Differentiable Stacks, Gerbes, and Twisted K-Theory

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Chapitre 1

Lie Groupoids and Differentiable Stacks

1.1 Groupoids

1.1.1 Definitions

Groupoids are a generalization of groups, where the multiplication may not always be defined for all pairs of points. The notion of groupoids was first introduced by W. Brandt in 1926 [**?**], which blends the concepts of space and group : they have both space-like and group-like properties that interact in a delicate way. See [11, 35] for nice surveys on groupoids. We also refer to [21, 20] for general theory of Lie groupoids.

In short, a groupoid is a small category where all morphisms are invertible. We may also define a groupoid more explicitly as follows.

Definition 1. *A groupoid consists of a set X_0, called the set of units, and a set X_1, called the set of morphisms, together with the following structure maps :*

(a) *A pair of maps $s,\ t : X_1 \to X_0$, called the source and the target map, respectively ;*

(b) *A multiplication $m : X_2 \to X_1$, where $X_2 = \{(x,y) | \forall x, y \in X_1,\ t(x) = s(y)\}$ is called the set of composable pairs. We usually denote $m(x,y)$ by $x \cdot y$. It is required to satisfy the following properties :*
   - *$s(x \cdot y) = s(x),\ t(x \cdot y) = t(y)\ \forall (x,y) \in X_2$, and*
   - *$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativity) whenever one side is defined ;*

(c) *An embedding $\varepsilon : X_0 \to X_1$, called the unit map, such that*

$$\varepsilon\left(s(x)\right) \cdot x = x \cdot \varepsilon\left(t(x)\right) = x,\ \forall x \in X_1.$$

(d) *An inverse map $\iota : X_1 \to X_1$, denoted also by $\iota(x) = x^{-1}$ such that*

$$x^{-1} \cdot x = \varepsilon\left(t(x)\right),\ x \cdot x^{-1} = \varepsilon\left(s(x)\right).$$

Such a groupoid is usually denoted $X_1 \rightrightarrows X_0$.

Let $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ be groupoids. Then $X_1 \times Y_1 \rightrightarrows X_0 \times Y_0$ admits a natural groupoid structure, called *the product groupoid.*

Charles Ehresmann was the first one who introduced smooth structures on groupoids, which led to the notion of Lie groupoids.

Definition 2. *A Lie groupoid is a groupoid $X_1 \rightrightarrows X_0$, where both X_1 and X_0 are smooth manifolds, the source and target maps are submersions, and all the structure maps are smooth.*

We ask that the source and target maps are submersions so that X_2 is a smooth manifold.

Definition 3. *A Lie groupoid is called* proper *if the map $(s,t) : X_1 \to X_0 \times X_0$ is a proper map. That is, the inverse image in X_1 of any compact subset of $X_0 \times X_0$ under the map (s,t) is still compact.*

Definition 4. *A Lie groupoid is called* étale *if both the source and target maps are local diffeomorphisms.*

Proper Lie groupoids play, in the context of Lie groupoids, a role similar to that of compact Lie groups in Lie theory.

Remark 5. *One can also consider topological groupoids in a similar fashion. See [24]*

1.1.2 Examples

Example 6 (Groups)**.** *A group is clearly a special case of groupoids, where the unit set is a point (a set with only one element), and both the source and target maps project any element of G to this point. Hence, a Lie group is a special case of Lie groupoid. More generally, any bundle of Lie groups is a Lie groupoid where the source map and the target map coincide.*

Example 7 (Sets)**.** *Let X be a set. Then $X \rightrightarrows X$ is a groupoid. Here $X_1 = X_0 = X$. The source, target and unit maps are the identity map, and the multiplication and the inverse are, respectively, $x \cdot x = x$, and $x^{-1} = x$. In particular, any smooth manifold is a Lie groupoid.*

Example 8 (Pair groupoids/Banal groupoids)**.** *Given a set X, let $X_1 = X \times X$, $X_0 = X$ and $s(x,y) = x$, $t(x,y) = y$. Composable pairs are $((x,y),(y,z))$ for any x,y,z in X. Define the multiplication, the unit, and the inverse, respectively, by*

$$(x,y) \cdot (y,z) = (x,z), \quad \varepsilon(x) = (x,x), \quad (x,y)^{-1} = (y,x).$$

Then $X \times X \rightrightarrows X$ is clearly a groupoid. It is a Lie groupoid if X is a smooth manifold.

In general, given a map $\phi : X \to M$, $X \times_M X \rightrightarrows X$ is a groupoid, called Banal groupoid. *It is a Lie groupoid when both X and M are smooth manifolds and ϕ is a submersions.*

Example 9. *[Transformation groupoids] Let M be a set, G a group acting on M from the right. Take $X_1 = M \times G$, and $X_0 = M$. The source and target maps are $s(x,g) = x$, $t(x,g) = xg$. Then composable pairs are $((x,g),(xg,h))$ for any g,h in G and x in M. The multiplication is defined by*

$$(m,g) \cdot (mg,h) = (m,gh),$$

the unit map is $\varepsilon(m) = (m, 1_G)$, and the inverse is $(m,g)^{-1} = (mg, g^{-1})$. This groupoid is usually called transformation or action groupoid, and denoted by $M \rtimes G \rightrightarrows M$. When M is a smooth manifold, and G is a Lie group acting on M smoothly from the right, the transformation groupoid $M \rtimes G \rightrightarrows M$ is a Lie groupoid.

Example 10. *[Gauge groupoids] Let G be a Lie group and let $P \xrightarrow{\pi} M$ be a principal* left *G-bundle. Set $X_1 := \frac{P \times P}{G}$ and $X_0 = M$, where G acts on $P \times P$ by the diagonal action. For any p, q in P, $[p, q]$ denotes the class of (p, q) in X_1. Define the source and target maps, respectively, by $s([p, q]) = \pi(p)$, and $t([p, q]) = \pi(q)$. Then $([p, q], [r, s])$ is a composable pair if and only if $\pi(q) = \pi(r)$, that is, if and only if there exists an element g in G such that $r = gq$. The multiplication is defined by*

$$[p, q] \cdot [gq, s] = [gp, gq] \cdot [gq, s] = [gp, s].$$

The unit is $\varepsilon(m) = [p, p]$ for any $p \in \pi^{-1}(m)$, and the inverse is $[p, q]^{-1} = [q, p]$. This is called the gauge groupoid of P. It is clear that a gauge groupoid is a Lie groupoid.

Example 11 (Fundamental groupoids)**.** *Let M be a topological space. By $\Pi(M)$, we denote the space of all base points preserving homotopy classes of continuous paths in M. Let $X_0 := M$ and $X_1 := \Pi(M)$. Then $X_1 \rightrightarrows X_0$ is a groupoid, called the fundamental groupoid of the topological space M. The structure maps are defined as follows.*

Denote by $r(x, y)$ a path from x to y, and $[r(x, y)]$ its base points preserving homotopy class. Set $s([r(x, y)]) = x$ and $t([r(x, y)]) = y$. Then composable pairs are of the form $([r(x, y)], [r'(y, z)])$. We define multiplication by

$$[r(x, y)] \cdot [r'(y, z)] = [(r \circ r')(x, z)],$$

the homotopy class of the concatenation of the two paths. The unit is $\varepsilon(x) = [r(x, x)]$ and the inverse is $[r(x, y)]^{-1} = [\bar{r}(y, x)]$, where $r(x, x)$ denotes the constant path at the point x, and $\bar{r}(y, x)$ the inverse path of $r(x, y)$.

In general, fundamental groupoids are not Lie groupoids, but are topological groupoids.

Example 12 (Holonomy groupoids)**.** *Let M be a smooth manifold, and $D \subseteq TM$ an integrable distribution. According to Frobenius theorem, D defines a foliation $\mathcal{F}$ on M. By a D-path, we mean a path in M whose tangent vectors lie in D. Fix a leaf $\mathcal{O}$ of the foliation $\mathcal{F}$. Let $r : [0, 1] \to \mathcal{O}$ be a D-path in $\mathcal{O}$. Choose transversal sections N_0 and N_1 at the points $r(0)$ and $r(1)$, respectively (i.e. submanifolds of M transversal to the leaves of the foliation with $r(0) \in N_0$ and $r(1) \in N_1$). For any point x in N_0 sufficiently close to $r(0)$, there exists a unique D-path from x to some point y in N_1 that is close to r. One can check that the the map $x \mapsto y$ is a local diffeomorphism from $(N_0, r(0))$ to $(N_1, r(1))$. Its germ $\mathrm{hol}^{N_0,N_1}(r)$ is a well-defined function of paths r in $\mathcal{O}$ and the transversal sections N_0 and N_1.*

If L_0 and L_1 is another pair of transversal sections at the points $r(0)$ and $r(1)$, respectively, then

$$\mathrm{hol}^{L_0,L_1}(r) = \mathrm{hol}^{L_0,N_0}(\widetilde{r(0)}) \circ \mathrm{hol}^{N_0,N_1}(r) \circ \mathrm{hol}^{N_1,L_1}(\widetilde{r(1)}), \tag{1.1}$$

where $\widetilde{r(0)}$ and $\widetilde{r(1)}$ denote the constant paths corresponding to the points $r(0)$ and $r(1)$, respectively. Two paths r and $\widetilde{r}$ in $\mathcal{O}$ with the same base points are said to be equivalent if they have the same holonomy. Due to Equation (1.1), this is indeed a well defined equivalence relation. By $[r]$, we denote the equivalence class of a D-path r.

Set

$$Hol(M, \mathcal{F}) = \{(x, [r], y) | x, y \in M, r \text{ is a } D - \text{path from } x \text{ to } y\}.$$

Define the source and target maps by $s(x, [r], y) = x$ and $t(x, [r], y) = y$, respectively, and the multiplication by the natural one induced by the concatenation of paths. There is also an obvious unit map and inverse map. One can check that $Hol(M, \mathcal{F}) \rightrightarrows M$ is indeed a groupoid, called the holonomy groupoid. It turns out that the holonomy groupoid $Hol(M, \mathcal{F}) \rightrightarrows M$ satisfies all the axioms of Lie groupoids except that $Hol(M, \mathcal{F})$ may not be Hausdorff. We refer the interested reader to [15, 29, 18] for a detailed exposition.

1.1.3 Some general constructions

Definition 13 (Lie groupoid morphisms)**.** *Let $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ be Lie groupoids. A* morphism of Lie groupoids *is a pair of smooth maps $\phi_1 : X_1 \to Y_1$, $\phi_0 : X_0 \to Y_0$ such that the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & Y_1 \\ {\scriptstyle s_X, t_X}\downarrow\uparrow{\scriptstyle \varepsilon_X} & & {\scriptstyle s_Y, t_Y}\downarrow\uparrow{\scriptstyle \varepsilon_Y} \\ X_0 & \xrightarrow[\phi_0]{} & Y_0 \end{array} \tag{1.2}$$

commutes, and

$$\phi_1(x \cdot y) = \phi_1(x) \cdot \phi_1(y), \quad \phi_1(x^{-1}) = \phi_1(x)^{-1}, \ \forall (x, y) \in X_2, \ x \in X_1.$$

We denote the morphism in (1.2) by $\phi_\bullet : X_\bullet \to Y_\bullet$.

Definition 14. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. A Lie groupoid $Y_1 \rightrightarrows Y_0$ is said to be a* Lie subgroupoid *of $X_1 \rightrightarrows X_0$ if Y_1 and Y_0 are submanifolds of X_1 and Y_0, respectively, and the natural inclusion map is a Lie groupoid morphism.*

Here, and in the sequel, a submanifold means an embedded submanifold.

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid with source map s and target map t, and Y_0 a submanifold of X_0. Assume that the map $(s, t) : X_1 \to X_0 \times X_0$ is transversal to the submanifold $Y_0 \times Y_0 \subset X_0 \times X_0$. Then $X_1|_{Y_0}^{Y_0} := s^{-1}(Y_0) \cap t^{-1}(Y_0)$ is a submanifold of X_1. It is simple to see that $X_1|_{Y_0}^{Y_0} \rightrightarrows Y_0$ is a Lie subgroupoid of $X_1 \rightrightarrows X_0$.

If $Y_0 = \{x_0\}$ is any point in X_0, then $X_1|_{x_0}^{x_0}$ is a group, called *the isotropy group* at x_0. Since the map $(s, t) : X_1 \to X_0 \times X_0$ is, in general, not a submision at (x_0, x_0), the transveral condition above does not hold in this situation. Thus, a priori, it is not clear whether the isotropy group $X_1|_{x_0}^{x_0}$ is a Lie group. However, one can prove the following

Proposition 15 (PRECISE REF [21])**.** *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Then for any $\{x_0\} \in X_0$, the isotropy group $X_1|_{x_0}^{x_0}$ is a Lie group.*

Definition 16 (Orbits and coarse moduli space)**.** *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Two elements x_0 and $y_0 \in X_0$ are said to be equivalent if there exists $x \in X_1$ such that $s(x) = x_0$ and $t(x) = y_0$. This defines an equivalence relation in X_0. The equivalence classes are called the orbits of the groupoid $X_1 \rightrightarrows X_0$. The space of orbits is called the* coarse moduli space *of the groupoid.*

In general, the coarse moduli space is only a topological space, and not necessarily a smooth manifold.

In the case of a transformation groupoid $M \rtimes G \rightrightarrows M$ (Example (9)), “isotropic groups" and the “orbit space" of the groupoid become the usual ones for the group action. This is exactly the reason where these terminologies come from.

Proposition 17. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Then*
- *For any $x_0 \in X_0$, the isotropic group $X_1|_{x_0}^{x_0}$ is a Lie group.*
- *If x_0 and y_0 are in the same groupoid orbit, their corresponding isotropic groups are isomorphic.*

PROOF. (1) Consider the map $t : s^{-1}(x_0) \to X_0$. One proves that this is a constant rank map. Therefore, $s^{-1}(x_0) \cap t^{-1}(x_0)$ is a smooth submanifold of $s^{-1}(x_0)$, and hence a Lie group.

(2) Assume that x_0 and y_0 are in the same groupoid orbit. Then there exists $x \in X_1$ such that $s(x) = x_0$ and $t(x) = y_0$. One checks that the map $y \to xyx^{-1}$ is an isomorphism of Lie groups from $X_1|_{y_0}^{y_0}$ to $X_1|_{x_0}^{x_0}$. □

1.1.4 Bisections

Definition 18. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid with source map s and target map t. A submanifold $L \subset X_1$ is called a bisection if it is a section for both the source and the target map.*

If L is a bisection of $X_1 \rightrightarrows X_0$, then $s|_L$ and $t|_L$ are diffeomorphisms from L to X_0. The product of two bisections is defined as follows. Denote by $U(X_\bullet)$ or $U(X_1 \rightrightarrows X_0)$ the set of all bisections of $X_1 \rightrightarrows X_0$. Then let, for $L_1, L_2 \in U(X_\bullet)$,

$$L_1 \cdot L_2 = \{x \cdot y \,|\, x \in L_1,\ y \in L_2,\ t(x) = s(y)\}.$$

This makes $U(X_\bullet)$ into a Lie group (usually infinite dimensional) with unit being the submanifold $\varepsilon(X_0) \subset X_1$.

Example 19.
- *For a Lie group G, $U(G \rightrightarrows \{*\}) \cong G$.*
- *If $M \times M \rightrightarrows M$ is the pair groupoid corresponding to a manifold M, then $L \subset M \times M$ is a bisection if and only if L is the graph of a diffeomorphism. Hence,*

$$U(M \times M \rightrightarrows M) \cong \mathit{Diff}(M),$$

the diffeomorphism group of M.
- *Let G be a Lie group, and $S^1 \times G \rightrightarrows S^1$ be the bundle of groups over S^1, considered as a Lie groupoid. Then a bisection is a smooth map form S^1 to G, and $U(S^1 \times G \rightrightarrows S^1) \cong \mathfrak{C}^\infty(S^1, G)$ is the loop group of G.*

For any bisection $L \in U(X_\bullet)$, one can define the adjoint action of L on $X_1 \rightrightarrows X_0$ in a usual fashion :

$$Ad_L x = L \cdot x \cdot L^{-1}, \quad \forall x \in X_1.$$

Proposition 20. *For any $L \in U(X_\bullet)$, Ad_L is an automorphism of the Lie groupoid $X_1 \rightrightarrows X_0$.*

Remark 21. *For a given Lie groupoid $X_1 \rightrightarrows X_0$, for any $x \in X_1$, there always exists a local bisection through the point x although a global bisection may not exist. However, when x is close enough to the unit space, a global bisection through x always exists [21].* Ping: REF ?

Ping: EX ?

1.2 Morita equivalence

This section is devoted to the discussion on an important equivalence relation of Lie groupoids, the so called *Morita equivalence*.

1.2.1 Lie groupoid torsors

We shall first generalize to Lie groupoids the notion of group actions and torsors.

Definition 22. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. A* left $X_1 \rightrightarrows X_0$ or $X_\bullet$-space *consists of a smooth manifold Z, together with a smooth map $Z \stackrel{J}{\to} X_0$, called the anchor map (or momentum map), and a smooth map $X_1 \times_{t,X_0,J} Z \to Z$, $(x,z) \mapsto x \cdot z$, called the action map, satisfying*

(a) $J(x \cdot z) = s(x)$, $\forall x \in X_1, z \in Z$;

(b) $(x_1 x_2) \cdot z = x_1 \cdot (x_2 \cdot z)$, *whenever one side is defined ;*

(c) $\varepsilon(J(z)) \cdot z = z$, $\forall z \in Z$

Here $X_1 \times_{t,X_0,J} Z = \{(x,z) \in X_1 \times Z \,|\, t(x) = J(z)\}$ denotes the fiber product of X_1 and Z over X_0.

Right $X_1 \rightrightarrows X_0$-spaces can be defined similarly.

Example 23. *[Transformation groupoids] The construction of transformation groupoids in Example 9 extends to groupoid actions. Consider $Y_1 = X_1 \times_{t,X_0,J} Z \to Z$ and $Y_0 = Z$. Define the source, target, and multiplication maps, respectively, as follows : $s(x,z) = x \cdot z$, $t(x,z) = z$ and $(x,z)\cdot(x',z') = (xx',z')$. One can also define the unit map and the inverse map accordingly and check easily that $Y_1 \rightrightarrows Y_0$ is indeed a groupoid, denoted $X_1 \ltimes Z \rightrightarrows Z$. It is simple to see that $X_1 \ltimes Z \rightrightarrows Z$ is also a Lie groupoid.*

Naturally, the action is said to be *free* if $x \cdot z = z$ implies that $x = \varepsilon(J(z))$, for any $z \in Z$, and it is said to be *proper* if the map $(\lambda, \mathrm{pr}_2) : X_1 \times_{t,X_0,J} Z \to Z \times Z$ is a proper map, i.e. the inverse image of any compact set is compact, where λ denotes the action map and pr_2 is the projection.

By Z/X_1 we denote the orbit space of Z under the action of the groupoid $X_1 \rightrightarrows X_0$. Note that in general Z/X_1 may not be a smooth manifold. The following proposition extends a classical result regarding Lie group action on a smooth manifold.

Proposition 24. *Assume that $Z \stackrel{J}{\to} X_0$ is a $X_\bullet$-space, and the groupoid action is free and proper. Then Z/X_1 is a smooth manifold.*

Examples 25.

— *If $X_1 \rightrightarrows X_0$ is a Lie group G, a left $X_\bullet$-space is the usual G-manifold.*
— *X_1 is a $X_\bullet$-space, where the momentum map is the source map $s : X_1 \to X_0$, and the action map is the groupoid multiplication.*
— *Let (Z, ω) be a symplectic G-space with an equivariant momentum map $J : Z \to \mathfrak{g}^*$ [2]. Let $X_1 \rightrightarrows X_0$ be the transformation groupoid $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$, where G acts on $\mathfrak{g}^*$ by the coadjoint action. Define a $X_\bullet$-action on Z by $(g, u) \cdot z = g \cdot z$ whenever $u = J(z)$. It is easy to check that $J : Z \to \mathfrak{g}^*$ is indeed a $X_\bullet$-space. In fact, the equation $J(x \cdot z) = s(x)$, $\forall x \in X_1, z \in Z$, is equivalent to the G-equivariance of the momentum map J. See [22] for details.*

Definition 26. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid, and M a smooth manifold. A* left $X_\bullet$-torsor *(or a* left $X_\bullet$-principle bundle*) over M is a $X_\bullet$-space $Z \xrightarrow{J} X_0$ with a surjective submersion $\pi : Z \to M$ such that, for any $z, z' \in Z$, $\pi(z) = \pi(z')$ if and only if there exists a $x \in X_1$ such that $x \cdot z$ is defined and $x \cdot z = z'$, and moreover such x is unique.*

Similarly, one can define right $X_\bullet$-torsors (or right $X_\bullet$-principle bundles).

The surjective submersion $\pi : Z \to M$ is called the *structure map*.

Definition 27. *Let $\pi : P \to S$ and $\rho : Q \to T$ be $X_\bullet$-torsors. A morphism of $X_\bullet$-torsors from Q to P is given by a commutative diagram of smooth maps*

$$\begin{array}{ccc} Q & \xrightarrow{\phi} & P \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \tag{1.3}$$

such that ϕ is $X_\bullet$-equivariant.

Note that the diagram (1.3) is necessarily a pullback diagram. The $X_\bullet$-torsors form a category with respect to this notion of morphism. In particular, we now know what it means for two $X_\bullet$-torsors to be isomorphic.

Example 28 (Trivial torsors)**.** *(1) Let $f : M \to X_0$ be a smooth map. One can define, in a canonical way, a $X_\bullet$-torsor over M, which is called the* trivial $X_\bullet$-torsor given by f.
For this purpose, let Z be the fibered product $Z = X_1 \times_{t, X_0, f} M$. The structure map $\pi : Z \to M$ is the second projection. The momentum map of the $X_\bullet$-action is the first projection followed by the source map s. The action is then defined by

$$x \cdot (y, m) = (x \cdot y, m) \, .$$

One checks that this is indeed a $X_\bullet$-torsor over M.

(2) In the above construction, one can take $M = X_0$ and *f the identity map. In this way, one obtains the* universal trivial $X_\bullet$-torsor, *whose total space is X_1 and the base is X_0. The structure morphism and the momentum map of the universal $X_\bullet$-torsor are, respectively, $t, s : X_1 \to X_0$. the action map is the groupoid multiplication.*

The following lemma can be verified directly.

Lemma 29. *Let $\pi : Z \to M$ be a $X_\bullet$-torsor over the manifold M. Then any section $\lambda : M \to Z$ of π induces an isomorphism between the $X_\bullet$-torsor Z and the trivial $X_\bullet$-torsor over Z given by $J \circ \lambda$, where $J : Z \to X_0$ is the momentum map of Z.*

Since every surjective submersion admits local sections, we see that every $X_\bullet$-torsor is *locally trivial*, i.e. for any m in M there exists a local section $s : U \to Z$ of π defined on an open neighborhood $U \subset M$ of m, such that $\pi^{-1}(U) \cong X_1 \times_{t,X_0,J\circ s} U$ as $X_\bullet$-torsors.

The following proposition can thus be verified directly.

Proposition 30. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. The following statements are equivalent :*

(a) *Z is a $X_\bullet$-torsor over M with momentum map $Z \xrightarrow{J} X_0$;*

(b) *$Z \xrightarrow{J} X_0$ is a $X_\bullet$-space with a free and proper action such that Z/X_1 is diffeomorphic to M.*

1.2.2 Morita equivalence from groupoid bitorsors

Definition 31. *Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are said to be* Morita equivalent, *denoted by $X_\bullet \sim Y_\bullet$, if there exists a manifold Z with a pair of surjective submersions $X_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} Y_0$ such that*

- *Z is a left $X_\bullet$-torsor over Y_0, and a right $Y_\bullet$-torsor over X_0 ;*
- *the $X_\bullet$-action on Z commutes with the $Y_\bullet$-action.*

In this case, the manifold Z is called an *equivalence bitorsor* or a *$X_\bullet$-$Y_\bullet$-bitorsor.*

Example 32. *Assume that a Lie group G acts freely and properly on a manifold M from the right. Then the quotient space M/G is a smooth manifold, which can be seen as a Lie groupoid $M/G \rightrightarrows M/G$. One can check that this groupoid $M/G \rightrightarrows M/G$ is in fact Morita equivalent to the transformation groupoid $G \ltimes M \rightrightarrows M$. The equivalence bitorsor is*

$$\begin{array}{ccccc} M/G & & M & & M \rtimes G \\ & \overset{\pi}{\swarrow} & & \overset{id}{\searrow} & \\ \downarrow\downarrow & & & & \downarrow\downarrow \\ M/G & & & & M \end{array}$$

with the left action $[m] \cdot m = [m]$, $(g, m) \cdot m = gm$ and the right action $m \cdot (m, g) = mg$, $\forall g \in G$ and $m \in M$, where $\pi : M \to M/G$ is the projection.

Example 33. *More generally, assume that M is a smooth G-manifold, where the Lie group G acts on M from the right. Assume that $H \subseteq G$ is a closed normal Lie subgroup whose action on M is free and proper. It is a classical theorem that G/H is a Lie group and M/H is a smooth right G/H-manifold. Then its transformation groupoid $M/H \rtimes G/H \rightrightarrows M/H$ is Morita equivalent to $M \rtimes G \rightrightarrows M$, where the equivalence bitorsor is*

$$\begin{array}{ccccc} M/H \rtimes G/H & & M & & M \rtimes G \\ & \overset{id}{\swarrow} & & \overset{\pi}{\searrow} & \\ \downarrow\downarrow & & & & \downarrow\downarrow \\ M/H & & & & M \end{array}$$

We leave the reader as an exercise to write down all the relevant structures.

Remark 34. *In case that the G-action is not free and proper, the quotient space (i.e. the coarse moduli space of the transformation groupoid) can be badly behaved even as a topological space. On the other hand, the transformation groupoid itself $M \rtimes G \rightrightarrows M$ is always a Lie groupoid. Hence one can study differential geometry of the quotient space*

"M/G" in terms of the transformation groupoid $M \rtimes G \rightrightarrows M$ (more precisely, its Morita equivalent classes, i.e., the corresponding differentiable stack $[M/G]$, called the quotient stack*).*

For instance, consider the two torus $\mathbb{T}^2$, equipped with an action of $\mathbb{R}$ by an irrational rotation :

$$[(x,y)] \cdot t = [(x+t, y+\theta t)], \forall t \in \mathbb{R}, \quad \textit{and } [(x,y)] \in \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2,$$

where θ is an irrational number. Any $\mathbb{R}$-orbit is dense in $\mathbb{T}^2$, and the quotient space $\mathbb{T}^2/\mathbb{R}$ is not even Hausdorff. On the other hand, $\mathbb{T}^2 \rtimes \mathbb{R} \rightrightarrows \mathbb{T}^2$ is a nice Lie groupoid.

Example 35. *Let $P \xrightarrow{\pi} M$ be a principal left G-bundle. Then the gauge groupoid $\frac{P\times P}{G} \rightrightarrows M$ (see Example 10) is Morita equivalent to $G \rightrightarrows \cdot$, where the equivalence bitorsor is given by*

$$\begin{array}{ccccc} G & & P & & \frac{P\times P}{G} \\ \downdownarrows & \swarrow & & \overset{\pi}{\searrow} & \downdownarrows \\ \cdot & & & & M \end{array}$$

Proposition 36. *Morita equivalence defines an equivalence relation for Lie groupoids.*

Proof. We successively prove Morita equivalence is reflexive, symmetric and transitive. For reflexivity, note that $X_\bullet \sim X_\bullet$ with the equivalence bitorsor

$$\begin{array}{ccccc} X_1 & & X_1 & & X_1 \\ \downdownarrows & \overset{s}{\swarrow} & & \overset{t}{\searrow} & \downdownarrows \\ X_0 & & & & X_0 \end{array}$$

where $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$ is equipped with the universal trivial $\Gamma_\bullet$-$\Gamma_\bullet$-bitorsor structure.

For symmetry, assume that the Lie groupoid $X_1 \rightrightarrows X_0$ is Morita equivalent to the Lie groupoid $Y_1 \rightrightarrows Y_0$ with equivalence bitorsor

$$\begin{array}{ccccc} X_1 & & Z & & Y_1 \\ \downdownarrows & \overset{\rho}{\swarrow} & & \overset{\sigma}{\searrow} & \downdownarrows \\ X_0 & & & & Y_0 \end{array}$$

Then $Y_1 \rightrightarrows Y_0$ is Morita equivalent to $X_1 \rightrightarrows X_0$ with the equivalence bitorsor

$$\begin{array}{ccccc} Y_1 & & Z & & X_1 \\ \downdownarrows & \overset{\sigma}{\swarrow} & & \overset{\rho}{\searrow} & \downdownarrows \\ Y_0 & & & & X_0 \end{array}$$

with the reversed left $Y_\bullet$-action and right $X_\bullet$-action.

Finally, for transitivity, assume that $X_1 \rightrightarrows X_0$ is Morita equivalent to $Y_1 \rightrightarrows Y_0$ with the equivalence bitorsor

$$\begin{array}{ccccc} X_1 & & Z & & Y_1 \\ \downdownarrows & \overset{\rho_1}{\swarrow} & & \overset{\sigma_1}{\searrow} & \downdownarrows \\ X_0 & & & & Y_0 \end{array}$$

and $Y_1 \rightrightarrows Y_0$ is Morita equivalent to $W_1 \rightrightarrows W_0$ with the equivalence bitorsor

$$\begin{array}{ccccc} Y_1 & & Z' & & W_1 \\ \downarrow\downarrow & \overset{\rho_2}{\swarrow} & & \overset{\sigma_2}{\searrow} & \downarrow\downarrow \\ Y_0 & & & & W_0 \end{array}$$

Let $\bar{Z} = \frac{Z \times_{Y_0} Z'}{Y_1}$, where $Y_1 \rightrightarrows Y_0$ acts on $Z \times_{Y_0} Z'$ by the diagonal action $y \cdot (z, z') = (zy^{-1}, yz')\ \forall (z, z') \in Z \times_{Y_0} Z'$. Since the $Y_1 \rightrightarrows Y_0$-action on $Z \times_{Y_0} Z'$ is free and proper, $\bar{Z}$ is a smooth manifold. One checks that

$$\begin{array}{ccccc} X_1 & & \bar{Z} & & W_1 \\ \downarrow\downarrow & \overset{\rho_3}{\swarrow} & & \overset{\sigma_3}{\searrow} & \downarrow\downarrow \\ X_0 & & & & W_0 \end{array}$$

is indeed an equivalence bitorsor between $X_1 \rightrightarrows X_0$ and $W_1 \rightrightarrows W_0$, where the maps ρ_3 and σ_3 are given, respectively, by

$$\rho_3([(z, z')]) = \rho_1(z),\ \ \sigma_3([(z, z')]) = \sigma_2(z'),$$

the $X_\bullet$-action on $\bar{Z}$ from the left is given by

$$x[(z, z')] = [(xz, z')],$$

and the $W_\bullet$-action on $\bar{Z}$ from the right is given by

$$[(z, z')]w = [(z, z'w)],$$

for all compatible $x \in X_1, w \in W_1, (z, z') \in Z \times_{Y_0} Z'$. $\square$

1.2.3 Gauge Lie groupoids

The construction of gauge groupoids as in Example 10 extends to torsors over Lie groupoids. In fact, as we see below, if $X_1 \rightrightarrows X_0$ is Morita equivalent to $Y_1 \rightrightarrows Y_0$, with equivalence bitorsor $X_0 \leftarrow Z \rightarrow Y_0$, then $Y_1 \rightrightarrows Y_0$ is Morita equivalent to the gauge groupoid associated to Z with respect to the action of $X_1 \rightrightarrows X_0$.

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid, and Z a left $X_\bullet$-torsor over Y_0 with momentum map $Z \xrightarrow{J} X_0$ and the structure map $\pi : Z \to Y_0$. Consider the quotient space $\frac{Z \times_{X_0} Z}{X_1}$, where $X_1 \rightrightarrows X_0$ acts on $Z \times_{X_0} Z$ diagonally : $x \cdot (z, z') = (xz, xz')$, for all compatible $x \in X_1$, $(z, z') \in Z \times_{X_0} Z$. The following proposition can be easily verified, and is left to the reader.

Proposition 37. $\frac{Z \times_{X_0} Z}{X_1} \rightrightarrows Y_0$ *with the source, target, and unit maps :*

$$s([z, z']) = \pi(z), \qquad t([z, z']) = \pi(z'), \qquad \varepsilon([y_0]) = [z, z],$$

where $z \in \pi^{-1}(y_0)$, *and the natural multiplication and inverse :*

$$[z, z'] \cdot [z', z''] = [z, z''], \quad [z, z']^{-1} = [z', z]$$

is a Lie groupoid.

This Lie groupoid is called the *gauge Lie groupoid* associated to the left $X_\bullet$-torsor Z.

Theorem 38. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid, and Z a left $X_\bullet$-torsor over Y_0 with momentum map $Z \xrightarrow{J} X_0$ and the structure map $\pi : Z \to Y_0$. Then, the gauge Lie groupoid $\frac{Z\times_{X_0}Z}{X_1} \rightrightarrows Y_0$ is Morita equivalent to $X_1 \rightrightarrows X_0$.*

Conversely, if $X_1 \rightrightarrows X_0$ is Morita equivalent to $Y_1 \rightrightarrows Y_0$ with the equivalence bitorsor $X_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} Y_0$, then $Y_1 \rightrightarrows Y_0$ is isomorphic to the gauge groupoid $\frac{Z\times_{X_0}Z}{X_1} \rightrightarrows Y_0$.

Proof. It is straightforward to check that $X_0 \xleftarrow{J} Z \xrightarrow{\pi} Y_0$ is a bitorsor :

$$\begin{array}{ccccc} X_1 & & X & & \frac{Z\times_{X_0}Z}{X_1} \\ \downdownarrows & \overset{J}{\swarrow} & & \overset{\pi}{\searrow} & \downdownarrows \\ X_0 & & & & Y_0 \end{array}$$

where the gauge groupoid $\frac{Z\times_{X_0}Z}{X_1} \rightrightarrows Y_0$ acts on Z naturally from the right by

$$z \cdot [z, z'] = z'.$$

Conversely, consider the map

$$\phi : Y_1 \to \frac{Z \times_{X_0} Z}{X_1}, \quad \phi(y) = [z, zy],$$

where z is any element in Z such that $\sigma(z) = s(y)$. To see that ϕ is well defined, let z' be another element of Z such that $\sigma(z') = s(y)$. Then there exists some $x \in X_1$ with $z' = xz$, and thus $[z, zy] = [z', z'y]$. Finally one checks that ϕ is indeed a Lie groupoid isomorphism. □

Note that in the above theorem the roles of $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are completely symmetric. That is, $X_1 \rightrightarrows X_0$ is also isomorphic to the gauge Lie groupoid of Z considered as right $Y_1 \rightrightarrows Y_0$-torsor.

1.2.4 Morita equivalence from Morita morphisms

In this section, we introduce the notion of Morita morphisms of Lie groupoids, and prove that it yields another useful criterion for Morita equivalence.

Definition 39. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid, Z a smooth manifold and $\phi : Z \to X_0$ a surjective submersion. The* pullback groupoid *of $X_1 \rightrightarrows X_0$ by ϕ is defined to be the groupoid $X_1[Z] \rightrightarrows Z$, where*

$$X_1[Z] := \{(z, x, z') \mid z, z' \in Z,\ x \in X_1 \text{ such that } \phi(z) = s(x), \phi(z') = t(x)\},$$

with the source map $s(z, x, z') = z$, target map $t(z, x, z') = z'$, the unit map $\varepsilon(z) = (z, \varepsilon(\phi(z)), z)$, the inverse $(z, x, z')^{-1} = (z', x^{-1}, z)$, and the multiplication

$$(z, x, z') \cdot (z', x', z'') = (z, xx', z'').$$

It is easy to check that the natural projection

$$\begin{array}{ccc} X_1[Z] & \longrightarrow & X_1 \\ \downdownarrows & & \downdownarrows \\ Z & \longrightarrow & X_0 \end{array} \tag{1.4}$$

is a Lie groupoid morphism, which we will call a Morita morphism from $X_1[Z] \rightrightarrows Z$ to $X_1 \rightrightarrows X_0$.

Below is essentially this same definition formulated in a different manner.

Definition 40. *Let $X_\bullet$ and $Y_\bullet$ be two Lie groupoids. A Lie groupoid morphism $\phi_\bullet : X_\bullet \to Y_\bullet$ is called a* Morita morphism*, if*

(a) $\phi_0 : X_0 \to Y_0$ is a surjective submersion,

(b) the associated diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{(s,t)} & X_0 \times X_0 \\ \downarrow & & \downarrow \phi_0\times\phi_0 \\ Y_1 & \xrightarrow{(s,t)} & Y_0 \times Y_0 \end{array}$$

is cartesian, i.e. a pull back diagram of differential manifolds (or X_1 is diffeomorphic to the fiber product of Y_1 with $X_0 \times X_0$ over $Y_0 \times Y_0$).

Example 41. *Let M be a manifold, and $\phi : X \to M$ a surjective submersion. The pullback of the groupoid $M \rightrightarrows M$ under the map ϕ is the Banal groupoid $X \times_M X \rightrightarrows X$ as in Example* **??**. *Hence we have the following Morita morphism*

$$\begin{array}{ccc} X \times_M X & \longrightarrow & M \\ \downdownarrows & & \downdownarrows \\ X & \xrightarrow{\phi} & M \end{array}$$

In particular, if (U_i) is an open cover of M, $X := \amalg U_i$, and $\phi : X \to M$ is the covering map, then $X \times_M X :\cong \amalg U_{ij}$, where $U_{ij} = U_i \cap U_j$. Thus we obtain the groupoid $\amalg U_{ij} \rightrightarrows \amalg U_i$, called Čech groupoid, *and a Morita morphism*

$$\begin{array}{ccc} \amalg U_{ij} & \longrightarrow & M \\ \downdownarrows & & \downdownarrows \\ \amalg U_i & \xrightarrow{\phi} & M \end{array}$$

Definition 42. *Let $\phi : X_\bullet \to Y_\bullet$ and $\psi : X_\bullet \to Y_\bullet$ be two morphisms of Lie groupoids. A* natural equivalence *from ϕ to ψ, notation $\theta : \phi \Rightarrow \psi$, is a C^∞ map $\theta : X_0 \to Y_1$ satisfying $s(\theta(x_0)) = \phi(x_0)$ and $t(\theta(x_0)) = \psi(x_0)$, $\forall x_0 \in X_0$, such that for every $x \in X_1$ we have*

$$\theta\big(s(x)\big) \cdot \psi(x) = \phi(x) \cdot \theta\big(t(x)\big) .$$

where $\cdot$ denotes the multiplication in Y_1.

mien/Michael : help putting a diagram here

It is helpful to see the equation above as a commutative diagram below :

For any fixed Lie groupoids $X_\bullet$ and $Y_\bullet$, the morphisms and natural equivalences form a category $\mathrm{Hom}(X_\bullet, Y_\bullet)$, which is a set-theoretic groupoid. With this notion of morphism groupoid, the Lie groupoids form a 2-category.

Proposition 43. *Let $\phi_\bullet : X_\bullet \to Y_\bullet$ be a Morita morphism of Lie groupoids. Assume that $\psi_0 : Y_0 \to X_0$ is a section of $\phi_0 : X_0 \to Y_0$. Then ψ_0 induces uniquely a Lie groupoid morphism $\psi_\bullet : Y_\bullet \to X_\bullet$ with the following properties*
— $\phi_\bullet \circ \psi_\bullet = \mathrm{id}_{Y_\bullet}$ *;*
— $\psi_\bullet \circ \phi_\bullet \cong \mathrm{id}_{X_\bullet}$ *; I.e., there exits a natural equivalence $\theta : \psi_\bullet \circ \phi_\bullet \Rightarrow \mathrm{id}_{X_\bullet}$.*

Note that without referring to smooth structures, for groupoids, this means that these two groupoids are indeed equivalent categories since section always exists.

Theorem 44. *Two Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are Morita equivalent if and only if there exists a third Lie groupoid $Z_1 \rightrightarrows Z_0$ and Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$.*

Démonstration. To prove the "if" part, it suffices to prove that a Morita morphism induces a Morita equivalence as in Definition 31, since we know that Morita equivalence is indeed an equivalence relation. Assume that we have a Morita morphism as in the diagram (1.4). It is easy to check that

$$\begin{array}{ccccc} X_1[Z] & & Z \times_{X_0,s} X_1 & & X_1 \\ \downarrow\downarrow & \overset{\rho}{\swarrow} & & \overset{\sigma}{\searrow} & \downarrow\downarrow \\ Z & & & & X_0 \end{array}$$

is a $X[Z]_\bullet$-$X_\bullet$-bitorsor. Here $\rho(z,x) = z$, $\sigma(z,x) = t(x)$, $\forall (z,x) \in Z \times_{X_0,s} X_1$. The left action of $X_1[Z] \rightrightarrows Z$ on $Z \times_{X_0,s} X_1$ is given by

$$(z,x,z') \cdot (z',x') = (z, xx'),$$

while the right action of $X_1 \rightrightarrows X_0$ on $Z \times_{X_0,s} X_1$ is given by

$$(z,x) \cdot x' = (z, xx'),$$

whenever composable.

Conversely, assume that we have the following equivalence $X_\bullet$-$Y_\bullet$-bitorsor as in Definition 31.

$$\begin{array}{ccccc} X_1 & & Z & & Y_1 \\ \downarrow\downarrow & \overset{\rho}{\swarrow} & & \overset{\sigma}{\searrow} & \downarrow\downarrow \\ X_0 & & & & Y_0 \end{array}$$

It is straightforward to check that both pull back groupoids $X_1[Z] \rightrightarrows Z$ and $Y_1[Z] \rightrightarrows Z$ are isomorphic to the transformation groupoid $(X_1 \times \bar{Y}_1) \ltimes Z \rightrightarrows Z$, where the product groupoid $X_1 \times \bar{Y}_1 \rightrightarrows X_0 \times Y_0$ acts on Z from left in a natural manner. Here $\bar{Y}_1 \rightrightarrows Y_0$ denotes the Lie groupoid $Y_1 \rightrightarrows Y_0$ with the opposite structures. As a consequence, we obtain the following Lie groupoid morphisms

$$\begin{array}{ccccc} X_1 & \overset{\widetilde{\rho}}{\leftarrow} & (X_1 \times Y_1) \ltimes Z & \overset{\widetilde{\sigma}}{\rightarrow} & Y_1 \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ X_0 & \leftarrow & Z & \rightarrow & Y_0 \end{array}$$

where $\widetilde{\rho} = \mathrm{pr}_1$, and $\widetilde{\sigma} = \iota \circ \mathrm{pr}_2$ are Morita morphisms. Here both pr_1 and pr_2 are natural projections. The conclusion thus follows. □

Both notions of Morita equivalence are useful in applications. In the sequel, we will use both of them interchangely.

1.2.5 Properties of Morita equivalent Lie groupoids

Assume that Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are Morita equivalent with equivalence $X_\bullet$-$Y_\bullet$-bitorsor :

$$\begin{array}{ccccc} X_1 & & Z & & Y_1 \\ \downarrow\downarrow & \overset{\rho}{\swarrow} & & \overset{\sigma}{\searrow} & \downarrow\downarrow \\ X_0 & & & & Y_0 \end{array}$$

We denote by s_1 and t_1 the source and target map of $X_1 \rightrightarrows X_0$, and by s_2 and t_2 the source and target map of $Y_1 \rightrightarrows Y_0$.

Definition 45. *Two elements $u \in X_0$ and $v \in Y_0$ are said to be* related*, which we denote by $u \sim v$, if $\rho^{-1}(u) \cap \sigma^{-1}(v) \neq \varnothing$.*

Proposition 46.

(a) *Let $u \in X_0$, consider $\mathcal{O}_u = \{v \in Y_0 | v \text{ is related to } u\}$, then $\mathcal{O}_u$ is a groupoid orbit of $Y_1 \rightrightarrows Y_0$.*

(b) *Let $v \in Y_0$, consider $\mathcal{O}_v = \{u \in X_0 | u \text{ is related to } v\}$, then $\mathcal{O}_v$ is a groupoid orbit of $X_1 \rightrightarrows X_0$.*

(c) *There exists a bijection between orbits of $X_1 \rightrightarrows X_0$ and orbits of $Y_1 \rightrightarrows Y_0$.*

(d) *If $u \in X_0$ and $v \in Y_0$ are related, then the isotropy groups I_u and I_v are isomorphic.*

Démonstration.

(a) Let $u \in X_0$. If $v, v' \in Y_0$ are such that $u \sim v$ and $u \sim v'$, then there exist z and z' in Z satisfying $\sigma(z) = v$, $\sigma(z') = v'$ and $\rho(z) = \rho(z') = u$. Since $Z \to X_0$ is a $Y_\bullet$-torsor, there exists $y \in Y_1$ with $s_2(y) = v'$ such that $z = z'y$. In particular, $v = \sigma(z) = \sigma(z'y) = t_2(y)$, and v and v' are in the same $Y_\bullet$-orbit. Conversely, let $v \in \mathcal{O}_u$ and $z \in Z$ such that $\sigma(z) = v$ and $\rho(z) = u$. Assume that $v' \in Y_0$ is in the same orbit of v. Then there exists an element $y \in Y_1$ with $s_2(y) = v$ and $t_2(y) = v'$. Then $\sigma(zy) = t_2(y) = v'$ and $\rho(zy) = \rho(z) = u$ so that $v' \in \mathcal{O}_u$ as well.

(b) is the symmetric of (a).

(c) Let $\mathcal{O} \subset X_0$ be a groupoid orbit of $X_1 \rightrightarrows X_0$, and u any element in $\mathcal{O}$. Then $\mathcal{O}_u \subset Y_0$ is a groupoid orbit of $Y_1 \rightrightarrows Y_0$ by (a) above, and is independent of the choice of u in $\mathcal{O}$. It is then easy to check that the map

$$\mathcal{O} \mapsto \mathcal{O}_u$$

yields the required bijection.

(d) Let $u \in X_0$ and $v \in Y_0$ be related. Take $z \in Z$ with $\rho(z) = u$ and $\sigma(z) = v$. The map $\varphi : I_u \to I_v$ is built as follows. Let $g \in I_u$. Then $t_1(g) = u = \rho(z)$. Hence gz is defined, and we have $\rho(gz) = t_1(g) = u$. In particular, since Z is a $Y_\bullet$−torsor, there must exist a unique $h \in Y_1$ such that zh is defined, and $zh = gz$. Define $\varphi(g) = h$. One easily checks that φ is indeed a group isomorphism.

□

The above proposition establishes a bijection between orbits and isotropic groups of Morita equivalent Lie groupoids. Indeed, such a bijection is a homeomorphism with respect to the topology of the coarse moduli spaces and the smooth structures of the isotropic Lie groups. To show this, it is more convenient to use Morita morphisms rather than equivalence bitorsors.

Proposition 47. *Let $\phi_\bullet : X_\bullet \to Y_\bullet$ be a Morita morphism of Lie groupoids. Then*

(a) *$\phi_0 : X_0 \to Y_0$ induces a homeomorphism of the coarse moduli spaces : $X_0/X_1 \to Y_0/Y_1$.*

(b) *ϕ_1 induces isomorphisms of corresponding isotropic Lie groups. Namely, for any $x_0 \in X_0$, $\phi_1 : X_1|_{x_0}^{x_0} \to Y_1|_{\phi_0(x_0)}^{\phi_0(x_0)}$ is an isomorphism of Lie groups.*

(c) $2\dim X_0 - \dim X_1 = 2\dim Y_0 - \dim Y_1$.

Proposition 48. *Let D_ρ and D_σ be the integrable distributions induced by the ρ and σ-fibers in Z, respectively. Then the following holds :*

(a) *$D_\rho + D_\sigma$ is a smooth integrable distribution in Z ;*

(b) *$\rho_*^{-1}(\mathcal{D}_{X_0}) = \sigma_*^{-1}(\mathcal{D}_{Y_0}) = D_\rho + D_\sigma$, where $\mathcal{D}_{X_0}$ is the singular foliation of the groupoid orbits on $X_1 \rightrightarrows X_0$ and $\mathcal{D}_{Y_0}$ is the singular foliation of the groupoid orbits on $Y_1 \rightrightarrows Y_0$.*
In particular, if L is a leaf of $D_\rho + D_\sigma$, then $\rho(L) \subset X_0$ and $\sigma(L) \subset Y_0$ are related orbits. That is, if $\mathcal{O} \subset X_0$ is a $X_\bullet$-orbit, then $\sigma(\rho^{-1}(\mathcal{O}))$ is a $Y_\bullet$-orbit.

Proof. Assume that $\mathcal{O}_1 \subset X_0$ and $\mathcal{O}_2 \subset Y_0$ are related orbits. If $z \in \rho^{-1}(\mathcal{O}_1)$, then there exists an element $z' \in Z$ such that $\rho(z') = \rho(z)$, and $\sigma(z') \in \mathcal{O}_2$. Since $\sigma(z)$ and $\sigma(z')$ are in the same orbit, we have $\sigma(z) \in \mathcal{O}_2$, i.e. $z \in \sigma^{-1}(\mathcal{O}_2)$. Therefore $\rho^{-1}(\mathcal{O}_1) \subset \sigma^{-1}(\mathcal{O}_2)$. □

1.2.6 Differentiable stacks

In this section, we briefly recall the categorical approach of stacks, and establish the dictionary between differentiable stacks and Lie groupoids. Readers may consult [6] for more details.

From now on, let us fix a Lie groupoid $X_1 \rightrightarrows X_0$. Let $\mathfrak{S}$ be the category of all C^∞-manifolds with C^∞-maps as morphisms, and $\mathfrak{X}$ the category of all left $(X_1 \rightrightarrows X_0)$-torsors. Consider the canonical functor

$$F : \mathfrak{X} \to \mathfrak{S} \tag{1.5}$$

given by mapping a torsor $\pi : Z \to M$ to the underlying manifold M.

The following proposition can be verified directly.

Proposition 49. *The functor $F : \mathfrak{X} \to \mathfrak{S}$ satisfies the following properties :*

(i) for every arrow $V \to U$ in $\mathfrak{S}$, and every object P of $\mathfrak{X}$ lying over U (i.e., $\pi(P) = U$), there exists an arrow $Q \to P$ in $\mathfrak{X}$ lying over $V \to U$;

(ii) for every commutative triangle $W \to V \to U$ in $\mathfrak{S}$ and arrows $R \to P$ lying over $W \to U$ and $Q \to P$ lying over $V \to U$, there exists a unique arrow $R \to Q$ lying over $W \to V$, such that the composition $R \to Q \to P$ equals $R \to P$.

The object Q over V, whose existence is asserted in (i), is unique up to a unique isomorphism by (ii). Any choice of such a Q is called a *pullback* of P via $f : V \to U$, denoted $Q = P|V$, or $Q = f^*P$. $U = \pi(x)$).

Properties (i)-(ii) are called *fibration axioms*, and a functor $F : \mathfrak{X} \to \mathfrak{S}$ satisfying fibration axioms is called a *category fibered in groupoids* or simply a *groupoid fibration*. Hence we may say that the category $\mathfrak{X}$ of left $(X_1 \rightrightarrows X_0)$-torsors is a groupoid fibration over $\mathfrak{S}$. Roughly speaking, one can consider groupoid fibrations as a categorical analogue of fiber bundles with fibers being groupoids.

Given a category fibered in groupoids $\mathfrak{X} \to \mathfrak{S}$ and an object U of $\mathfrak{S}$, its *fiber* of $\mathfrak{X}$ over U, i.e., the category of all objects of $\mathfrak{X}$ lying over U and all morphisms of $\mathfrak{X}$ lying over id_U, notation $\mathfrak{X}_U$, is a (set-theoretic) groupoid. This follows from Property (ii), above. In our situation, it is the category of all left $(X_1 \rightrightarrows X_0)$-torsors over a fixed base manifold U, which is clearly a groupoid. Note that the groupoid fibrations over $\mathfrak{S}$ form a 2-category (see [?]).

Indeed the functor (1.5) satisfies three more properties, which is normally called *stack* axioms. To explain this, one needs to endow $\mathfrak{S}$ with a Grothendieck topology. We endow $\mathfrak{S}$ with the Grothendieck topology given by the following notion of covering family. Call a family $\{U_i \to M\}$ of morphisms in $\mathfrak{S}$ with target M a *covering family of* M, if all maps $U_i \to M$ are étale and the total map $\coprod_i U_i \to M$ is surjective. One checks that the conditions for a Grothendieck topology (see Exposé II in [1]) are satisfied. (Note that, in the terminology of [1], we have actually defined a *pretopology*. This pretopology gives rise to a Grothendieck topology, as explained in [1].) We call this topology the *étale* topology on $\mathfrak{S}$. One can also work with the topology of open covers. In this topology, all covering families are open covers $\{U_i \to M\}$, in the usual topological sense.

A *site* is a category endowed with a Grothendieck topology. So if we refer to $\mathfrak{S}$ as a site, we emphasize that we think of $\mathfrak{S}$ together with its étale topology.

Proposition 50. *The functor $F : \mathfrak{X} \to \mathfrak{S}$ in* (1.5) *satisfies the following three properties :*

(i) for any C^∞-manifold $M \in \mathfrak{S}$, any two objects $P, Q \in \mathfrak{X}$ lying over M, and any two isomorphisms $\phi, \psi : P \to Q$ over M, such that $\phi|U_i = \psi|U_i$, for all U_i in a covering family $U_i \to M$, we have that $\phi = \psi$;

(ii) for any C^∞-manifold $M \in \mathfrak{S}$, any two objects $P, Q \in \mathfrak{X}$ lying over M, a covering family $U_i \to M$ and, for every i, an isomorphism $\phi_i : P|U_i \to Q|U_i$, such that $\phi_i|U_{ij} = \phi_j|U_{ij}$, for all i, j, there exists an isomorphism $\phi : P \to Q$, such that $\phi|U_i = \phi_i$, for all i ;

(iii) for every C^∞-manifold M, every covering family $\{U_i\}$ of M, every family $\{P_i\}$ of objects P_i in the fiber $\mathfrak{X}_{U_i}$ and every family of morphisms $\{\phi_{ij}\}$, $\phi_{ij} : P_i|U_{ij} \to P_j|U_{ij}$, satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, in the fiber $\mathfrak{X}_{U_{ijk}}$, there exists an object P over M, together with isomorphisms $\phi_i : P|U_i \to P_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ (over U_{ij}).

Note that the isomorphism ϕ, whose existence is asserted in (ii) is unique, by (i). Similarly, the object P, whose existence is asserted in (iii), is unique up to a unique isomorphism, because of (i) and (ii). The object P is said to be obtained by *gluing* the objects P_i according to the gluing data ϕ_{ij}.

Remark 51. *The properties listed in both Proposition 49 and Proposition 50 should be considered as properties of $(X_1 \rightrightarrows X_0)$-torsors, and can be easily verified directly. In fact, they extend the classical facts regarding group torsors or principle bundles.*

A category fibered in groupoids $\mathfrak{X} \to \mathfrak{S}$ is called a *stack* over $\mathfrak{S}$ if the three additional axioms in Proposition 50 are satisfied. In particular, for any Lie groupoid $X_1 \rightrightarrows X_0$, the category $\mathfrak{X}$ of all left $(X_1 \rightrightarrows X_0)$-torsors is a stack, called *differentiable* or a C^∞*-stack.*

Two stacks $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{S}$ are said to be *isomorphic* if they are equivalent as categories over $\mathfrak{S}$. This means that there exist morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{X}$ and 2-isomorphisms $\theta : f \circ g \Rightarrow \mathrm{id}_{\mathfrak{Y}}$ and $\eta : g \circ f \Rightarrow \mathrm{id}_{\mathfrak{X}}$.

The following theorem is proved in [6].

Theorem 52. *Let $X_\bullet$ and $Y_\bullet$ be Lie groupoids. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the associated differentiable stacks, i.e., $\mathfrak{X}$ is the stack of $X_\bullet$-torsors and $\mathfrak{Y}$ the stack of $Y_\bullet$-torsors. Then the following are equivalent :*

(i) the differentiable stacks $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic ;

(ii) the Lie groupoids $X_\bullet$ and $Y_\bullet$ are Morita equivalent ;

Our viewpoint in this book is to avoid as much as possible the categorical approach for stacks as was done originally [1], at least for those differentiable or C^∞ ones. Instead, we will use Lie groupoids. The advantage is that we may use tools in differential geometry and noncommutative geometry to study these objects. Of course, the price we have to pay is that they are not intrinsic.

We are now ready to introduce

Definition 53. *A differentiable or C^∞-stack is a Morita equivalence class of Lie groupoids.*

In a certain sense, Lie groupoids are like "local charts" on a differentiable stack. Given a Lie groupoid $X_1 \rightrightarrows X_0$, its corresponding differentiable stack $\mathfrak{X}$ is denoted by $BX_\bullet$, or $[X_0/X_1]$. Such a Lie groupoid $X_1 \rightrightarrows X_0$ is called a presentation of the stack $\mathfrak{X}$.

Definition 54.
- *A differentiable stack is said to be* separated *or* Hausdorff *if it can be represented by a proper Lie groupoid.*
- *An orbifold is a differentiable stack which can be represented by a proper and étale Lie groupoid.*
- *A quotient stack, denoted $[M/G]$, is a differentiable stack which can be represented by a transformation groupoid $M \rtimes G \rightrightarrows M$.*

It is easy to see that a quotient stack $[M/G]$ is separated if the Lie group G is compact, or more generally, the action is proper. Note that "properness" is Morita invariant, while étale groupoids and transformation groupoids are not.

Definition 55. *If $\mathfrak{X}$ is a differentiable stack and $X_1 \rightrightarrows X_0$ a Lie groupoid presenting $\mathfrak{X}$, then we call $\dim \mathfrak{X} = 2 \dim X_0 - \dim X_1$ the* dimension *of $\mathfrak{X}$.*

Note that $2 \dim X_0 - \dim X_1$ is equal to the dimension of unit space minus the dimension of source fibers of the Lie groupoid $X_1 \rightrightarrows X_0$. It is also the codimension of a orbit minus the isotropy group dimension. Also $\dim \mathfrak{X}$ can be negative. In particular, if G is a Lie group of dimension n, the stack $[\cdot/G]$ is of dimension $-n$. We see that, from Proposition 47, $\dim \mathfrak{X}$ is independent of the presentation of $\mathfrak{X}$, and therefore is well-defined.

1.2.7 Generalized morphisms and Hilsum-Skandalis maps

In this section, we will review some basic facts concerning generalized morphisms, which were initially introduced by Hilsum-Skandalis [16]. We confine ourselves to Lie groupoids although most of the discussion can easily be adapted to general locally compact topological groupoids.

The notion of strict morphism of Lie groupoids (in the sense of Definition 13) is often too strong in that two Lie groupoids are rarely strictly isomorphic but are much more frequently Morita equivalent. Roughly speaking, generalized morphisms are maps between Lie groupoids *up to Morita equivalence*, and Hilsum-Skandalis maps are equivalence classes of generalized morphisms. In fact, Hilsum-Skandalis maps between Lie groupoids correspond exactly to maps between their associated differentiable stacks.

Let us recall the definition below [15, 16, 25].

Definition 56. *Let $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ be Lie groupoids.*

i) A generalized morphism *from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ consists of a smooth manifold Z, together with two smooth maps $X_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} Y_0$, a left $X_\bullet$-action and a right $Y_\bullet$-action on Z such that the two actions commute, and Z is a right $Y_\bullet$-torsor over X_0.*

ii) Generalized morphisms $X_0 \xleftarrow{\rho_1} Z_1 \xrightarrow{\sigma_1} Y_0$ and $X_0 \xleftarrow{\rho_2} Z_2 \xrightarrow{\sigma_2} Y_0$ from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ are said to be equivalent if there exists a $X_\bullet$-$Y_\bullet$-biequivariant diffeomorphim $Z_1 \to Z_2$. Denote by $[Z]$ the equivalent class of such generalized morphisms.

iii) A Hilsum-Skandalis map *from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ is an equivalent class of generalized morphisms from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$.*

We write $F : X_\bullet \rightsquigarrow Y_\bullet$ to denote a Hilsum-Skandalis map from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$.

Lemma 57. *A morphism of Lie groupoids induces a generalized morphism in a canonical way.*

Démonstration. Assume that $f\colon X_\bullet \to Y_\bullet$ is a strict morphism of Lie groupoids. Let $Z_f = X_0 \times_{f,Y_0,s} Y_1$. Define a map $\rho : Z_f \to X_0$ and a map $\sigma : Z_f \to Y_0$, respectively, by $\rho(x_0, y) = x_0$, $\sigma(x_0, y) = t(y)$. Also, define a left $X_\bullet$-action on Z_f by

$$x \cdot (t(x), y) = (s(x), f(x)y),$$

and an right $Y_\bullet$-action on Z_f by

$$(x_0, y) \cdot y' = (x_0, yy').$$

It is simple to check that $X_0 \xleftarrow{\rho} Z_f \xrightarrow{\sigma} Y_0$ is indeed a generalized morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$. □

The corresponding Hilsum-Skandalis map $[Z_f] : X_\bullet \rightsquigarrow Y_\bullet$ is called the associated Hilsum-Skandalis map of the strict morphism $f : X_\bullet \to Y_\bullet$.

Lemma 58. *Let ϕ and $\psi\colon X_\bullet \to Y_\bullet$ be strict Lie groupoid morphisms. Their associated generalized morphisms are equivalent if and only if there is a natural equivalence from ϕ to ψ in the sense of Definition 42. That is, there exists a smooth map $\theta\colon X_0 \to Y_1$ such that $\psi(x) = \theta(s(x))^{-1} \cdot \phi(x) \cdot \theta(t(x)), \forall x \in X_1$.*

Démonstration. Assume that $\tau : Z_\phi \to Z_\psi$ is a $X_\bullet$-$Y_\bullet$-biequivariant diffeomorphim. Then τ must be of the form $(x_0, y) \mapsto (x_0, \theta(x_0)y), \forall (x_0, y) \in Z_\phi$ since τ is $X_\bullet$-equivariant. On the other hand, since τ is $Y_\bullet$-equivariant, it follows that

$$\big(s(x),\ \theta(s(x)) \cdot \psi(x)\big) = \big(s(x),\ \phi(x) \cdot \theta(t(x))\big), \quad \forall x \in X_1.$$

Hence, the conclusion follows. The converse can be proved by working backwards. □

As we see below, generalized morphisms can be composed just like the usual strict groupoid morphisms.

Proposition 59. *Let $X_0 \overset{\rho}{\leftarrow} Z \overset{\sigma}{\to} Y_0$ be a generalized morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$, and $Y_0 \overset{\rho'}{\leftarrow} Z' \overset{\sigma'}{\to} W_0$ a generalized morphism from $Y_1 \rightrightarrows Y_0$ to $W_1 \rightrightarrows W_0$. Then*

$$Z'' := \frac{Z \times_{Y_0} Z'}{Y_1},$$

where $Y_1 \rightrightarrows Y_0$ acts on $Z \times_{Y_0} Z'$ diagonally : $(z, z') \cdot y = (zy, y^{-1}z')$, for all compatible $(z, z') \in Z \times_{Y_0} Z'$, and $y \in Y_1$, together with those obvious structure maps, defines a generalized morphism from $X_1 \rightrightarrows X_0$ to $W_1 \rightrightarrows W_0$.

Proof. The proof is similar to that of the transitivity of Proposition 36, and is left to the reader. □

The resulting generalized morphism above is called the *composition of Z and Z'*, and denoted $Z'' := Z \circ Z'$. It follows from a straightforward verification that the composition of generalized morphisms is compatible with equivalence, i.e., the compositions of equivalent generalized morphisms are also equivalent.

Theorem 60. *There is a well defined category $\mathcal{G}$, whose objects are Lie groupoids, and whose morphisms are Hilsum-Skandalis maps.*

Note that, from the proof of Theorem 44, we see that isomorphisms in the category $\mathcal{G}$ are exactly the Morita equivalences [27, 38, 6].

1.2.8 An alternative definition

We now describe an equivalent notion of generalized morphisms and Hilsum-Skandalis maps, which is more categoretic in nature, but conceptually clearer and useful later on.

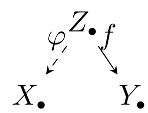
Let $X_0 \overset{\rho}{\leftarrow} Z \overset{\sigma}{\to} Y_0$ be a generalized morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$. Let $X_1[Z] \rightrightarrows Z$ be the pullback groupoid of $X_1 \rightrightarrows X_0$ by $\rho : Z \to X_0$, which is a surjective submersion by assumption. By φ, we denote the Morita morphism from $X_1[Z] \rightrightarrows Z$ to $X_1 \rightrightarrows X_0$. Now we define a map $f : X_1[Z] \to Y_1$, $f(z, x, z') = y$, by the equation $z \cdot y = x \cdot z'$, where $(z, x, z') \in Z \times_{\rho, X_0, s} X_1 \times_{t, X_0, \rho} Z \cong X_1[Z]$. It is simple to check that f is a well-defined map, which, together with the map $f : Z \to Y_0$, $f(z) = \sigma(z), \forall z \in Z$ on the unit spaces, denoted the same symbol by abuse of notations, is indeed a Lie groupoid morphism from $X_1[Z] \rightrightarrows Z$ to $Y_1 \rightrightarrows Y_0$. According to Lemma 57, we have generalized morphisms Z_φ from $X_1[Z] \rightrightarrows Z$ to $X_1 \rightrightarrows X_0$, and Z_f from $X_1[Z] \rightrightarrows Z$ to $Y_1 \rightrightarrows Y_0$. Since φ is a Morita morphism, it follows that Z_φ is a generalized isomorphism, and therefor Z_φ^{-1} is a generalized morphism from $X_1 \rightrightarrows X_0$ to $X_1[Z] \rightrightarrows Z$. Consider the composition $Z_f \circ Z_\varphi^{-1}$. Then $Z_f \circ Z_\varphi^{-1}$ is a generalized morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$. By a direct verification, we prove the following

Proposition 61. *As generalized morphisms from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$, $Z_{f} \circ Z_{\varphi}^{-1}$ is equivalent to the given one $X_0 \overset{\rho}{\leftarrow} Z \overset{\sigma}{\rightarrow} Y_0$.*

This proposition motivates the following

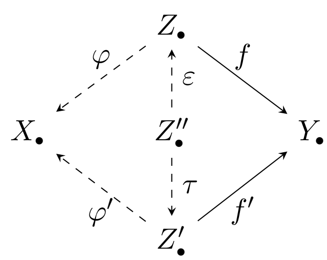
Definition 62. *Let $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ be Lie groupoids.*

(a) *A* roof with tip $Z_\bullet$ between $X_\bullet$ and $Y_\bullet$ *is a diagram of the form :*



where $f : Z_\bullet \to Y_\bullet$ is a Lie groupoid morphism and $\varphi : Z_\bullet \to X_\bullet$ is a Morita morphism. We will denote the above roof by $(\varphi, f) : X_\bullet \leftarrow Z_\bullet \to Y_\bullet$.

(b) *Two roofs $(\varphi, f) : X_\bullet \leftarrow Z_\bullet \to Y_\bullet$ and $(\varphi', f') : X_\bullet \leftarrow Z'_\bullet \to Y_\bullet$ are said to be* equivalent *if there is another Lie groupoid $Z''_\bullet$ and Morita morphisms $\varepsilon : Z''_\bullet \to Z_\bullet$ and $\tau : Z''_\bullet \to Z'_\bullet$ such that the diagram :*

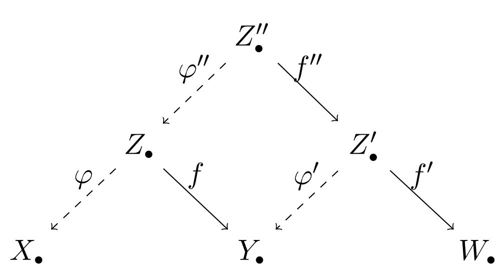


(1.6)

commutes. We denote by $[\varphi, f] : X_\bullet \rightsquigarrow Y_\bullet$ the equivalence class of the roof $(\varphi, f) : X_\bullet \leftarrow Z_\bullet \to Y_\bullet$.

Roofs can also be composed, as we shall see below.

Proposition 63. *Let $(\varphi, f) : X_\bullet \leftarrow Z_\bullet \to Y_\bullet$ and $(\varphi', f') : Y_\bullet \leftarrow Z'_\bullet \to W_\bullet$ be two roofs. Then there exists a Lie groupoid $Z''_1 \rightrightarrows Z''_0$, a Morita morphism $\varphi'' : Z''_\bullet \to Z_\bullet$, and a Lie groupoid morphism $f'' : Z''_\bullet \to Z'_\bullet$ such that the middle square in the diagram :*



(1.7)

commutes.

PROOF. Let $Z_1'' = Z_1 \times_{Y_1} Z_1'$ and $Z_0'' = Z_0 \times_{Y_0} Z_0'$. Since φ' is Morita morphism, hence it is a surjective submersion on both objects and arrows. Therefore, it follows that $Z_1'' \rightrightarrows Z_0''$ is a Lie groupoid. Here the groupoid structure on $Z_1'' \rightrightarrows Z_0''$ is naturally induced from the those on $Z_\bullet$, $Y_\bullet$ and $Z_\bullet'$ since both f and φ' are groupoid morphisms. It can be considered as a fibered product in the category of groupoids. In fact, one can check that $Z_1'' \cong Z_0' \times_{Y_0} Z_1 \times_{Y_0} Z_0'$, and $Z_1'' \rightrightarrows Z_0''$ is isomorphic to the pullback groupoid $Z_1 \rightrightarrows Z_0$ under the projection $Z_0'' = Z_0 \times_{Y_0} Z_0' \to Z_0$. The rest of the claim can be checked directly. □

Finally, one defines the composition of roofs to be

$$(\varphi, f)\circ(\varphi', f') = (\varphi''\circ\varphi, f'\circ f'') : X_\bullet \leftarrow Z_\bullet'' \to W_\bullet.$$

As before, it is straightforward to check that composition is stable under equivalence of roofs. Indeed we have the following result which can be proved by a tedious but straightforward verification.

Theorem 64. (a) *Lie groupoids, together with arrows being equivalence classes of roofs, becomes a well defined category, denoted $\mathcal{G}'$.*

(b) *The categories $\mathcal{G}'$ and $\mathcal{G}$ are isomorphic.*

From now on, we will call "Hilsum-Skandalis map from $X_\bullet$ to $Y_\bullet$" either an isomorphism class of generalized morphisms from $X_\bullet$ to $Y_\bullet$ or an equivalence class of roofs from $X_\bullet$ to $Y_\bullet$. When the discussion allows, we shall not distinguish between "equivalence classes" and representatives, leaving to the reader the (obvious) check that this is possible.

We remark that, the construction of the category $\mathcal{G}'$ with morphisms the equivalence classes of roofs, is known in category theory as the process of *localization*. Namely, $\mathcal{G}'$ is the category obtained after localization at the class of Morita morphisms [?] from the category of Lie groupoids with morphisms being strict Lie groupoid morphisms.

Given Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$, a Hilsum-Skandalis map from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ naturally induces a morphism, or a C^∞-map their associated differential stacks $\mathfrak{X} \to \mathfrak{Y}$. Conversely, to any C^∞-map from $\mathfrak{X}$ to $\mathfrak{Y}$, there exists a unique Hilsum-Skandalis map from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$. Therefor we have a canonical functor from the category $\mathcal{G}$ to the category of differential stacks, which is indeed fully faithful and essentially surjective, and therefore is equivalence of categories (see Proposition 1.3.13 in [17]).

Theorem 65. *The category $\mathcal{G}$ is equivalent to that of differentiable stacks.*

From now on, we will use the category $\mathcal{G}$ as a *replacement* of that of differentiable stacks, which is easier to manage from the differential geometry point of view.

1.2.9 Principal G-bundles over Lie groupoids

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Associated to any $X_\bullet$-space $J : P_0 \to X_0$, there is a natural groupoid $P_1 \rightrightarrows P_0$, called the *transformation groupoid*, which is defined as follows. We let $P_1 = X_1 \times_{t,X_0,J} P_0$, and the source and target maps are, respectively, $s(x,p) = p$, $t(x,p) = x \cdot p$, and the multiplication

$$(x,p)\cdot(y,q) = (x\cdot y, q), \quad \text{where } p = y\cdot q. \tag{1.8}$$

It is simple to check that the first projection defines a strict homomorphism of groupoids from $P_1 \rightrightarrows P_0$ to $X_1 \rightrightarrows X_0$.

Definition 66. *Let G be a Lie group. An right* principal G-bundle *over $X_1 \rightrightarrows X_0$ is a principal right G-bundle $P_0 \xrightarrow{J} X_0$, which, at the same time, is also a $X_\bullet$-space such that the following compatibility condition is satisfied : for all $g \in G$, all $p \in P_0$, and $x \in X_1$ such that $t(x) = J(p)$*

$$(x \cdot p) \cdot g = x \cdot (p \cdot g). \tag{1.9}$$

In this case $P_1 \to X_1$ also becomes a principal right G-bundle.

Examples 67. *Let $X_1 \rightrightarrows X_0$ be the transformation groupoid $H \ltimes M \rightrightarrows M$, where Lie group H acts on M from the left. Then an right principal G-bundle over $X_1 \rightrightarrows X_0$ corresponds exactly to an H-equivariant principal (right) G-bundle over M.*

A principal right G-bundle over a Lie groupoid $X_1 \rightrightarrows X_0$ can also be equivalently considered as a generalized morphism from $X_1 \rightrightarrows X_0$ to $G \rightrightarrows \cdot$. As a consequence of Proposition 59, we see that principal bundles can be pulled back, as in the classical case, under a "generalized morphism" in the following sense.

Proposition 68. *Let f be a generalized morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ given by $X_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} Y_0$. Assume that $P_0 \to Y_0$ is a right principal G-bundle over $Y_1 \rightrightarrows Y_0$. Then $= Z \times_{Y_0} P_0 \to X_0$, with the natural structure maps, is a right principal G-bundle over $X_1 \rightrightarrows X_0$, denoted f^*P_0. As a consequence, if $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are Morita equivalent Lie groupoids, then the category of right principal G-bundles over $X_1 \rightrightarrows X_0$ and the category of right principal G-bundles over $Y_1 \rightrightarrows Y_0$ are equivalent.*

Given an right principal G-bundle $J : P_0 \to X_0$ over $X_1 \rightrightarrows X_0$, let $\frac{P_0 \times P_0}{G} \rightrightarrows X_0$ be the gauge groupoid. Denote by (p_1, p_2) an element of $P_0 \times P_0$ and by $\overline{(p_1, p_2)}$ its class in $\frac{P_0 \times P_0}{G}$. We introduce a map from X_1 to $\frac{P_0 \times P_0}{G}$ by

$$x \mapsto \overline{(xp, p)},$$

where p is any element that satisfies $J(p) = t(x)$. Thus we obtain the following groupoid homomorphism :

$$\begin{array}{ccc} X_1 & \longrightarrow & \frac{P_0 \times P_0}{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_0 & \longrightarrow & X_0 \end{array} \tag{1.10}$$

Since any transitive groupoid is Morita equivalent to its isotropy group, the groupoid $\frac{P_0 \times P_0}{G} \rightrightarrows X_0$ is Morita equivalent to $G \rightrightarrows \cdot$. It is not hard to check that the homomorphism (1.10), and the right principal G-bundle $P_0 \to X_0$ define equivalent generalized morphisms from $X_1 \rightrightarrows X_0$ to $G \rightrightarrows \cdot$.

1.3 Cohomology theory

1.3.1 Simpilicial manifolds

In this section, we recall some basic constructions regarding simplicial manifolds. We follow closely the notations of [13].

Definition 69. *A set is a sequence $(S_n)_{n\in\mathbb{N}}$ of sets together two sequences of maps :*

$$\varepsilon_i^n : S_n \to S_{n-1},\ i = 0, \dots, n,$$

called face maps*,*

$$\eta_i^n : S_n \to S_{n+1},\ i = 0, \dots, n,$$

called degeneracy maps*, which satisfy the following identities :*

(a) $\varepsilon_i^{n-1}\varepsilon_j^n = \varepsilon_{j-1}^{n-1}\varepsilon_i^n, \quad i < j,$

(b) $\eta_i^{n+1}\eta_j^n = \eta_{j+1}^{n+1}\eta_i^n, \quad i \leq j,$

(c)

$$\varepsilon_i^{n+1}\eta_j^n = \begin{cases} \eta_{j-1}^{n-1}\varepsilon_i^n, & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \eta_j^{n-1}\varepsilon_{i-1}^n, & i > j+1, \end{cases}$$

By abuse of notations, the simplicial set $(S_n)_{n\in\mathbb{N}}$ is also denoted by $S_\bullet$.

Denote by Δ^n the standard n-simplex :

$$\Delta^n = \{(t_0, \dots, t_n) | \sum_{i=0}^n t_i = 1, t_0 \geq 0, \dots, t_n \geq 0\}.$$

Consider the two sequences of maps

$$\widetilde{\varepsilon}_n^i : \Delta^{n-1} \to \Delta^n \ \text{ and } \widetilde{\eta}_n^i : \Delta^{n+1} \to \Delta^n, \quad i = 0, 1, \cdots, n, \tag{1.11}$$

defined by

$$\widetilde{\varepsilon}_n^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}), \tag{1.12}$$

$$\widetilde{\eta}_n^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1}). \tag{1.13}$$

Geometrically, $\widetilde{\varepsilon}_n^i$ is the affine map identifying the standard $(n-1)$-simplex with the i-th face of the standard n-simplex, while $\widetilde{\eta}_n^i$ is the affine map collapsing the standard $(n+1)$-simplex onto the standard n-simplex by identifying its i-th and $i+1$-th vertices.

Let M be a smooth manifold. A smooth *singular n-simplex* in M is a smooth map $\sigma : \Delta^n \to M$, where Δ^n is the standard n-simplex. Denote by $S_n^\infty(M)$ the set of all smooth singular n-simplices. Let

$$\varepsilon_i^n : S_n^\infty(M) \to S_{n-1}^\infty(M), \quad \varepsilon_i^n(\sigma) = \sigma\circ\widetilde{\varepsilon}_n^i, \quad i = 0, \dots, n,$$

and

$$\eta_i^n : S_n^\infty(M) \to S_{n+1}^\infty(M), \quad \eta_i^n(\sigma) = \sigma\circ\widetilde{\eta}_n^i i = 0, \dots, n.$$

It is simple to check that $S_\bullet^\infty(M)$ is indeed a simplicial set with face maps ε_i^n and degeneracy maps η_i^n.

Definition 70. (a) *A* simplicial manifold *is a simplicial set $X_\bullet = (X_n)_{n\in\mathbb{N}}$, where, for every $n \in \mathbb{N}$, X_n is a smooth manifold and all the face and degeneracy maps are smooth maps.*

(b) *A smooth or C^∞ simplicial map $\phi : X_\bullet \to Y_\bullet$ between simplicial manifolds $X_\bullet$ and $Y_\bullet$ consists of a family of smooth maps $\phi : X_n \to Y_n$ commuting with all the face and degeneracy maps.*

Given a simplicial manifold $M_\bullet$, its *fat realization* is a topological space $||M_\bullet||$ [13, 31] given by

$$||M_\bullet|| = \left(\coprod_{n\geq 0} M_n \times \Delta^n\right) / \sim,$$

with the equivalence relation :

$$\big(x, \widetilde{\varepsilon}^i_n(t)\big) \sim \big(\varepsilon^n_i(x), t\big), \quad x \in M_n,\ t \in \Delta^{n-1}, i = 0, \cdots, n,\ n = 1, 2, \cdots \tag{1.14}$$

Its *geometric realization* $|M_\bullet|$ is the topological space resulted by further requiring

$$\big(x, \widetilde{\eta}^i_n(t)\big) \sim \big(\eta^n_i(x), t\big), \quad x \in M_n,\ t \in \Delta^{n+1}, i = 0, \cdots, n,\ n = 1, 2, \cdots \tag{1.15}$$

Here ε^n_i and η^n_i are the face and degeneracy maps of $M_\bullet$, while $\widetilde{\varepsilon}^i_n$ and $\widetilde{\eta}^i_n$ are the inclusion and projection maps defined by Eqs. (1.12-1.13).

It is known that the natural map $||M_\bullet|| \to |M_\bullet|$ is a homotopy equivalence [13]. Note that both $||\cdot||$ and $|\cdot|$ are functors from the category of simplicial manifolds to the category of topological spaces. Any simplicial set can be considered as a simplicial topological space by endowing the discrete topology. Hence the geometric realization $|\cdot|$ assigns any simplicial set a topological space [**?**].

1.3.2 Nerve of a Lie groupoid

An important class of simplicial manifolds arise from Lie groupoids [31]. Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Set

$$X_n = \{(x_1, x_2, \dots, x_n) \in X_1^n | t(x_i) = s(x_{i+1}), i = 1, \dots, n-1\},$$

the manifold consisting of all composable n-tuples. Define the face maps $\varepsilon^n_i : X_n \to X_{n-1}$ by, for $n > 1$

$$\varepsilon^n_0(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n) \tag{1.16}$$

$$\varepsilon^n_n(x_1, x_2, \dots, x_n) = (x_1, \dots, x_{n-1}) \tag{1.17}$$

$$\varepsilon^n_i(x_1, \dots, x_n) = (x_1, \dots, x_i x_{i+1}, \dots, x_n), \quad 1 \leq i \leq n-1, \tag{1.18}$$

and for $n = 1$ by, $\varepsilon^1_0(x) = t(x)$, $\varepsilon^1_1(x) = s(x)$. Also define the degeneracy maps by

$$\eta^0_0 = \varepsilon : X_0 \to X_1$$

(ε being the unit map of the groupoid), and $\eta^n_i : X_n \to X_{n+1}$ by :

$$\eta^n_0(x_1, \dots, x_n) = ((\varepsilon\circ s)(x_1), x_1, \dots, x_n) \tag{1.19}$$

$$\eta^n_i(x_1, \dots, x_n) = (x_1, \dots, x_i, (\varepsilon\circ t)(x_i), x_{i+1}, \dots, x_n), \quad 1 \leq i \leq n. \tag{1.20}$$

The following proposition can be easily verified.

Proposition 71. *For any Lie groupoid $X_1 \rightrightarrows X_0$, the manifolds $(X_n)_{n\geq 1}$ with the structure maps defined above becomes a simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X_1 \rightrightarrows X_0\,. \tag{1.21}$$

Such a simplicial manifold is called the *nerve of the Lie groupoid $X_1 \rightrightarrows X_0$*.

Example 72. *Let M be a manifold, and let $M \rightrightarrows M$ be its corresponding Lie groupoid. Its nerve is the simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} M \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} M \rightrightarrows M\,, \tag{1.22}$$

where all the face and degeneracy maps are the identity map. Then its geometric realization is homeomorphic to M, while the fat realization is $||N(\cdot)|| \times M$. where

$$||N(\cdot)|| = (\Delta^0 \cup \Delta^1 \cup \cdots)/\sim$$

with the appropriate quotient relation as in Eq. (1.14)

Example 73. *Let G be a Lie group. Being considered as a Lie groupoid over a point, its nerve is the standard simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} G \times G \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G \rightrightarrows * \tag{1.23}$$

Its fat realization is called the classing space *of the Lie group G, denoted BG.*

Example 74. *Consider the transformation groupoid $M \rtimes G \rightrightarrows M$, where the Lie group G acts on a smooth manifold M from the right. The nerve of the groupoid is the simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} M \times G \times G \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} M \times G \rightrightarrows M\,. \tag{1.24}$$

Its fat realization reduces to the Borel construction $M \times_G EG$, where EG is the total space of the universal principal G-bundle in the sense of Milnor [23, 13].

Example 75. *Consider the pair groupoid $M \times M \rightrightarrows M$, where M is a smooth manifold. Its nerve is the simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} M \times M \times M \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} M \times M \rightrightarrows M\,, \tag{1.25}$$

where the face maps are projections and the degeneracy maps are inclusions. One proves that its fat realization is contractible.

Example 76. *Let M be a manifold, and $(U_i)_{i\in I}$ an open covering of M. Consider the Čech groupoid $\amalg U_{i_1 i_2} \rightrightarrows \amalg U_i$. Its nerve is the simplicial manifold :*

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \amalg U_{i_1 i_2 i_3} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \amalg U_{i_1 i_2} \rightrightarrows \amalg U_i\,. \tag{1.26}$$

where $\amalg U_{i_1,\cdots,i_k}$ is the disjoint union of intersections of those U_i taken over all k-tuples $(i_1,\cdots,i_k)$. The face and degeneracy maps are given by natural inclusions. One checks that its fat realization is indeed homotopy equivalent to M.

1.3.3 de Rham cohomology of simplicial manifolds

As in the case of ordinary manifolds, one can also define de Rham cohomology for any simplicial manifold.

Consider a simplicial manifold :

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X_1 \rightrightarrows X_0 \,. \tag{1.27}$$

Its de Rham cohomology is the hypercohomology of the following double complex $\Omega^\bullet(X_\bullet)$:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow d & & \uparrow d & & \uparrow d & & \\ \Omega^1(X_0) & \xrightarrow{\partial} & \Omega^1(X_1) & \xrightarrow{\partial} & \Omega^1(X_2) & \xrightarrow{\partial} & \cdots \\ \uparrow d & & \uparrow d & & \uparrow d & & \\ \Omega^0(X_0) & \xrightarrow{\partial} & \Omega^0(X_1) & \xrightarrow{\partial} & \Omega^0(X_2) & \xrightarrow{\partial} & \cdots \end{array} \tag{1.28}$$

Here, the vertical differential $d : \Omega^k(X_p) \to \Omega^{k+1}(X_p)$ is the usual de Rham differential, and the horizontal differential $\partial : \Omega^k(X_p) \to \Omega^k(X_{p+1})$ is the alternating sum of the pull-back of the face maps of the corresponding simplicial manifolds $X_\bullet$. That is, $\partial : \Omega^\bullet(X_{p-1}) \to \Omega^\bullet(X_p)$ is given by

$$\partial = \sum_{i=0}^{p} (-1)^i (\varepsilon_i^p)^*.$$

The following lemma can be verified by a straightforward computation.

Lemma 77. *The differential ∂ satisfies the following properties*

(a) $\partial^2 = 0$.

(b) $\partial \circ d = d \circ \partial$.

Denote by

$$D = (-1)^p d + \partial$$

the total differential of the double complex (1.28). The corresponding k-th hypercohomology group, i.e. the k-th cohomology group of the cochain complex $(\Omega^\bullet(X_\bullet), D)$ is called the k-th *de Rham cohomology* of the simplicial manifold $X_\bullet$, and denoted $H^k_{dR}(X_\bullet)$, where $\Omega^k(X_\bullet) := \oplus_{p+q=k}\Omega^q(X_p)$. In particular, by the *de Rham cohomology* of a Lie groupoid $X_1 \rightrightarrows X_0$, we mean the de Rham cohomology of its nerve $X_\bullet$, which is denoted by the same symbol $H^\bullet_{dR}(X_\bullet)$ by abuse of notation.

Example 78. *Let M be a smooth manifold. Consider M as a Lie groupoid $M \rightrightarrows M$. It is simple to see that its de Rham cochain complex (or more precisely the de Rham cochain complex of its nerve) is quasi-isomorphic to the usual de Rham cochain complex of the manifold $(\Omega^\bullet(M), d)$. Therefore, its de Rham cohomology, being considered as a Lie groupoid, is exactly the same as the ordinary de Rham cohomology of the manifold.*

Example 79. *Let G be a connected Lie group. Then $(\Omega^\bullet(G_\bullet), D)$ is the standard de Rham-bar bicomplex [8, 9], which appears extensively in the theory of characteristic classes [8, 9] and symplectic structure on moduli spaces [34]. There is a natural map, called the* Bott-Shulman *map, from the ring $(S\mathfrak{g}^\vee)^G$ of invariant polynomials on the Lie algebra $\mathfrak{g}$ to the total cocycles of $\Omega^\bullet(G_\bullet)$, which induces an isomorphism on the level of cohomology : $H^{2k}_{dR}(G_\bullet) \cong (S^k\mathfrak{g}^\vee)^G$, $H^{2k+1}_{dR}(G_\bullet) = 0$, $\forall k = 0, 1, \cdots$, when G is a compact connected Lie group [8].*

Example 80. *Let M be a manifold and (U_i) a good open covering of M, i.e. all U_i and their finite intersections are contractible. Let $X_1 \rightrightarrows X_0$ denote the Čech groupoid $\amalg U_{i_1 i_2} \rightrightarrows \amalg U_i$. It is simple to see that $H^\bullet_{dR}(X_\bullet)$ is isomorphic to both the de Rham cohomology $H^\bullet_{dR}(M)$ and the Čech cohomology $H^\bullet_{Cech}(M)$. Indeed, this is the original Weil's argument in proving the de Rham theorem that the de Rham cohomology of a smooth manifold M is isomorphic to its Čech cohomology [33, 10] $H^\bullet_{dR}(M) \cong H^\bullet_{ech}(M)$.*

Define the cup-product as a degree 0 bilinear map

$$\cup : \Omega^\bullet(X_\bullet) \otimes \Omega^\bullet(X_\bullet) \to \Omega^\bullet(X_\bullet)$$

as follows. For any $\omega_1 \in \Omega^k(X_n)$ and $\omega_2 \in \Omega^l(X_m)$, $\omega_1 \cup \omega_2 \in \Omega^{k+l}(X_{n+m})$ is defined by

$$\omega_1 \cup \omega_2 = (-1)^{km} p_1^* \omega_1 \wedge p_2^* \omega_2,$$

where $p_1 : X_{p+q} \to X_p$ is defined by

$$\begin{cases} p_1(x_1, \ldots, x_{p+q}) & = & (x_1, \ldots, x_p) & \text{if } p \geq 1 \\ p_1(x_1, \ldots, x_q) & = & s(x_1) & \text{if } p = 0 \text{ and } q \geq 1 \\ p_1 & = & \mathrm{id} & \text{if } p = q = 0 \end{cases}$$

and $p_2 : X_{p+q} \to X_q$ is defined by

$$\begin{cases} p_2(x_1, \ldots, x_{p+q}) & = & (x_{p+1}, \ldots, x_p) & \text{if } q \geq 1 \\ p_2(x_1, \ldots, x_p) & = & t(x_p) & \text{if } q = 0 \\ p_2 & = & \mathrm{id} & \text{if } p = q = 0 \end{cases}$$

Alternatively, the cup product can also be written in terms of face maps :

$$\omega_1 \cup \omega_2 = (-1)^{km} (\varepsilon^{n+m}_{n+m})^* \circ \cdots \circ (\varepsilon^{n+i}_{n+i})^* \circ \cdots \circ (\varepsilon^{n+1}_{n+1})^* (\omega_1) \wedge (\varepsilon^{m+n}_0)^* \circ \cdots \circ (\varepsilon^{m+i}_0)^* \circ \cdots \circ (\varepsilon^{m+1}_0)^* (\omega_2), \tag{1.29}$$

$\forall \omega_1 \in \Omega^k(X_n),\ \omega_2 \in \Omega^l(X_m)$.

Lemma 81. *For all $\omega_1 \in \Omega^k(X_n)$, $\omega_2 \in \Omega^l(X_m)$ and $\omega_3 \in \Omega^s(X_p)$, we have*

(a) $(\omega_1 \cup \omega_2) \cup \omega_3 = \omega_1 \cup (\omega_2 \cup \omega_3)$.

(b) $D(\omega_1 \cup \omega_2) = D\omega_1 \cup \omega_2 + (-1)^{k+n} \omega_1 \cup D\omega_2$.

(c) *There exists a degree (-1) bilinear map : $\cup_1 : \Omega^\bullet(X_\bullet) \otimes \Omega^\bullet(X_\bullet) \to \Omega^\bullet(X_\bullet)$ such that*

$$\omega_1 \cup \omega_2 - (-1)^{(k+n)(l+m)} \omega_2 \cup \omega_1 = D\omega_1 \cup_1 \omega_2 - (-1)^{|\omega_1|} \omega_1 \cup_1 D\omega_2 - D(\omega_1 \cup_1 \omega_2).$$

As an immediate consequence, we have the following

Theorem 82. *The de Rahm cohomology $(H^\bullet_{dR}(X_\bullet), \cup)$ is a graded commutative algebra under the cup-product.*

1.3.4 Singular cohomology of simplicial manifolds

Let M be a smooth manifold. Recall that a smooth singular n-simplex in M is a smooth map $\sigma : \Delta^n \to M$, where Δ^n is the standard n-simplex. By $S_n^\infty(M)$, we denote the set of all smooth singular n-simplices. For a fixed ring Λ, say PID (principle ideal domain), let $C_n(M,\Lambda)$ be the space of singular n-chains with coefficients in Λ, i.e. elements of the finite formal sums $\sum_{\sigma \in S_n^\infty(M)} a_\sigma \sigma$, where $a_\sigma \in \Lambda$. The boundary operator

$$\delta : C_n(M,\Lambda) \to C_{n-1}(M,\Lambda)$$

is given by

$$\delta(\sigma) = \sum_i (-1)^i \, \sigma \circ \widehat{\varepsilon}_i^n ,$$

where $\widehat{\varepsilon}_i^n : \Delta^{n-1} \to \Delta^n$ are the face maps as in (1.12). The chain complex $(C_\bullet(M,\Lambda), \delta)$ is called the smooth singular chain complex.

Dually the space of smooth singular n-cochains with coefficients in Λ is

$$C^n(M,\Lambda) = \mathrm{Hom}_{\Lambda-\mathrm{mod}}(C_n(M,\Lambda),\Lambda)$$

with the coboundary operator $d = \delta^* : C^n(M,\Lambda) \to C^{n+1}(M,\Lambda)$. The cochain complex $(C^\bullet(M,\Lambda), d)$ is called the smooth singular cochain complex.

Now consider a simplicial manifold :

$$\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} X_2 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} X_1 \rightrightarrows X_0 \, . \tag{1.30}$$

We have a double cochain complex $C_\bullet(X_\bullet,\Lambda)$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\
C_1(X_0,\Lambda) & \xleftarrow{\partial} & C_1(X_1,\Lambda) & \xleftarrow{\partial} & C_1(X_2,\Lambda) & \xleftarrow{\partial} & \cdots \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\
C_0(X_0,\Lambda) & \xleftarrow{\partial} & C_0(X_1,\Lambda) & \xleftarrow{\partial} & C_0(X_2,\Lambda) & \xleftarrow{\partial} & \cdots
\end{array}$$

where $\partial : C_k(X_p,\Lambda) \to C_k(X_{p-1},\Lambda)$ is the alternating sum of the chain maps induced by the face maps

$$\partial = \sum_i (-1)^i (\varepsilon_i^p)_\# .$$

Here $(\varepsilon_i^p)_\# : C_k(X_p,\Lambda) \to C_k(X_{p-1},\Lambda)$ is the chain map induced by the ith face map $\varepsilon_i^p : X_p \to X_{p-1}$. The total differential is denoted by $D = (-1)^p\delta + \partial$, and the corresponding hyperhomology is denoted by $H_\bullet^{\sin}(X_\bullet,\Lambda)$, called the smooth singular homology of the simplicial manifold $X_\bullet$.

Dually, the smooth singular cochain complex of the simplicial manifold $X_\bullet$ is the total complex of the double complex :

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^1(X_0,\Lambda) & \xrightarrow{\partial} & C^1(X_1,\Lambda) & \xrightarrow{\partial} & C^1(X_2,\Lambda) & \xrightarrow{\partial} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^0(X_0,\Lambda) & \xrightarrow{\partial} & C^0(X_1,\Lambda) & \xrightarrow{\partial} & C^0(X_2,\Lambda) & \xrightarrow{\partial} & \cdots
\end{array} \tag{1.31}$$

where

$$\partial = \sum_{i=0}^{p} (-1)^i (\varepsilon_i^p)^{\#} : C^k(X_{p-1}, \Lambda) \to C^k(X_p, \Lambda)$$

is the alternating sum of the cochain maps $(\varepsilon_i^p)^{\#} : C^k(X_{p-1}, \Lambda) \to C^k(X_p, \Lambda)$ induced by the face maps $\varepsilon_i^p : X_p \to X_{p-1}$. The total differential is denoted by $D = (-1)^p d + \partial$, by abuse of notation, and the corresponding hypercohomology is denoted by $H^\bullet_{\text{sin}}(X_\bullet, \Lambda)$, called the smooth singular cohomology of the simplicial manifold $X_\bullet$.

There is a natural pairing between $C_\bullet(X_\bullet, \mathbb{R})$ and $\Omega^\bullet(X_\bullet)$ given as follows. For any smooth singular k-simplex $\sigma : \Delta_k \to X_p$ in X_p

$$\langle \sigma,\ \omega \rangle = \begin{cases} \int_{\Delta^k} \sigma^* \omega & \text{if } \omega \in \Omega^k(X_p) \\ 0 & \text{otherwise.} \end{cases} \tag{1.32}$$

The equation above extends linearly to a map

$$C_\bullet(X_\bullet, \mathbb{R}) \otimes \Omega^\bullet(X_\bullet) \to \mathbb{R}, \quad \langle \sigma, \omega \rangle \to \int_\sigma \omega. \tag{1.33}$$

The following lemma can be easily verified. The interested reader is refereed to [13] for a detailed proof.

Lemma 83. *The pairing* 1.33 *satisfies the following identities :*

$$\int_{\delta\sigma} \omega = \int_\sigma d\omega$$
$$\int_{\partial\sigma} \omega = \int_\sigma \partial\omega$$

Note that the first identity follows from the classical Stokes theorem for manifolds. As an immediate consequence, we have the following Stokes theorem for simplicial manifolds.

Corollary 84. *For any $\sigma \in C_\bullet(X_\bullet, \mathbb{R})$ and $\omega \in \Omega^\bullet(X_\bullet)$, we have*

$$\int_{D\sigma} \omega = \int_\sigma D\omega.$$

Lemma 85. *If $\phi : X_\bullet \to Y_\bullet$ is a C^∞-simplicial map, then for any $\sigma \in C_\bullet(X_\bullet, \mathbb{R})$ and $\omega \in \Omega^\bullet(Y_\bullet)$,*

$$\int_{\phi_\# \sigma} \omega = \int_\sigma \phi^* \omega. \tag{1.34}$$

Proposition 86. *There is a natural non-degenerate pairing :*

$$H_k^{\sin}(X_\bullet, \mathbb{R}) \otimes H_{DR}^k(X_\bullet) \to \mathbb{R}, \quad \langle [\sigma], [\omega] \rangle \to \int_\sigma \omega, \tag{1.35}$$

compatible with the pull back map $\phi^* : H_{DR}^\bullet(Y_\bullet) \to H_{DR}^\bullet(X_\bullet)$ and push forward map $\phi_* : H_\bullet^{\sin}(X_\bullet, \mathbb{R}) \to H_\bullet^{\sin}(Y_\bullet, \mathbb{R})$ induced from a C^∞-simplicial map $\phi : X_\bullet \to Y_\bullet$

Proof. It follows from Corollary 84 and Lemma 85 that the pairing (1.35) is indeed well-defined and is compatible with the induced maps from a C^∞-simplicial map. For the non-degeneracy, we refer the readers to Proposition 6.1 in [13] for details of the proof. □

Indeed, Proposition 86 implies the following simplicial de Rham theorem.

Theorem 87. *Let $X_\bullet$ be any simplicial manifold.*

(a) *The smooth singular cohomology $H^\bullet_{\sin}(X_\bullet, \mathbb{R})$ is canonically isomorphic to the de Rham cohomology $H^\bullet_{DR}(X_\bullet)$.*

(b) *Both cohomologies are isomorphic to the cohomology of its fat realization $H^\bullet(||X_\bullet||, \mathbb{R})$.*

Now we are ready to define integral differential forms on a simplicial manifold.

Definition 88. *A D-closed differential form $\omega \in \Omega^\bullet(X_\bullet)$ on a simplicial manifold $X_\bullet$ is said be* integral *if its class $[\omega]$ lies in the image under the map $H^\bullet_{\sin}(X_\bullet, \mathbb{Z}) \to H^\bullet_{\sin}(X_\bullet, \mathbb{R}) \cong H^\bullet_{DR}(X_\bullet)$.*

By $Z^k_{dR}(X_\bullet, \mathbb{Z})$ we denote the space of all integral D-closed forms on a simplicial manifold $X_\bullet$. Immediately, we have the following

Corollary 89.

$$Z^k_{dR}(X_\bullet, \mathbb{Z}) = \left\{ \omega \in Z^k_{dR}(X_\bullet) \middle| \int_\sigma \omega \in \mathbb{Z} \text{ for all cycles } \sigma \in Z_k(X_\bullet, \mathbb{Z}) \right\}. \tag{1.36}$$

1.3.5 de Rham and singular cohomology of differentiable stacks

Definition 90 (Exercise 2(b) Chapter 2 [13]). *Let $X_\bullet$ and $Y_\bullet$ be simplicial manifolds. Two C^∞ simplicial maps $\phi, \psi : X_\bullet \to Y_\bullet$ are said to be* homotopic *if for each p there are smooth maps $h_i : X_p \to Y_{p+1}$, $i = 0, 1, \cdots, p$, such that*

(a) $\varepsilon_0 h_0 = \phi$, $\varepsilon_{p+1} h_p = \psi$

(b)

$$\varepsilon_i h_j = \begin{cases} h_{j-1}\varepsilon_i, & i < j \\ h_j \varepsilon_{i-1}, & i > j+1 \\ \varepsilon_j h_{j-1} & i = j \end{cases}$$

(c)

$$\eta_i h_j = \begin{cases} h_{j+1}\eta_i, & i \leq j \\ h_j \eta_{i-1}, & i > j \end{cases}$$

Introduce a map $h : \Omega^q(Y_{p+1}) \to \Omega^q(X_p)$ by

$$h = \sum_{i=0}^{p} (-1)^i h_i^*$$

where $h_i^* : \Omega^q(Y_{p+1}) \to \Omega^q(X_p)$ is the pull back map induced by $h_i : X_p \to Y_{p+1}$.

Lemma 91. (a) *$\psi^* - \phi^* = \partial h + h\partial : \Omega^q(Y_\bullet) \to \Omega^q(X_\bullet)$, $\forall q$.*
(b) *$\psi^* - \phi^* = Dh + hD : \Omega^\bullet(Y_\bullet) \to \Omega^\bullet(X_\bullet)$.*

Similar conclusions hold for singular cochains as well.

Corollary 92. *If C^∞ simplicial maps $\phi, \psi : X_\bullet \to Y_\bullet$ are homotopic, then*

$$\psi^* = \phi^* \quad : \quad H^\bullet_{DR}(Y_\bullet) \to H^\bullet_{DR}(X_\bullet) \quad \text{and} \tag{1.37}$$
$$\psi^* = \phi^* \quad : \quad H^\bullet_{\sin}(Y_\bullet, \Lambda) \to H^\bullet_{\sin}(X_\bullet, \Lambda) \tag{1.38}$$

Now we turn our attention to Lie groupoids.

Lemma 93. (a) *A Lie groupoid morphism naturally induces a C^∞ simplicial map of their corresponding nerves;*
(b) *Any two naturally equivalent Lie groupoid morphisms induce homotopy equivalent C^∞ simplicial maps of their corresponding nerves.*

Proof. (1) is obvious. For (2), assume that $\phi, \psi : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ are morphisms of Lie groupoids, and $\theta : X_0 \to Y_1$ a natural equivalence from ϕ to ψ (see Definition 42). Define $h_i : X_p \to Y_{p+1}, \quad i = 1, \cdots, p$ by

$$h_i(x_1, \cdots, x_p) = \big(f(x_1), \cdots, f(x_i), \theta(t(x_i)), g(x_{i+1}), \cdots, g(x_p)\big)$$

and

$$h_0(x_1, \cdots, x_p) = \big(\theta(s(x_1)), g(x_1), \cdots, g(x_p)\big)$$

It follows from a direct verification that $(h_i)_{i=0,\cdots,p}$ indeed defines a homotopy of the induced simplicial maps $\phi, \psi : X_\bullet \to Y_\bullet$. □

Corollary 94. (a) *Any Lie groupoid morphism $f : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ induces a morphism of cohomology $f^* : H^\bullet_{DR}(Y_\bullet) \to H^\bullet_{DR}(X_\bullet)$ and $f^* : H^\bullet_{\sin}(Y_\bullet, \Lambda) \to H^\bullet_{\sin}(X_\bullet, \Lambda)$.*

(b) *If* $\phi \simeq \psi : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ *are naturally equivalent morphisms of Lie groupoids, then* $\phi^* = \psi^* : H^\bullet_{DR}(Y_\bullet) \to H^\bullet_{DR}(X_\bullet)$ *and* $\phi^* = \psi^* : H^\bullet_{\text{sin}}(Y_\bullet, \Lambda) \to H^\bullet_{\text{sin}}(X_\bullet, \Lambda)$.

As a consequence, if $f : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ is a Morita morphism admitting a section, then f^* induces an isomorphism on both de Rham and singular cohomologies. In general one can prove the following result using bisimplicial manifolds and double fibration argument. Readers may consult [5, 4] for details. An alternative approach is to use Noohi's theorem that the fat realizations of Morita equivalent Lie groupoids are weak homotopy equivalent [28].

Proposition 95. *Any Morita morphism of Lie groupoids* $f : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ *induces an isomorphism on de Rham cohomology* $f^* : H^\bullet_{DR}(Y_\bullet) \,\text{iso}\, H^\bullet_{DR}(X_\bullet)$ *and an isomorphism on singular cohomology* $f^* : H^\bullet_{\text{sin}}(Y_\bullet, \Lambda) \,\text{iso}\, H^\bullet_{\text{sin}}(X_\bullet, \Lambda)$.

Corollary 96. *Any Hilsum-Skandalis map* $F : X_\bullet \rightsquigarrow Y_\bullet$ *from* $X_1 \rightrightarrows X_0$ *to* $Y_1 \rightrightarrows Y_0$ *induces a well defined morphism*

$$F^* : H^\bullet_{DR}(Y_\bullet) \to H^\bullet_{DR}(X_\bullet) \quad \text{and} \tag{1.39}$$

$$F^* : H^\bullet_{\text{sin}}(Y_\bullet, \Lambda) \to H^\bullet_{\text{sin}}(X_\bullet, \Lambda). \tag{1.40}$$

Proof. Assume that F is represented by a roof $(\varphi, f) : X_\bullet \leftarrow Z_\bullet \to Y_\bullet$. Then we define $F^* = (\varphi^*)^{-1} \circ f^*$. According to Corollary 94, F^* is independent of the choice of the roof, and therefore is well defined. □

According to Proposition 95, both de Rham cohomology and singular cohomology of Lie groupoids are invariant under Morita equivalence, and hence well defined for differentiable stacks.

Definition 97. *Let* $\mathfrak{X}$ *be a differentiable stack. Define*

$$H^n_{DR}(\mathfrak{X}) = H^n_{DR}(X_\bullet) \quad \text{and} \tag{1.41}$$

$$H^n_{\text{sin}}(\mathfrak{X}, \Lambda) = H^\bullet_{\text{sin}}(X_\bullet, \Lambda), \tag{1.42}$$

where $X_1 \rightrightarrows X_0$ *is any Lie groupoid representing the stack* $\mathfrak{X}$.

Theorem 98. *Any* C^∞*-map* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *of differentiable stacks induces morphisms on cohomologies*

$$\phi^* : H^\bullet_{DR}(\mathfrak{Y}) \to H^\bullet_{DR}(\mathfrak{X}) \quad \text{and} \tag{1.43}$$

$$\phi^* : H^\bullet_{\text{sin}}(\mathfrak{Y}, \Lambda) \to H^\bullet_{\text{sin}}(\mathfrak{X}, \Lambda). \tag{1.44}$$

1.3.6 Sheaf cohomology

Sheaf cohomology of differentiable stacks can be defined as derived functors of the global section functor similar to the classical approach of Grothendieck. Here we describe a down-to-earth approach in terms of nervers of Lie groupoids following Behrend [4, 5], which serves our purpose better in differential geometry.

First of all, one needs to make precise what a sheaf over a differentiable stack means. Again, this is quite involved and we refer the interested readers to [6] for details. For our purpose, we only need a special type of sheaves over stacks, called *big sheaves*.

Definition 99. *A big sheaf is a functor*

$$F: \quad \text{(smooth manifolds)} \to \text{(abelian groups)}$$

such that if we restrict F to any given smooth manifold and its open subsets, we obtain a sheaf on the manifold in the ordinary sense.

Example 100. *The following is a list of big sheaves that we are interested in this Note.*

- *For any $k \geq 0$, let Ω^k denote the contravariant functor that assigns, to any manifold M, the space of all differential k-forms on M as an abelian group. Then Ω^k is a big sheaf;*
- *Let $\mathcal{R}$, and $\mathcal{S}^1$ respectively, denote the contravariant functor that assigns, to any manifold M, the space of all $\mathbb{R}$-valued, and respectively, S^1-valued smooth functions on M as abelian groups. Then both $\mathcal{R}$ and $\mathcal{S}^1$ are big sheaves. Note that $\mathcal{R}$ is the exactly the same as Ω^0 by definition.*
- *Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}/\mathbb{Z}$ respectively denote the contravariant functor that assigns, to any manifold M, the space of locally constant $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}/\mathbb{Z}$-valued functions on M as abelian groups. Then $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}/\mathbb{Z}$ are big sheaves*

Next we will introduce sheaf cohomology for differentiable stacks when the sheaf is a big sheaf. Note that a big sheaf induces a sheaf in the ordinary sense on any smooth manifold X, which is called the *induced small sheaf* over X.

First of all, let us consider the following simplest situation. Assume that F is a big sheaf such that its induced small sheaf F_X over any manifold X is acyclic, i.e. $H^i(X, F_X) = 0$, for all $i > 0$. These include $\mathcal{R}$ and Ω^k since both small sheaves $\mathcal{R}_X$ and Ω^k_X are fine.

Let $\mathfrak{X}$ be a differentiable stack, and $X_1 \rightrightarrows X_0$ a Lie groupoid representing $\mathfrak{X}$. Consider the nerver of $X_1 \rightrightarrows X_0$ as given by Diagram (1.21). Indeed Diagram (1.21) induces a diagram

$$F(X_0) \rightrightarrows F(X_1) \Rrightarrow F(X_2) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots \tag{1.45}$$

which can, in fact, be refined to a cosimplicial set.

Let

$$\partial : F(X_{p-1}) \to F(X_p)$$

be the alternating sum of the maps of Diagram (1.45) :

$$\partial = \sum_{i=0}^{p} (-1)^i (\varepsilon_i^p)^* : \quad F(X_{p-1}) \to F(X_p).$$

Here, for each i, $(\varepsilon_i^p)^* : F(X_{p-1}) \to F(X_p$ is the pull back map of $\varepsilon_i^p : X_p \to X_{p-1}$. We obtain a cochain complex of abelian groups :

$$F(X_0) \xrightarrow{\partial} F(X_1) \xrightarrow{\partial} F(X_2) \xrightarrow{\partial} \cdots,$$

whose cohomology groups are denoted by

$$\check{H}^i(X_\bullet, F) = h^i\big(F(X_\bullet)\big)$$

When F is the big sheaf $\mathcal{R}$, i.e. Ω^0, $\check{H}^i(X_\bullet, \Omega^0)$ is also called *groupoid cohomology* of the Lie groupoid $X_1 \rightrightarrows X_0$ with trivial coefficients [36].

Lemma 101. *Assume that for any manifold X the induced small sheaf F_X over X is acyclic, i.e. $H^i(X, F_X) = 0$, for all $i > 0$. Then $\check{H}^i(X_\bullet, F)$ is Morita invariant. That is, if $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ are Morita equivalent, then $\check{H}^i(X_\bullet, F) \cong \check{H}^i(Y_\bullet, F)$.*

Ping: SHOULD WE SKETCH A PROOF

Definition 102. *Let F be a big sheaf such that for any manifold X the induced small sheaf F_X over X is acyclic, i.e. $H^i(X, F_X) = 0$, for all $i > 0$. For a differentiable stack $\mathfrak{X}$,*

$$H^i(\mathfrak{X}, F) = \check{H}^i(X_\bullet, F),$$

where $X_1 \rightrightarrows X_0$ is any Lie groupoid presenting $\mathfrak{X}$.

The following theorem is due to Abad-Crainic [[3] Corollary 4.2].

Theorem 103. *If $\mathfrak{X}$ is a separated differentiable stack, i.e. it is represented by a proper Lie groupoid $X_1 \rightrightarrows X_0$, then*

$$H^i(\mathfrak{X}, \Omega^k) = 0, \quad \text{if } i > k.$$

In particular, we have

Corollary 104. *For a separated differentiable stack $\mathfrak{X}$, $H^\bullet(\mathfrak{X}, \Omega^0)$ is acyclic.*

Now we move to an arbitrary big sheaf F. Recall that any sheaf $\mathcal{S}$ on a topological space X has a canonical flabby [1] resolution [37].

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathfrak{C}^0(\mathcal{S}) \longrightarrow \mathfrak{C}^1(\mathcal{S}) \longrightarrow \mathfrak{C}^2(\mathcal{S}) \longrightarrow \dots, \tag{1.46}$$

where $\mathfrak{C}^0(\mathcal{S})$ is the sheaf of discontinuous sections of $\mathcal{S}$ over X. Apply the construction above to the small sheaf F_X, for all smooth manifold X, because of the canonical nature of the construction of the resolution (1.46), we in fact obtain a resolution of F by big sheaves :

$$0 \longrightarrow F \longrightarrow \mathfrak{C}^0(F) \xrightarrow{d} \mathfrak{C}^1(F) \xrightarrow{d} \mathfrak{C}^2(F) \xrightarrow{d} \dots, \tag{1.47}$$

where, for each i, $\mathfrak{C}^i(F)$ is a big sheaf and its induced small sheaf $\mathfrak{C}^i(F)|_X$ is flabby and hence acyclic.

Imitating the construction of de Rham cohomology, we consider the double complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow d & & \uparrow d & & \uparrow d & & \\ \mathfrak{C}^1(F)(X_0) & \xrightarrow{\partial} & \mathfrak{C}^1(F)(X_1) & \xrightarrow{\partial} & \mathfrak{C}^1(F)(X_2) & \xrightarrow{\partial} & \cdots \\ \uparrow d & & \uparrow d & & \uparrow d & & \\ \mathfrak{C}^0(F)(X_0) & \xrightarrow{\partial} & \mathfrak{C}^0(F)(X_1) & \xrightarrow{\partial} & \mathfrak{C}^0(F)(X_2) & \xrightarrow{\partial} & \cdots \end{array} \tag{1.48}$$

Here, again, the horizontal differential $\partial : \mathfrak{C}^k(F)(X_p) \to \mathfrak{C}^k(F)(X_{p+1})$ is the alternating sum of the induced morphisms of the face maps of the corresponding simplicial manifolds $X_\bullet$.

1. A sheaf $\mathcal{S}$ over a topological space X is called *flabby* if $\mathcal{S}(X) \to \mathcal{S}(U)$ is surjective for all open subsets U in X. Flabby sheaves must be soft, and therefore acyclic.

Definition 105. *Let F be a big sheaf and $\mathfrak{X}$ a differentiable stack. We define*

$$H^i(\mathfrak{X}, F) = h^i\big(\operatorname{tot}\mathfrak{C}^\bullet(F)(X_\bullet)\big),$$

the total cohomology of the double complex (1.48)*, where $X_1 \rightrightarrows X_0$ is any Lie groupoid presenting $\mathfrak{X}$.*

The total cohomology $h^i\big(\operatorname{tot}\mathfrak{C}^\bullet(F)(X_\bullet)\big)$ is also denoted $\check{H}^i(X_\bullet, \mathfrak{C}^\bullet(F))$. Of course, one needs to prove that this definition is well defined, i.e., Morita invariant, which can be handled along the lines similar to the case of de Rham cohomology. It is simple to see that if $\mathfrak{X}$ is a smooth manifold X, $H^i(\mathfrak{X}, F) \cong H^i(X, F_X)$. Sheaf cohomology of differentiable stacks satisfies many properties as the ordinary sheaf cohomology over a topological space (see Theorem 3.11 [37]). In what follows, we list a number of important ones that will be useful later on.

Definition 106. *We say a sequence of big sheaves $0 \to A \to \mathfrak{B} \to \mathfrak{C} \to 0$ is a* short exact sequence *of big sheaves if for each $d \geq 0$, we have the following exact sequence of small sheaves :*

$$0 \longrightarrow A^d_{\mathbb{R}} \longrightarrow \mathfrak{B}^d_{\mathbb{R}} \longrightarrow \mathfrak{C}^d_{\mathbb{R}} \longrightarrow 0$$

Theorem 107. (a) *Any short exact sequence of big sheaves $0 \to A \to \mathfrak{B} \to \mathfrak{C} \to 0$ induces a long exact sequence*

$$\cdots \to H^i(\mathfrak{X}, A) \to H^i(\mathfrak{X}, \mathfrak{B}) \to H^i(\mathfrak{X}, \mathfrak{C}) \to H^{i+1}(\mathfrak{X}, A) \to \cdots$$

(b) *A commutative diagram of exact sequences of big sheaves*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathfrak{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & \mathfrak{B}' & \longrightarrow & \mathfrak{C}' & \longrightarrow & 0 \end{array} \tag{1.49}$$

induces a commutative diagram of long exact sequences :

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^i(\mathfrak{X}, A) & \longrightarrow & H^i(\mathfrak{X}, \mathfrak{B}) & \longrightarrow & H^i(\mathfrak{X}, \mathfrak{C}) & \longrightarrow & H^{i+1}(\mathfrak{X}, A) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^i(\mathfrak{X}, A') & \longrightarrow & H^i(\mathfrak{X}, \mathfrak{B}') & \longrightarrow & H^i(\mathfrak{X}, \mathfrak{C}') & \longrightarrow & H^{i+1}(\mathfrak{X}, A') & \longrightarrow & \dots \end{array} \tag{1.50}$$

As in the classical case, the following theorem is very useful in computing sheaf cohomology.

Theorem 108. *Let F be a big sheaf and let*

$$0 \longrightarrow F \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^3 \longrightarrow \dots, \tag{1.51}$$

be a resolution of F such that, for each i, F^i is a big sheaf and its induced small sheaf $F^i|_X$ is acyclic. Then there is a natural isomorphism :

$$h^i\big(\operatorname{tot}F_\bullet(X_\bullet)\big) \longrightarrow H^i(\mathfrak{X}, F),$$

where $F^\bullet(X_\bullet)$ is the double complex similar to (1.48) *and $h^i\big(\operatorname{tot}F_\bullet(X_\bullet)\big)$ denotes its total cohomology.*

Now we consider two particular cases. Consider the constant big sheaf $\mathbb{R}$. It is obvious that the de Rham complex of big sheaves $\Omega^\bullet$ is a resolution of $\mathbb{R}$ satisfying the condition as in Theorem 108 :

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \dots, \tag{1.52}$$

As a consequence, we have

Corollary 109. *For any differentiable stack, the sheaf cohomology $H^\bullet(\mathfrak{X}, \mathbb{R})$ is isomorphic to the de Rham cohomology $H^\bullet_{DR}(\mathfrak{X})$.*

Consider the constant big sheaf $\mathbb{Z}$. For any manifold M, let $\mathcal{C}^i_\infty(\mathbb{Z})(M)$ be the abelian group of smooth singular i-cochains in M with coefficients in $\mathbb{Z}$ (see Section 1.3.4). Then the assignment $M \to \mathcal{C}^i_\infty(\mathbb{Z})(M)$ defines a big sheaf $\mathcal{C}^i_\infty(\mathbb{Z})$. Since, for each i, $\mathcal{C}^i_\infty(\mathbb{Z})$ is a flabby sheaf, therefore it is acyclic. One checks that

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}^0_\infty(\mathbb{Z}) \longrightarrow \mathcal{C}^1_\infty(\mathbb{Z}) \longrightarrow \mathcal{C}^2_\infty(\mathbb{Z}) \longrightarrow \dots \tag{1.53}$$

is a resolution of $\mathbb{Z}$ satisfying the condition as in Theorem 108. Therefore

Corollary 110. *For any differentiable stack, the sheaf cohomology $H^\bullet(\mathfrak{X}, \mathbb{Z})$ is isomorphic to the singular cohomology $H^\bullet_{\sin}(\mathfrak{X}, \mathbb{Z})$.*

Consider a Lie group G acting smoothly on a manifold X from the right. We have the transformation groupoid $M \rtimes G \rightrightarrows M$ and the associated differentiable stack is the quotient stack $[M/G]$. Consider the universal principle bundle $G \to EG \to BG$, where BG is the classifying space of G [23]. The equivariant cohomology, by definition, is the cohomology of $(EG \times M)/G$, the homotopy quotient of M by G, where G acts on $EG \times M$ diagonally. That is, $H^\bullet_G(M, \mathbb{Z}) = H^\bullet((EG \times M)/G, \mathbb{Z})$.

Proposition 111. *Assume a Lie group G acts smoothly on a manifold M. Then*

$$H^\bullet_G(M, \mathbb{Z}) \cong H^\bullet([M/G], \mathbb{Z})$$

Since an orbifold is locally a quotient space by a finite group, its cohomology as a stack differs from the cohomology of its coarse moduli space by torsion. We refer the reader to [4] for the proof of the following

Proposition 112. *Let $\mathfrak{X}$ be an orbifold with coarse moduli space $\bar{\mathfrak{X}}$. Then the canonical morphisms $\mathfrak{X} \to \bar{\mathfrak{X}}$ (as topological stacks) induces isomorphisms on $\mathbb{Q}$-valued cohomology*

$$H^\bullet(\bar{\mathfrak{X}}, \mathbb{Q}) \xrightarrow{\sim} H^\bullet(\mathfrak{X}, \mathbb{Q}).$$

Chapitre 2

Chapter 4 : Twisted K-theory

2.1 Reduced C^*-algebras of Lie groupoid S^1-central extensions

2.1.1 Preliminary on fields of C^*-algebras

Definition 113. *Let X be a Hausdorff topological space. An upper semicontinuous, (resp. continuous) field of Banach spaces over X consists of a family $(E_x)_{x\in X}$ of Banach spaces together with a topology on the total space $\widetilde{E} := \coprod_{x\in X} E_x$ satisfying the following conditions :*

(i) the topology on E_x induced from that on $\widetilde{E}$ is the norm-topology of the Banach space ;
(ii) the projection $\pi\colon \widetilde{E} \to X$ is a continuous and open map ;
(iii) both the addition $(e, e') \in \widetilde{E} \times_X \widetilde{E} \mapsto e + e' \in \widetilde{E}$, and the scalar multiplication $(\lambda, e) \in \mathbb{C} \times \widetilde{E} \to \lambda e \in \widetilde{E}$ are continuous maps ;
(iv) the norm $\widetilde{E} \to \mathbb{R}_+$ is an upper semicontinuou continuous (resp. continuous) map ;
(v) if $\|e_i\| \to 0$ and $\pi(e_i) \to x$, then $e_i \to 0_x$;
(vi) for any $e \in E_x$, there always exists a continuous section ξ of $\widetilde{E}$ such that $\xi(x) = e$.

A field of Banach spaces can be constructed as follows [14] :

Proposition 114. *Let X be a Hausdorff topological space. Assume that $(E_x)_{x\in X}$ is a family of Banach spaces and $\widetilde{E} := \coprod_{x\in X} E_x$. Let $\mathbb{E}$ be a $C(X)$-module consisting of sections of $\widetilde{E} \to X$ satisfying the following conditions :*

(i) for every $\xi \in \mathbb{E}$, the function $x \mapsto \|\xi(x)\|$ is upper semicontinuous (resp. continuous) ;
(ii) for any $x \in X$, the set $\{\xi(x) |\, \xi \in \mathbb{E}\}$ is dense in E_x,

then there is a unique topology on $\widetilde{E}$ making $\widetilde{E} \to X$ into an upper semicontinuous (resp. continuous) field of Banach spaces such that elements of $\mathbb{E}$ are exactly continuous sections.

One defines *fields of Banach algebras* and *fields of C^*-algebras* in a similar fashion. Given a field of Banach spaces E over X, denote by $C(X, \widetilde{E})$, $C_0(X, \widetilde{E})$ and $C_c(X, \widetilde{E})$ the space of continuous sections, the space of continuous sections vanishing at infinity, the space of compactly supported continuous sections of the bundle $\widetilde{E} \to X$, respectively.

Definition 115. *Let $\coprod_{x\in X} E_x \to X$ be an upper semicontinuous (resp. continuous) field of Banach spaces over X, and let $f\colon Y \to X$ be a continuous map. Then the upper semicontinuous (resp. continuous) field f^*E over Y is the field over Y with the fiber $E_{f(y)}$ at $y \in Y$, and whose total space is $Y \times_X \widetilde{E}$ with the induced topology from $Y \times \widetilde{E}$.*

If E is determined by a $C(X)$-module $\mathbb{E}$ as in Proposition 114, then f^*E is determined by $f^*\mathbb{E} = \{\xi\circ f \mid \xi \in \mathbb{E}\}$.

Definition 116. *Let X be a locally compact topological space. A $C_0(X)$-algebra is a C^*-algebra A together with a $*$-homomorphism from $C_0(X)$ to $Z(M(A))$, the center of the multiplier algebra $M(A)$ of A such that $C_0(X)A = A$.*

Proposition 117. *For a locally compact topological space X, there is a one-one correspondence between $C_0(X)$-algebras and upper semicontinuous fields of C^*-algebras over X.*

Proof. This essentially follows from [7, Proposition 2.12 a)]. The correspondence goes as follows.

For any $x \in X$, denote by $C_0(X)_x$ the ideal of $C_0(X)$ consisting of functions that vanish at x. Given a $C_0(X)$-algebra A, let $A_x = A/(C_0(X)_x A)$. Denote by $\pi_x\colon A \to A_x$ the projection map, and $\widetilde{A} = \coprod_{x\in X} A_x$. Then there is a unique upper semicontinuous field of C^*-algebra structure on $\widetilde{A} \to X$ such that the map

$$\begin{array}{rcl} A & \to & C_0(X,\widetilde{A}) \\ a & \mapsto & (x \mapsto \pi_x(a)) \end{array}$$

is an isomorphism of C^*-algebras.

Conversely, assume that $\widetilde{A} := \coprod_{x\in X} A_x \to X$ is a upper semicontinuous field of C^*-algebras over X, and $A = C_0(X,\widetilde{A})$ is the space of continuous sections vanishing at infinity. Then A is obviously a $C_0(X)$-algebra, and the evaluation map $A \to A_x$ induces a $*$-isomorphism $A_x \to A_x$. □

Definition 118 ([19])**.** *Assume that $X_1 \rightrightarrows X_0$ is a topological groupoid, and A is a $C_0(X_0)$-algebra. By a $X_1 \rightrightarrows X_0$-action on A, we mean there is an isomorphism of $C_0(X_1)$-algebras $\alpha\colon t^*A \to s^*A$ such that $\alpha_{xy} = \alpha_x\alpha_y$ for all $(x,y) \in X_2$, where $\alpha_x\colon (t^*A)_x \cong A_{t(x)} \to (s^*A)_x \cong A_{s(x)}$ is the induced isomorphism.*

Definition 119. *Let $X_1 \rightrightarrows X_0$ be a topological groupoid, and $\widetilde{A} = \coprod_{m\in X_0} A_m \to X_0$ an upper semicontinuous field of C^*-algebras. By a groupoid $X_1 \rightrightarrows X_0$ action on $\widetilde{A} \to X_0$, we mean that there exists an isomorphism $\alpha\colon t^*\widetilde{A} \to s^*\widetilde{A}$ of fields of C^*-algebras over X_1 such that*

$$\alpha_{xy} = \alpha_x\alpha_y$$

for all $(x,y) \in X_2$.

From Proposition 117, immediately we have the following

Proposition 120. *Let $X_1 \rightrightarrows X_0$ be a topological groupoid. There is a one-one correspondence between $C_0(X_0)$-algebras with $X_1 \rightrightarrows X_0$-actions, and $X_1 \rightrightarrows X_0$-actions on fields of C^*-algebras over X_0.*

Now we turn to C^*-modules over $C_0(X)$-algebras.

Proposition 121. *Let $\widetilde{A} = \coprod_{x\in X} A_x \to X$ be an upper semicontinuous field of C^*-algebras over X, and $A = C_0(X, \widetilde{A})$. Assume that $\mathcal{E}$ is an A-Hilbert module. Then there is an unique upper semicontinuous field of Banach spaces $\widetilde{\mathcal{E}} := \coprod_{x\in X} \mathcal{E}_x \to X$ such that $\mathcal{E} \cong C_0(X, \widetilde{\mathcal{E}})$.*

PROOF. The proof essentially follows from Proposition 114. Here $\mathcal{E}_x := \mathcal{E} \otimes_A A_x$, and $\widetilde{\mathcal{E}} := \coprod_{x\in X} \mathcal{E}_x \to X$. The isomorphsim between $\mathcal{E}$ and $C_0(X, \widetilde{\mathcal{E}})$ is given by $\xi \mapsto (x \mapsto \pi_x(\xi))$, where $\pi_x \colon \mathcal{E} \to \mathcal{E}_x$ is the canonical map. □

In particular, any $C_0(X)$-Hilbert module can be identified with the space of continuous sections vanishing at infinity of a continuous field of Hilbert spaces.

Consider an upper semicontinuous field of C^*-algebras $\widetilde{A} = \coprod_{x\in X} A_x \to X$. Let $A = C_0(X, \widetilde{A})$. Assume that $\mathcal{E}$ is an A-Hilbert module with $\widetilde{\mathcal{E}} \to X$ its corresponding upper semicontinuous field of Banach spaces. One may introduce a topology on

$$\mathcal{L}(\widetilde{\mathcal{E}}) := \coprod_{x\in X} \mathcal{L}(\mathcal{E}_x) \tag{2.1}$$

satisfying the property that for every net $T_i \in \mathcal{L}(\mathcal{E}_{x_i})$ and $T \in \mathcal{L}(\mathcal{E}_x)$, T_i converges to T if and only if for every $\xi \in C(X, \widetilde{\mathcal{E}})$, the following conditions are satisfied :

(i) $x_i \to x$;
(ii) $T_i\xi(x_i) \to T\xi(x)$; and
(iii) $T_i^*\xi(x_i) \to T^*\xi(x)$.

Then the bundle $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ satisfies all the properties in Definition 113 except that the norm is not necessarily upper semicontinuous , and the induced topology on $\mathcal{L}(\mathcal{E}_x)$ is not the norm-topology.

Definition 122. *(i) A section T of $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ is strongly continuous if for every $\xi \in C(X, \widetilde{A})$, the section $x \mapsto T_x\xi(x)$ belongs to $C(X, \widetilde{A})$;*
(ii) A section T of $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ is $*$-strongly continuous if both T and T^* are strongly continuous.

The following lemma can be easily verified.

Lemma 123. *A section of $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ is $*$-strongly continuous if and only if it is a continuous section of the bundle $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ with respect to the topology as defined in Equation* (2.1).

Denote by $C_b(X, \mathcal{L}(\widetilde{\mathcal{E}}))$ the space of norm-bounded continuous sections.

Proposition 124. *There is an isomorphism*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{E}) & \to & C_b(X, \mathcal{L}(\widetilde{\mathcal{E}})) \\ T & \mapsto & (x \mapsto T_x), \end{array}$$

where $T_x = T \otimes_A \mathrm{Id} \in \mathcal{L}(\mathcal{E} \otimes_A A_x) = \mathcal{L}(\mathcal{E}_x)$.

PROOF. This follows directly from Proposition 121 and the fact that $\mathcal{L}(\mathcal{E})$ is, by definition, the space of maps from $\mathcal{E}$ to $\mathcal{E}$ admitting an adjoint. □

The analogue of the above proposition for $\mathcal{K}(\mathcal{E})$ is less obvious. In what follows, we will consider a particular case.

Proposition 125. *Let $\widetilde{\mathcal{H}} = \coprod_{x\in X} \mathcal{H}_x \to X$ be a continuous field of Hilbert spaces, and $\mathcal{H} = C_0(X, \widetilde{\mathcal{H}})$ its associated $C_0(X)$-Hilbert module. Then there exists a unique topology on $\mathcal{K}(\widetilde{\mathcal{H}}) := \coprod_{x\in X} \mathcal{K}(\mathcal{H}_x)$ such that*
(i) the field $\mathcal{K}(\widetilde{\mathcal{H}}) \to X$ is a continuous field of C^*-algebras ;
(ii) for every $\xi, \eta \in C_0(X, \widetilde{\mathcal{H}})$, the section $x \mapsto T_{\xi(x),\eta(x)}$ belongs to $C_0(X, \mathcal{K}(\widetilde{\mathcal{H}}))$.
Moreover, the map

$$T_{\xi,\eta} \mapsto (x \mapsto T_{\xi(x),\eta(x)})$$

extends uniquely to an isomorphism of C^*-algebras $\mathcal{K}(\mathcal{H}) \overset{\sim}{\to} C_0(X, \mathcal{K}(\widetilde{\mathcal{H}}))$.

PROOF. We sketch the proof in the case that the field is countably generated. In this case, by the stabilization theorem (see [12, 32]), we may assume that the field is trivial $\widetilde{\mathcal{H}} \cong X \times \mathbb{H}$. It is known that $\mathcal{K}(\mathcal{H})$ is isomorphic to $C_0(X, \mathcal{K}(\mathbb{H}))$, where $\mathcal{K}(\mathbb{H})$ is endowed with the norm-topology. It thus follows from Proposition 114 that $C_0(X, \mathcal{K}(\mathbb{H}))$ coincides with the space of continuous sections vanishing at infinity of a continuous field of C^*-algebras over X with fibers isomorphic to $\mathcal{K}(\mathbb{H})$. □

2.1.2 Fell bundles over a groupoid

We start with the following definition due to Yamagami [39].

Definition 126. *Let $X_1 \rightrightarrows X_0$ be a locally compact groupoid. An upper semicontinuous (resp. continuous) Fell bundle over $X_1 \rightrightarrows X_0$ is an upper semicontinuous (resp. continuous) field of Banach spaces $(E_x)_{x\in X_1}$ over X_1 equipped with the following structures :*
(a) an associative bilinear product

$$E_x \times E_y \mapsto E_{xy}, (e_1, e_2) \mapsto e_1e_2, \forall (x, y) \in X_2,$$

and
(b) an antilinear involution

$$E_x \mapsto E_{x^{-1}}, e \mapsto e^*$$

such that the following properties hold : $\forall (x, y) \in X_2$ and $(e_1, e_2) \in E_x \times E_y$,

(i) $\|e_1e_2\| \leqslant \|e_1\|\|e_2\|$;
(ii) $(e_1e_2)^* = e_2^*e_1^*$;
(iii) $\|e_1^*e_1\| = \|e_1\|^2$;
(iv) $e_1^*e_1$ is a positive element of the C^*-algebra $E_{t(x)}$;
(v) both the product $m^*(E) \to E : (e_1, e_2) \mapsto e_1e_2$, and the involution $E \to E : e \mapsto e^*$, are continuous maps, where $m : X_2 \to X_1$ is the groupoid multiplication map ;
(vi) $\forall (x, y) \in X_2$, the image of the product $E_x \times E_y \to E_{xy}$ spans a dense subspace of E_{xy}.

Remark 127. *Note that (i)–(iii) imply that E_m, $\forall m \in X_0$, is indeed a C^*-algebra. Therefore (iv) makes sense.*

Examples 128. *Let $X_1 \rightrightarrows X_0$ be a locally compact groupoid, which acts on a $C_0(X_0)$-algebra A. Let $\mathcal{A}$ be its associated upper semicontinuous field of C^*-algebras as in Proposition 117. Then $E = t^*A$ is a upper semicontinuous Fell bundle over the groupoid $X_1 \rightrightarrows X_0$, where the product is given by*

$$\begin{aligned} &E_x \times E_y (\cong A_{t(x)} \times A_{t(y)}) \mapsto E_{xy} (\cong A_{t(y)} \\ &(a, b) \mapsto \alpha_{y^{-1}}(a)b, \end{aligned}$$

and the involution is given by

$$\begin{aligned} &E_x (\cong A_{t(x)}) \mapsto E_{x^{-1}} (\cong A_{s(x)}) \\ &a \mapsto \alpha_x(a^*) \end{aligned}$$

Therefore upper semicontinuous Fell bundles over a groupoid extend the notion of actions of a groupoid on C^*-algebras. In fact, according to Muhly [26], they can be considered as "actions of groupoids on C^*-algebras by Morita equivalences".

Now let us return to the discussion on a general upper semicontinuous Fell bundle E. Define an $A_{t(x)}$-valued scalar product on E_x by

$$\langle e_1, e_2 \rangle = e_1^* e_2.$$

Then E_x becomes an $A_{t(x)}$-Hilbert module, and the left multiplication by elements of $A_{s(x)}$ defines a *-homomorphism $A_{s(x)} \to \mathcal{L}(E_x)$. In other words, E_x is an $A_{s(x)}$-$A_{t(x)}$-correspondence.

Note that the product $E_x \times E_y \to E_{xy}$ induces an isomorphism of $A_{s(x)} - A_{t(y)}$ bimodules $E_x \otimes_{A_{t(x)}} E_y \to E_{xy}$. Indeed, to check that this map is isometric, note that $\forall \xi_i \in E_x$, $\eta_i \in E_y$,

$$\begin{aligned} \langle \sum_i \xi_i \otimes \eta_i, \sum_i \xi_i \otimes \eta_i \rangle &= \sum_{i,j} \langle \eta_i, \langle \xi_i, \xi_j \rangle \eta_j \rangle \\ = \quad \sum_{i,j} (\xi_i \eta_i)^* (\xi_j \eta_j) &= \langle \sum_i \xi_i \eta_i, \sum_i \xi_i \eta_i \rangle. \end{aligned}$$

The following proposition justifies the reason that we require the field to be upper semicontinuous :

Proposition 129. *If E is an upper semicontinuous Fell bundle over the groupoid $X_1 \rightrightarrows X_0$, then sections of the form $(x, y) \mapsto \sum_i \xi_i(x)\eta_i(y)$, where ξ_i, $\eta_i \in C_0(X_1, E)$, are dense in $C_0(X_2, m^*E)$.*

2.1.3 Reduced C^*-algebras

We first recall the definition of the reduced C^*-algebra associated to an upper semicontinuous Fell bundle over a groupoid. See [30, Chapter 2] for Haar measures and the C^*-algebra of a groupoid.

Assume that $X_1 \rightrightarrows X_0$ is a locally compact groupoid with a Haar system $(\lambda^m)_{m\in X_0}$, and E an upper semicontinuous Fell bundle over the groupoid $X_1 \rightrightarrows X_0$. Denote by $C_c(X_1, E)$ the space of its compactly supported continuous sections. Define the convolution product and the involution on $C_c(X_1, E)$ as follows. For any $\xi, \eta \in C_c(X_1, E)$,

$$(\xi * \eta)(x) = \int\limits_{y\in X_1^{s(x)}} \xi(y)\eta(y^{-1}x)\lambda^{s(x)}(dy)$$

$$\xi^*(x) = \xi(x^{-1})^*$$

To see that $\xi * \eta$ is well-defined, from (v) in Definition 126, it follows that the map $(x, y) \mapsto \xi(y)\eta(y^{-1}x)$ can be expressed as the uniform limit of those of the form $\sum_i f_i(y, y^{-1}x)\zeta_i(y(y^{-1}x))$, where $f_i \in C_c(X_2)$ and $\zeta_i \in C_c(X_1, E)$. The latter can be written as the sum of elements of the form $g(y)h(x)\zeta(x)$, where $g(x)$ and $h(x) \in C_c(X_1)$ and $\zeta \in C_c(X_1, E)$. Moreover, the function $h(x)$ can be assumed to be supported on a fixed compact subset of X_1.

It is clear that

$$\Big(\int\limits_{y\in X_1^{s(x)}} g(y)\,\lambda^{s(x)}(dy)\Big)h(x)\zeta(x)$$

is the product of $\zeta(x)$ by an element of $C_c(X_1)$, and therefore belongs to $C_c(X_1, E)$. As a consequence, $\xi * \eta$ can be uniformly approximated by elements in $C_c(X_1, E)$.

For any $\xi \in C_c(X_1, E)$, let

$$\|\xi\|_1 = \sup_{m\in X_0} \int\limits_{X_1^m} \|\xi(x)\|\,\lambda^m(dx), \quad \text{and } \|\xi\|_I = \max(\|\xi\|_1, \|\xi^*\|_1).$$

Proposition 130. *The completion of $C_c(X_1, E)$ with respect to the norm $\|\cdot\|_I$ is a Banach $*$-algebra, and denoted by $L^1(X_1, E)$.*

By $C^*(X_1, E)$, we denote the enveloping C^*-algebra [?] of $L^1(X_1, E)$, which is called the *C^*-algebra of the field E*. In most situations, we are interested in reduced C^*-algebra, whose definition we recall below.

Let $A = C_0(X_0, E)$, the restriction of $C_0(X_1, E)$ to the unit space X_0. It is clear that A is a $C_0(X_0)$-algebra, and $A_m = E_x$ for all $m \in X_0$ according to Proposition 117. Denote by $L^2(X_1, E)$ the Hilbert A-module obtained by completing $C_c(X_1, E)$ with respect to the A-valued scalar product :

$$\langle \xi, \eta\rangle(m) = \int\limits_{y\in X_1|_m} \langle \xi(y), \eta(y)\rangle\,\lambda_m(dy) \in A_m, \quad \forall m \in X_0$$

HE FORMULA OVE ; $\lambda_m(dy)$?

Then for any $\xi \in C_c(X_1, E)$, the map

$$\pi_l(\xi)\colon \eta \mapsto \xi * \eta$$

defines an operator in $\mathcal{L}(L^2(X_1, E))$, and $\xi \mapsto \pi_l(\xi)$ extends to a representation of $L^1(X_1, E)$, called the left regular representation.

Definition 131. *Let E be a Fell bundle over a locally compact groupoid $X_1 \rightrightarrows X_0$. The reduced C^*-algebra of the field E is defined to be the completion of the image of the left regular representation*

$$\overline{\pi_l(L^1(X_1,E))} = \overline{\pi_l(C_c(X_1,E))} \subset \mathcal{L}(L^2(X_1,E)),$$

and denoted by $C_r^*(X_1,E)$.

2.1.4 Reduced C^*-algebras of S^1-central extensions of groupoids

Given an S^1-central extension of Lie groupoids $\widetilde{X}_1 \to X_1 \rightrightarrows X_0$, let $L = \widetilde{X}_1 \times_{S^1} \mathbb{C}$ be the associated complex line bundle of the S^1-principal bundle $\widetilde{X}_1 \to X_1$. Then $L \to X_1$ can be considered as a Fell bundle of C^*-algebras over the groupoid $X_1 \rightrightarrows X_0$, and therefore one can construct a C^*-algebra as in Section 2.1.3.

Definition 132. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid and $\widetilde{X}_1 \to X_1 \rightrightarrows X_0$ a Lie groupoid S^1-central extension. Then its reduced C^*-algebra $C_r^*(X_1,\widetilde{X}_1)$ is defined to be $C_r^*(X_1,L)$, where $L = \widetilde{X}_1 \times_{S^1} \mathbb{C}$ is the associated complex line bundle considered as a Fell bundle of C^*-algebras over $X_1 \rightrightarrows X_0$.*

Below we describe an equivalent definition of $C_r^*(X_1,\widetilde{X}_1)$. Consider

$$C_c(\widetilde{X}_1)^{S^1} = \{f \in C_c(\widetilde{X}_1) |\ f(\lambda\widetilde{x}) = \lambda^{-1} f(\widetilde{x}),\ \forall \lambda \in S^1,\ x_1 \in \widetilde{X}_1\}.$$

Then $C_c(\widetilde{X}_1)^{S^1}$ is stable under both the convolution and the adjoint, and the map

$$\Psi : C_c(\widetilde{X}_1)^{S^1} \;\to\; C_c(X_1,L), \tag{2.2}$$

$$f \;\to\; \widetilde{x} \mapsto [(\widetilde{x}, f(\widetilde{x}))] \in L_x = \widetilde{X}_1|_x \times_{S^1} \mathbb{C} \tag{2.3}$$

is indeed well-defined, which establishes an isomorphism of convolution algebras.

Define

$$C_r^*(\widetilde{X}_1)^{S^1} := \overline{C_c(\widetilde{X}_1)^{S^1}} \subset C_r^*(\widetilde{X}_1). \tag{2.4}$$

In other words, $C_r^*(\widetilde{X}_1)^{S^1}$ is the norm-closure of $C_c(\widetilde{X}_1)^{S^1}$ in the reduced groupoid C^*-algebra $C_r^*(\widetilde{X}_1)$

Proposition 133. *For an S^1-central extension of Lie groupoids $\widetilde{X}_1 \to X_1 \rightrightarrows X_0$,*

$$C_r^*(X_1,\widetilde{X}_1) \cong C_r^*(\widetilde{X}_1)^{S^1}. \tag{2.5}$$

Proof. Eq. (2.2) extends to an isometric isomorphism of Hilbert $C_0(X_0)$-modules :

$$L^2(\widetilde{X}_1)^{S^1} \to L^2(X_1,L). \tag{2.6}$$

For any $f \in C_c(\widetilde{X}_1)^{S^1}$, let $\eta = \Psi(f)$. Then the norm of f, as a convolution operator acting on $L^2(\widetilde{X}_1)$, is equal to the norm of η, as a convolution operator acting on $L^2(X_1,L)$.

Now we have

JL : is this correct argument ?

$$\|f\|_{C_r^*(\widetilde{X}_1)} = \sup_{\|\varphi\|_{L^2(\widetilde{X}_1)}=1} \|f * \varphi\| = \sup_{\varphi \in L^2(\widetilde{X}_1)^{S^1},\ \|\varphi\|_{L^2(\widetilde{X}_1)}=1} \|f * \varphi\| = \sup_{\|\psi\|_{L^2(X_1,L)}=1} \|\eta * \psi\|_{L^2(X_1,L)} = \|\eta\|_{C_r^*(X_1,L)}.$$

It thus follows that

$$C_r^*(X_1,L) \cong C_r^*(\widetilde{X}_1)^{S^1}. \tag{2.7}$$

□

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