{\Omega} |\xi_0(x)/a(x)|}(|\ln\varepsilon| + O(|\ln|\ln\varepsilon||)) \\ &= H_{c_1} + O(|\ln|\ln\varepsilon||) = H'_{c_1}. \end{aligned}$$

From the above discussions, we see that if $h_{ex} < H_{c_1'}$, then $d_i = 0$ for all $i \in I$. Moreover,

$$J^0 \geq J_a(u) \geq \sum_{i\in I} F_a(u, B_i) + J^0 + o(1)\ ,$$

and therefore

$$\sum_{i\in I} F_a(u, B_i) \leq o(1)\ . \tag{17}$$

The last claim implies $|u| \geq 3/4$ as in [2]. Otherwise, it follows from a well-known result of [2] that if there exists x_0 such that $|u(x_0) \leq 3/4|$, then there exist constants $\lambda, \mu > 0$ such that

$$\forall \varepsilon > 0,\quad \frac{1}{\varepsilon^2}\int_{B(x_0,\lambda\varepsilon)} (1 - |u|^2)^2 \geq \mu > 0$$

which contradicts (17).

Therefore u is vortex-less solution. We hence have

$$J_a(u) = J^0 + F_a(u) + o(1). \tag{18}$$

This means $F_a(u) = o(1) \leq M|\ln\varepsilon|$ and then $u \in \overline{D_M^a}$.

For $h_{ex} < H'_{c_1}$, by the uniqueness for the vortex-less minimizer stated in the lemma 1.1, the proof of Theorem 1.1 is complete.

3. **Global Minimizers for** $H_{c_1} << h_{ex} << H_{c_2}$**.** In this section, we begin to prove Theorem 1.2 and 1.3. Let the applied magnetic field h_{ex} satisfies $H_{c_1} << h_{ex} << H_{c_2}$.

Minimizers and even critical points of $J_a(u)$ are expected to exhibit a vortex structure for these values of h_{ex} and small ε. Away from the vortex, it is expected that $|u|$ is about 1.

In the previous section, when we discuss the Meissner solutions for $h_{ex} \leq H'_{c_1}$, we have removed the restriction $F_a < M|\ln\varepsilon|$; that is, we have proved that in that case, the Meissner solution given in [4] are actually global minimizers of the energy.

The goal of this section is to describe as precisely as possible the vortex structure of the minimizers or critical points including their number, position, degree, $\cdots$). However, for $h_{ex} >> H_{c_1}$, the approaches in section 2 are no longer suitable, because the number of vortices of minimizers is expected to be proportional to h_{ex}, and it may diverge. We therefore can only get very limited information on the vortices as we may see in the following.

From now, we always assume

$$H_{c_1} << h_{ex} << H_{c_2} = c/\varepsilon^2.$$

3.1. **Proof of Theorem 1.2.** First of all, we know that the inf $J_a(u)$ over $H^1(\Omega, \mathbb{R}^2)$ is achieved and that any critical point of $J_a(u)$ satisfies the equation (5) with $\mathbf{A}_0$ be given by (2).

Locally we write $u = \rho e^{i\varphi}$, $\rho = |u|$. It is clear that the solution of the equation (5) satisfies

$$|u| \leq 1, \quad |\nabla u| \leq \frac{C}{\varepsilon}, \quad \text{in } \Omega.$$

It is not difficult to see that the equation (5) can be rewritten as

$$-\nabla\cdot(a(x)\nabla u) + 2ia(x)\mathbf{A}_0\cdot\nabla u + a(x)|\mathbf{A}_0|^2 u = \frac{a(x)}{\varepsilon^2}u(1-|u|^2)$$

and for any solution of (5) we have

$$a(x)|\nabla_{\mathbf{A}_0}u|^2 = a(x)(|\nabla\rho|^2 + \rho^2|\nabla\varphi - \mathbf{A}_0|^2)\ ,$$

$$J_a(u) = \frac{1}{2}\int_\Omega a(x)(|\nabla\rho|^2 + \rho^2|\nabla\varphi - \mathbf{A}_0|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2).$$

In order to prove Theorem 1.2, we first give the upper bound for the energy by constructing a suitable comparison function in the following lemma.

Lemma 3.1. (upper bound of the energy). *For any function $h_{ex}(\varepsilon) \leq C\varepsilon^{-2}$, there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there is a function $u \in H^1(\Omega, \mathbb{R}^2)$ satisfying*

$$J_a(u) \leq \frac{1}{2}A_\Omega h_{ex}\ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1+o(1)). \tag{19}$$

where $A_\Omega = \int_\Omega a(x)dx$.

Proof. Let K be a square with side-length $\delta = \sqrt{2\pi/h_{ex}}$ centered at the origin. Define $\mu(x)$ as follows

$$\begin{cases} \mu(x) = 0, \text{ in } K\setminus B(0,\varepsilon) \\ \mu(x) = \frac{2}{\varepsilon^2}, \text{ in } B(0,\varepsilon) \end{cases}$$

such that

$$\int_K \mu = 2\pi.$$

Let h be the unique solution of the following problem

$$\begin{cases} -\Delta h + h = \mu, & \text{in } K \\ \frac{\partial h}{\partial \mathbf{n}} = 0, & \text{on } \partial K, \end{cases}$$

Then, it follows from [11] that $\overline{h} = \frac{1}{|K|}\int_K h = h_{ex}$ where $|K|$ is the area of K, and

$$\frac{1}{2}\int_K |\nabla h|^2 + |h - h_{ex}|^2 \leq \pi \ln \frac{1}{\varepsilon\sqrt{h_{ex}}} + O(1). \tag{20}$$

Extending μ and h by periodicity to $\mathbb{R}^2$, we get that h is periodic with respect to the lattice, continuous and belongs to $H^1_{\text{loc}}(\mathbb{R}^2)$. Note that though $\mathbf{A}_0(x)$ is not periodic, it does satisfy $\mathbf{A}_0 \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and we can extend it to $H^1_{\text{loc}}(\mathbb{R}^2)$.

Define a vector $\vec{H}(x)$ for any $x = (x_1, x_2)$ by

$$\vec{H}(x_1, x_2) = (-\int_0^{x_2} (h(x_1, s) - h_{ex})ds, 0)$$

such that

$$\mathbf{curl}\vec{H} = h(x) - h_{ex}.$$

With such an h, we define φ in $\mathbb{R}^2 \setminus B(a_i, \varepsilon)$ by

$$\varphi(x) = \int_{x_0}^{x} (\nabla^{\perp} h + (\mathbf{A}_0 + \vec{H}))\ , \text{ in } \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$$

or

$$\nabla\varphi = \nabla^{\perp} h + \mathbf{A}_0 + \vec{H}\ , \text{ in } \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$$

where a_i's denote the center of the squares K_i that tile $\mathbb{R}^2$. In the definition for any point $x_0 \in \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$, define for any $x \in \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$,

$$\varphi(x) = \int_\gamma -\frac{\partial h}{\partial \mathbf{n}} + (\mathbf{A}_0 + \vec{H}) \cdot \tau$$

where $\gamma \subset \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$ is any curve joining x_0 to x, τ is the unit tangential to γ and $(\tau, \mathbf{n})$ is a direct orthonormal frame of $\mathbb{R}^2$. It follows from [11] that this definition is independent of the choice of γ. Indeed, the values of φ given by two different curves differ only by a multiple of 2π, or equivalently that for any closed curve $\Gamma \subset \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$ and enclosing a domain V, we have

$$\begin{aligned} \int_\Gamma \frac{\partial \varphi}{\partial \tau} &= \int_\Gamma -\frac{\partial h}{\partial \mathbf{n}} + \mathbf{A}_0 \cdot \tau + \vec{H} \cdot \tau \\ &= \int_V -\Delta h + h = \sum_{i\in I, B(a_i,\varepsilon)\subset V} \int_{B(a_i,\varepsilon)} \mu \\ &= 2\pi \text{Card}(\{i \in I; B(a_i, \varepsilon) \subset V\})\ . \end{aligned} \tag{21}$$

Now, $e^{i\varphi}$ is well-defined on $\mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, \varepsilon)$ with degree one around each a_i, we then define

$$\rho \equiv 0, \quad \text{on } \cup_{i\in I} B(a_i, \varepsilon); \tag{22}$$

$$\rho \equiv 1, \quad \text{on } \mathbb{R}^2 \setminus \cup_{i\in I} B(a_i, 2\varepsilon); \tag{23}$$

such that

$$\int_{B(a_i,\varepsilon)} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1 - \rho^2)^2 \leq C; \quad \forall i \in I.$$

Let $u = \rho e^{i\varphi}$. Note that

$$\left| a(x) - \frac{A_K}{|K|} \right| \leq C\delta, \; x \in K.$$

With the above construction, we get

$$\begin{aligned}
J_a(u,K) &= \frac{1}{2}\int_K a(x)[|\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2] \\
&= \frac{1}{2}\int_K a(x)[|\nabla \rho|^2 + \rho^2 |\nabla\varphi - \mathbf{A}_0|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2] \\
&\leq \frac{1}{2}\int_{K\setminus B(a,\varepsilon)} a(x)|\nabla\varphi - \mathbf{A}_0|^2 + C \\
&\leq \frac{1}{2}\int_K a(x)|\nabla h|^2 + C \\
&\leq \frac{1}{2}\cdot\frac{A_K}{|K|}\int_K |\nabla h|^2 + C + C\delta\pi \ln\frac{1}{\varepsilon\sqrt{h_{ex}}} \\
&\leq \frac{A_K}{|K|}\pi \ln\frac{1}{\varepsilon\sqrt{h_{ex}}} + C
\end{aligned}$$

where we have used the fact:

$$|\nabla\varphi - \mathbf{A}_0|^2 = |\nabla h|^2 + 2\nabla h \cdot \vec{H} + |\vec{H}|^2$$

and the fact that, from the definition of $\vec{H}$, for $x \in K$ we have

$$\int_K \rho^2(2\nabla h \cdot \vec{H} + |\vec{H}|^2) \leq o(1)\ , \tag{24}$$

since

$$|\vec{H}| \leq \delta \|h - h_{ex}\|_{L^\infty(K)}\ ,$$

$$\|h - h_{ex}\|^2_{L^\infty(K)} \leq \|h - h_{ex}\|^2_{H^1(K)} \leq \pi \ln\frac{1}{\varepsilon\sqrt{h_{ex}}}\ .$$

Repeating the above computation, we see from (20) that

$$E_0 = \frac{1}{2}\int_K [|\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2] \leq \pi\ln\frac{1}{\varepsilon\sqrt{h_{ex}}} + C. \tag{25}$$

Denote

$$e(x) = |\nabla_{\mathbf{A}_0} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2.$$

By periodicity, we have

$$\int_\Omega a(x)\int_{y\in K} e(x+y)dydx = A_\Omega E_0.$$

By Fubini Theorem we have $y_0 \in K$ such that

$$\int_\Omega a(x)e(x+y_0)dx = A_\Omega E_0/|K|\ . \tag{26}$$

By defining $u_1(x)$ to be the restriction of $u(x+y_0)$ on Ω and using $|K| = 2\pi/h_{ex}$, we get

$$\begin{aligned}
J_a(u_1) &= \int_\Omega a(x)e(x+y_0)dx \leq A_\Omega E_0|K| \\
&\leq \frac{1}{2}A_\Omega h_{ex}\ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1+o(1)).
\end{aligned} \tag{27}$$

The function u_1 is the comparison function we want to find. Lemma 3.1 is proved. □

Note that if u is a minimizer of $J_a(u)$, then (19) and (22-23) are true.

We next give the lower bound for the energy of the minimizers to match the upper bound obtained above. To do this, we need to construct vortices for given minimizers. That is to construct disjoint balls that contain the set $\{x : |u(x)| \leq 1/2\}$. However, u may have a line of zeros crossing Ω. This is possible with a cost in the energy no bigger than C/ε. In viewing the upper bound, if $h_{ex} \geq C/\varepsilon$, a line of zeros can not be excluded.

Thus it is not possible to construct disjoint balls to cover the set $\{x : |u(x)| \leq 1/2\}$. But as in [11], we may obtain lower bound by localizing the energy, that is, cutting Ω into squares K_i of side length tending to zero and classify them into "nice" squares and "bad" squares.

In the sequel we always denote the minimizer by u.

LEMMA 3.2. (Lower bound of the energy). *Suppose that* $|\ln\varepsilon| << h_{ex}(\varepsilon) << 1/\varepsilon^2$ *and* u *is a minimizer of* $J_a(u)$ *over* $H^1(\Omega, \mathbb{R}^2)$. *Then*

$$J_a(u) \geq \frac{1}{2}A_\Omega h_{ex} \ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1+o(1)). \tag{28}$$

Proof. Let $J(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \rho^2|\nabla\varphi - \mathbf{A}_0|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2$. Then exactly following the proof of [11] we have

$$J(u,K) \geq \frac{1}{2}h_{ex}\delta^2 \ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1-o(1)).$$

It follows from (21) and (24) that

$$J_a(u,K) = \frac{A_K}{|K|}J(u,K) - C\delta J(u,K)\ .$$

Since u is a minimizer of $J_a(u)$, it follows from the above discussions that

$$J(u,K) \leq \pi\ln\frac{1}{\varepsilon\sqrt{h_{ex}}} + C\ .$$

Hence

$$\begin{aligned} J_a(u,K) &\geq \frac{A_K}{|K|}J(u,K) - o(1) \\ &= \delta^{-2}A_K J(u,K) - o(1) \\ &\geq \frac{1}{2}A_K h_{ex}\ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1-o(1)). \end{aligned} \tag{29}$$

It follows from (25) that

$$\begin{aligned} J_a(u) &= \sum_{i=1}^{N(\varepsilon)} J_a(u,K_i) \\ &= \frac{1}{2}h_{ex}\ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1-o(1))\sum_{i=1}^{N(\varepsilon)} A_{K_i} \\ &= \frac{1}{2}A_\Omega h_{ex}\ln\frac{1}{\varepsilon\sqrt{h_{ex}}}(1-o(1)) \end{aligned} \tag{30}$$

where $N(\varepsilon)$ is the number of the squares included in Ω which makes $\sum_{i=1}^{N(\varepsilon)} A_{K_i}$ going to A_Ω as $\varepsilon \to 0$. Lemma 3.2 is proved. □

Combining Lemma 3.1 with Lemma 3.2, the proof of Theorem 1.2 is completed.

3.2. **Energy concentration: Proof of Theorem 1.3.** The idea of the proof of theorem 1.3 is: first we prove that the energy is concentrated in the so called "nice" squares, and then the energy on every nice square is concentrated on some smaller balls, these properties then enable us to apply the arguments in [11] to complete the proof.

In the previous section we we have constructed squares such that $\Omega \subset \cup_i \overline{K_i}$. We know from (25) that for any K_i,

$$E_i := J_a(u, K_i) \geq \frac{1}{2} A_{K_i} h_{ex} L(1 - o(1)). \tag{31}$$

Let I be the set of the indices such that $K_i \subset \Omega$ and

$$\sum_{i \in I} E_i \leq \frac{1}{2} A_\Omega h_{ex} L(1 + o(1)).$$

On the other hand, it follows from (26) that

$$\sum_{i \in I} E_i \geq \frac{1}{2} \sum_{i \in I} A_{K_i} h_{ex} L(1 + o(1)).$$

Combining (27), we have

$$\sum_{i \in I} \left| E_i - \frac{1}{2} A_{K_i} h_{ex} L \right| \leq A_\Omega h_{ex} L f(\varepsilon)$$

where $f(\varepsilon) = o(1)$ as in (27). Define the square K_i as a nice square if

$$E_i \leq \frac{1}{2} A_{K_j} h_{ex} L(1 + \sqrt{f(\varepsilon)})$$

and denote the set of indices of nice squares by

$$\mathcal{J} = \{i \in I; K_i \text{ is a nice square}\}.$$

The rest of the squares are defined as bad squares.

Lemma 3.3. *As* $\varepsilon \to 0$,

$$\sum_{i \in \mathcal{J}} E_i = J_a(u) + o(1).$$

Proof. Let $\mathcal{J}' = I \setminus \mathcal{J}$, $M = \text{Card}\mathcal{J}'$. Then by the definition of bad squares and (27) we have

$$\frac{1}{2} h_{ex} L \sqrt{f(\varepsilon)} \sum_{i=1}^{M} A_{K_i} \leq \sum_{i \in \mathcal{J}'} (E_i - \frac{1}{2} h_{ex} A_{K_i} L) \leq \frac{1}{2} h_{ex} A_\Omega L f(\varepsilon)$$

and hence

$$\sum_{i=1}^{M} A_{K_i} \leq A_\Omega \sqrt{f(\varepsilon)} = o(1) .$$

Then it follows from (29) and (30) that

$$\sum_{i \in \mathcal{J}'} E_i \leq \frac{1}{2} h_{ex} L A_\Omega o(1) .$$

Using (31), we get

$$J_a(u) \geq \sum_{i\in\mathcal{J}} E_i \geq \sum_{i\in I} E_i - \sum_{i\in\mathcal{J}'} E_i \geq \frac{1}{2} A_\Omega h_{ex} L(1-o(1)) \ .$$

This combined with Theorem 1.2 gives the conclusion of the Lemma 3.3. □

The above lemma shows that the energy on Ω is concentrated on the nice squares. Following [11], the next lemma can be proved:

LEMMA 3.4. *Let $f(\varepsilon) = o(1)$ be as before, K be a nice square, that is*

$$J_a(u,K) \leq \frac{1}{2} A_K h_{ex} \ln \frac{1}{\varepsilon\sqrt{h_{ex}}}(1+(f(\varepsilon))^{\frac{1}{2}}).$$

Then there exist disjoint balls $B_1, B_2, \cdots, B_k$ with

$$\sum_{i=1}^{k} r_i \leq h_{ex}^{-\frac{1}{2}}$$

such that

$$\sum_{i=1}^{k} J_a(u,B_i) = J_a(u,K)(1-o(1)).$$

Moreover, $|u| > 1/2$ on each ∂B_i, $\{d_i = \deg(\frac{u}{|u|}, \partial B_i)\}$ are well-defined and

$$2\pi \sum_{i=1}^{k} d_i = h_{ex} A_K(1+o(1)) \ ,$$

$$2\pi \sum_{i=1}^{k} |d_i| = h_{ex} A_K(1+o(1)) \ .$$

The above lemma verifies that the energy on every nice square should be concentrated on some smaller balls. By now, we have presented all the necessary ingredients needed for deriving the Theorem 1.3. Finally, by repeating the argument provided in [11] page 38-41, the proof of the Theorem 1.3 can then be completed.

4. **Conclusion.** Drawing ideas from [11] and [4, 7], we have proved here that for h_{ex} that is near but smaller than H_{c_1}, the global minimizer to a Ginzburg-Landau model developed in [3] for the superconducting thin films with no vortex is unique and it is precisely those found in [4]. We have also shown that the the density of the vortices of the global minimizer is proportional to the applied field for field well in between the upper and lower critical fields. Such results enable us to rigorously justify some conclusions based on the heuristic arguments or support related physical observations. A complete description of the global minimizer in the latter case is yet unavailable. Further investigations are needed also for the case of the applied field being close to the upper critical field.

Acknowledgement. The authors are partially supported by the Natural Science Foundation of China No.19971030, the Natural Science Foundation of Guangdong No.000671, the State Key basic research project G199003280 and a grant from HKRGC.

REFERENCES

[1] A. Abrikosov, *On the magnetic properties of superconductivity of the second type, Soviet Phys. JETP*, 5, (1957), 1174-1182.

[2] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices,* Birkhäuser, (1994).

[3] S. Chapman, Q. Du, and M. Gunzburger, *A model for variable thickness superconducting thin films, ZAMP*, 47 (1996), pp410-431,

[4] S. Ding, Q. Du, *Critical magnetic field and asymptopic behavior of superconducting thin films,* to appear in *SIAM J. Math Anal*, (2002).

[5] Q. Du, M. Gunzburger and J. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, *SIAM Review*, 34 (1992), pp54-81.

[6] G. Lasher, *Mixed states of type-I superconducting films in a perpendicular magnetic field*, *Phys. Rev.*, 154 (1967), pp345-348.

[7] F. Lin, *Solutions of Ginzburg-Landau equations and critical points of the renormalized energy*, *Analyse non Liméuire, IHP*, 12 (1995), pp599-622.

[8] F. Lin and Q. Du, *Ginzburg-Landau vortices: dynamics, pinning and hysteresis*, *SIAM J. Math. Anal.*, 28 (1997), pp1265-1293.

[9] K. Maki, *Fluxoid structure in superconducting films*, *Ann. Phys.*, 34 (1965), pp363-376.

[10] E. Sandier, S. Serfaty, *Global minimizers for the Ginzburg-Landau functional below the first critical magnetic field, Analyse non Liméuire, IHP*, 17, (2000), 119-145.

[11] E. Sandier, S. Serfaty, *On the energy of type-II superconductors in the mixed phase, Revs. Math. Phys.*, 12, (2000), 1219-1257.

[12] M. Tinkham, Introduction to Superconductivity, 2nd edition, New York, McGraw-Hill, 1994.

Received June 2001; revised December 2001.

E-mail address: dingsj@scnu.edu.cn
E-mail address: madu@ust.hk