# DERIVED BRACKETS AND COURANT ALGEBROIDS

Abstract.

Dedicated to Jim Stasheff on the occasion of his 65th birthday

## 1. Derived brackets

In this Section we recall the definition of *derived brackets* [12], and show that derived brackets naturally satisfy axioms of *Courant algebroid*.

1.1. **Derived brackets and the de Rham differential.** The definition of the derived bracket is motivated by the following basic example [2] due to Vinogradov. Let $P$ be a manifold, and $\mathfrak{X}(P) \cong \Gamma(TP)$ be the Lie algebra of vector fields on $P$. The idea is to extend the Lie bracket on $\mathfrak{X}(P)$ to sections of the bundle $E = TP \oplus T^*P$.

Observe that sections of $E$ act on the space of differential forms $\Omega(P)$ by exterior multiplication and by contraction, respectively,

$$(X + \xi) \cdot \mu := \iota(X)\mu + \xi \wedge \mu, \tag{1}$$

where $\xi \in \Gamma(T^*P) = \Omega^1(P)$, $X \in \Gamma(TP) = \mathfrak{X}(P)$ and $\mu \in \Omega(P)$. This action turns $\Omega(P)$ into a Clifford module of the Clifford bundle $\mathcal{C}(E)$, where $E$ is equipped with the stanadard bilinear form:

$$(X_1 + \xi_1, X_2 + \xi_2) = \frac{1}{2}(\langle \xi_1, X_2\rangle + \langle \xi_2, X_1\rangle), \tag{2}$$

and the Clifford generating relation is $xy + yx = 2(x, y)$.

Using the de-Rham differential $d : \Omega^*(P) \to \Omega^{*+1}(P)$ one can form the *derived bracket* [12] on sections of $E$:

$$e_1 \triangleleft e_2 := [[d, e_1], e_2], \quad \forall e_1, e_2 \in \Gamma(TP \oplus T^*P), \tag{3}$$

where both sides are viewed as operators on $\Omega(P)$. Moreover, a straightforward calculuation yields that

$$\begin{aligned}
&X_1 \triangleleft X_2 = [X_1, X_2] \quad \forall X_1, X_2 \in \mathfrak{X}(P)\\
&\qquad \text{(the usual Lie bracket on vector fields)}\\
&X \triangleleft \xi = L_X\xi, \quad \forall X \in \mathfrak{X}(P),\ \xi \in \Omega^1(P)\\
&\xi \triangleleft X = -L_X\xi + d\langle \xi, X\rangle, \quad \forall X \in \mathfrak{X}(P),\ \xi \in \Omega^1(P)\\
&\xi_1 \triangleleft xi_2 = 0 \quad \forall \xi_1,\ \xi_2 \in \Omega^1(P).
\end{aligned}$$

Define the (skew-symmetrized) derived bracket:

$$\{e_1, e_2\} := \frac{1}{2}(e_1 \triangleleft e_2 - e_1 \triangleleft e_2) = \frac{1}{2}([[d, e_1], e_2] - [[d, e_2], e_1]), \tag{4}$$

Then, one obtains:

$$\{X, \xi\} = L_X\xi - \frac{1}{2}d\langle \xi, X\rangle, \quad \forall X \in \mathfrak{X}(P),\ \xi \in \Omega^1(P),$$

while the bracket between one forms and between vector fields remain unchanged. This skew-symmetrized bracket coincides with the bracket obtained by Courant in his study of Dirac structures [3].

1.2. **Generating operators.** Motivated by the above example, we consider a real vector bundle $E \to P$ with fiber $V$ of dimension $2n$ equipped with a symmetric nondegenerate bilinear form $(\cdot,\cdot)$ of signature $(n,n)$. One can associate to $E$ a bundle of Clifford algebras $\mathcal{C}(E) \to P$ with fiber $\mathcal{C}(V)$, the Clifford algebra of $(V,\ (\cdot,\cdot))$, with the generating relation $xy + yx = 2(x,y),\ \forall x,\ y \in V$. Let $S \to P$ be an irreducible Clifford module, that is, a bundle vector bundle with fiber the irreducible representation of $\mathcal{C}(V)$. The Clifford action of section of $\mathcal{C}(E)$ on sections of $S$ induces an isomorphism $\Gamma(\mathcal{C}(E)) \cong \mathrm{End}(\Gamma(S))$. The natural $\mathbb{Z}_2$-grading of $\Gamma(\mathcal{C}(E))$ induces a $\mathbb{Z}_2$-grading on the operators on $S$. Functions on $P$ are even with respect to this grading, and they act on $\Gamma(S))$ by multiplication, $f : s \mapsto fs$, where $f \in C^\infty(P)$ and $s \in \Gamma(S)$. The graded commutator $[\cdot,\cdot]$ on the space of (graded) operators on $\Gamma(S)$ is defined by formula,

$$[L_1, L_2] = L_1 \circ L_2 - (-1)^{l_1 l_2} L_2 \circ L_1,$$

where $L_1$ and $L_2$ are operators of degree $l_1$ and $l_2$ respectively and $l_{1,2} = 0, 1$.

In our consideration, the key role is played by certain first order linear differential operators on $\Gamma(S)$ which we call *generating operators*.

**Definition 1.1.** A first order odd differential operator $D$ on $\Gamma(S)$ is called a *generating operator* if it satisfies the following properties,

- For any function, $f \in C^\infty(P)$,

$$[D, f] \in \Gamma(\mathcal{C}_1(E)) \cong \Gamma(E). \tag{5}$$

- For any two sections $e_1, e_2 \in \Gamma(E)$,

$$[[D, e_1], e_2] \in \Gamma(E). \tag{6}$$

- The square of the operator $D$ is a smooth function on $P$,

$$D^2 \in C^\infty(P). \tag{7}$$

**Remark 1.1.** Properties of a generating operator $D$ in Definition 1.1 resemble those of a Dirac operator on Spin-bundle of a Riemannian manifold. For instance, similar to (5), the commutator of a Dirac operator with a function on the base is given by formula $[D, f] = \mathbf{c}(df)$, where $\mathbf{c}(df)$ stands for the operator of the Clifford action of the 1-form $df$ on spinors.

**Example 1.1.** An example of a generating operator is given by the de-Rham differential $d$ as in Section 2.1, where $E = TP \oplus T^*P$ and $S \cong \wedge^* T^*P$ with the Clifford action (1). Then, $[d, f] = df \in \Gamma(E)$ and $d^2 = 0$ are obvious, and a straightforward calculation shows that indeed $[[d, e_1], e_2] \in \Gamma(E)$. In other words, $d$ is a Dirac generating operator in the sense of Definition 1.1.

Given any section $a \in \mathcal{C}(E)$ of the Clifford bundle we denote

$$Da := [D, a].$$

The following lemma lists some useful formulas.

**Lemma 1.1.**

$$D(Da) = 0, \quad \forall a \in \mathcal{C}(E); \tag{8}$$

$$D[e_1, e_2] = [De_1, e_2] - [e_1, De_2], \quad \forall e_1, e_2 \in \Gamma(E); \tag{9}$$

$$[Df, e] = [De, f], \quad \forall f \in C^\infty(P), e \in \Gamma(E). \tag{10}$$

*Proof.* Note that $D(Da) = [D, [D, a]] = \frac{1}{2}[D^2, a] = 0, \forall a \in \mathcal{C}(E)$, which implies Equation (8). Equation (9) follows from the graded Jacobi identities for the operators $e_1, e_2$ and $d$. For Equation (10), we note that $[f, e] = 0$ and, hence, $0 = D[f, e] = [Df, e] + [f, De]$. □

Given a bundle $E$ and a generating operator $D$ one can construct a derived bracket on sections of $E$ [12] and an bundle map $\rho : E \to TP$.

**Definition 1.2.** [12] The derived bracket $\{\cdot, \cdot\tilde{\}}$ on sections of $E$ associated to a generating operator $D$ is defined by formula,

$$e_1 \triangleleft e_2 \tilde{=} [De_1, e_2], \tag{11}$$

the map $\rho : E \to TP$ is given by,

$$\rho(e)f = 2(Df, e) = [Df, \ e], \tag{12}$$

where $e \in \Gamma(E)$.

Definition **??** is inspired by the example of Section 1.1. By Equation (10), the anchor map can be alternatively written as $\rho(e)f = [De, f]$.

1.3. **Derived brackets.** The main result of this Section is the following:

**Theorem 1.3.** *The vector bundle $E$ equipped with the scaler product $(\cdot, \cdot)$, the anchor map $\rho$ as in Equation* (12) *and the bracket on section of $\Gamma(E)$ as in Equation* (11) *satisfy the following properties:*

- *For any $e_1, e_2, e_3 \in \Gamma(E)$,*

$$e \triangleleft (e_1 \triangleleft e_2) = (e \triangleleft e_1) \triangleleft e_2 + e_1 \triangleleft (e \triangleleft e_2); \tag{13}$$

- *for any $e_1, e_2 \in \Gamma(E)$,*

$$\rho(e_1 \triangleleft e_2) = \{\rho e_1, \rho e_2\}; \tag{14}$$

- *for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(P)$,*

$$e_1 \triangleleft (f e_2) = f(e_1 \triangleleft e_2) + (\rho(e_1)f)e_2; \tag{15}$$

- *for any $e \in \Gamma(E)$,*

$$e \triangleleft e = D(e, e); \tag{16}$$

- *for any $e, e_1, e_2 \in \Gamma(E)$,*

$$\rho(e)(e_1, e_2) = (e \triangleleft e_1, e_2) + (e_1, e \triangleleft e_2). \tag{17}$$

*Proof.* We start with (13). The left hand side reads,

$$e \triangleleft (e_1 \triangleleft e_2) = [De, [De_1, e_2]].$$

The first term on the right hand side can be written as,

$$(e \triangleleft e_1) \triangleleft e_2 = [D[De, e_1], e_2] = [[De, De_1], e_2],$$

and for the second term we get,

$$e_1 \triangleleft (e \triangleleft e_2) = [De_1, [De, e_2]].$$

Equation (13) follows by the graded Jacobi identity.

Next we turn to the properties of the map $\rho$,

$$\rho(e_1)\rho(e_2)f = [De_1, \rho(e_2)f] = [De_1, [Df, e_2]],$$

and for the commutator of $\rho(e_1)$ and $\rho(e_2)$ we obtain,

$$\begin{aligned} [\rho(e_1), \rho(e_2)]f &= [De_1, [Df, e_2]] - [De_2, [Df, e_1]] \\ &= [De_1, [Df, e_2]] + [[[Df, De_1], e_2] = [Df, [De_1, e_2]] = \rho(e_1 \triangleleft e_2)f. \end{aligned}$$

Here we have used the graded Jacobi identity and equation (10).

Equation (15) follows by the following calculation,

$$e_1 \triangleleft (fe_2) = [De_1, fe_2]f[De_1, e_2] + [De_1, f]e_2 = f(e_1 \triangleleft e_2) + (\rho(e_1)f)e_2.$$

Equation (16) follows by

$$e \triangleleft e = [De, e] = \frac{1}{2}D[e, e] = D(e, e).$$

Finally, using Equation (**??**) and (**??**), we obtain,

$$\rho(e)(e_1, e_2) = \frac{1}{2}[De, [e_1, e_2]] = \frac{1}{2}([[De, e_1], e_2] + [e_1, [De, e_2]]) = (e \triangleleft e_1, e_2) + (e_1, e \triangleleft e_2)$$

which coincides with (17). This completes the proof of the Theorem. □

## 2. Courant algebroids

2.1. **Courant algebroids.** According to Theorem 1.3, the derived bracket is a Loday (also called Lebiniz) bracket in terms of [12].

Now consider the skew symmetrized derived bracket

$$\{e_1, e_2\} = \frac{1}{2}(\{e_1, e_2\tilde{\}} - \{e_1, e_2\tilde{\}}) = \frac{1}{2}([De_1, e_2] - [De_2, e_1]). \tag{18}$$

According to Proposition 2.6.5 [17], Theorem 1.3 implies that $E$ is indeed a Courant algebroid.

Let us recall the definition of Courant algebroids below.

**Definition 2.1.** [14] *A* Courant algebroid *is a vector bundle $E \longrightarrow P$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ of signature $(n, n)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$ and a bundle map $\rho : E \longrightarrow TP$ satisfying the following properties:*

- *For any $e_1, e_2, e_3 \in \Gamma(E)$,*

$$\mathrm{Cycl}_{123}\{\{e_1, e_2\}, e_3\} = DT(e_1, e_2, e_3); \tag{19}$$

- *for any $e_1, e_2 \in \Gamma(E)$,*

$$\rho\{e_1, e_2\} = [\rho e_1, \rho e_2]; \tag{20}$$

- *for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(P)$,*

$$\{e_1, fe_2\} = f\{e_1, e_2\} + (\rho(e_1)f)e_2 - (e_1, e_2)Df; \tag{21}$$

- *for any* $f, g \in C^\infty(P)$, $(Df, Dg) = 0$, *i.e.*,

$$\rho \circ D = 0; \tag{22}$$

- *for any* $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)(e_1, e_2) = (\{e, e_1\} + D(e, e_1), e_2) + (e_1, \{e, e_2\} + D(e, e_2)), \tag{23}$$

*where* $T(e_1, e_2, e_3) \in C^\infty(P)$ *is defined by:*

$$T(e_1, e_2, e_3) = \frac{1}{3}\operatorname{Cycl}_{123}(\{e_1, e_2\}, e_3), \tag{24}$$

*and* $D : C^\infty(P) \longrightarrow \Gamma(E)$ *is the map defined by*

$$(Df, e) = \frac{1}{2}\rho(e)f. \tag{25}$$

**Remark 2.2.** • The following equivalent formula for $T$ will be useful (see Equation (11) in [14]) in our later discussion.

$$T(e_1, e_2, e_3) = ([e_1, e_2], e_3) - \frac{1}{2}\rho(e_1)(e_2, e_3) + \frac{1}{2}\rho(e_2)(e_3, e_1). \tag{26}$$

- Consider the unskew-symmetrized version:

$$\widetilde{\{e_1, e_2\}} = \{e_1, e_2\} + D(e_1, e_2).$$

Then the Courant algebroid axioms are equivalent to Equations (13)-(17). More precisely, Equations (13), (15) and (17) are equivalent to Equations (19), (21) and (23) respectively, and Equation (14) is equivalent to the combination of Equations (20) and (22).

As a consequence, Theorem 1.3 implies the following

**Corollary 2.3.** *For any Dirac generating operator, the vector bundle $E$ equipped with the scaler product $(\cdot, \cdot)$, the ancor map $\rho$ as in Equation* (12) *and the skew-symmetrized derived bracket on $\Gamma(E)$ as in Equation* (18) *becomes a Courant algebroid.*

2.2. **Examples.** Next we will give a few interesting examples of Dirac generating operators and their induced Courant algebroids.

The simplest one is the following:

**Example 2.1.** Let $\mathfrak{d}$ be a Lie algebra with a non-degenerate ad-invariant symmetric pairing $(\cdot, \cdot)$. Take $D = -\frac{1}{24}\sum f_{ijk}e^i e^j e^k$, where $\{e_1, \cdots, e_n\}$ is a basis of $\mathfrak{d}$ with dual basis $\{e^1, \cdots, e^n\}$, and $f_{ijk} = ([e_i, e_j], e_k)$. Indeed, if $C \in \wedge^3\mathfrak{d}^*$ denotes the Cartan element $C(u, v, w) = ([u, v], w)$, $\forall u, v, w \in \mathfrak{d}$, then $D = -C$ under the standard identification $\mathrm{Cliff}\mathfrak{d} \cong \wedge^*\mathfrak{d}$. Hence, for any $u, v, w \in \mathfrak{d}$, $([[D, u], v], w) = \langle \iota_v \iota_u C, w\rangle = C(u, v, w) = ([u, v], w)$. Thus it follows that $[[D, u], v] = [u, v]$. A simple calculation yields that $D^2 = ?f_{ijk}f^{ijk}$. Hence $D$ is indeed a Dirac generating operator in our sense, whose induced bracket coincides with the orginal one we start with. This is of course what one expects.

In particular, if $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is the double of a Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$, and $\wedge\mathfrak{g}^*$ is taken as the spinor of $\mathrm{Cliff}\mathfrak{d}$, it is simple to see that $D$ admits the following

$$D = d + \partial + \frac{1}{2}(\xi_0 + X_0).$$

Here $d : \wedge^*\mathfrak{g}^* \longrightarrow \wedge^{*+1}\mathfrak{g}^*$ is the Lie algebra cohomology differential of $\mathfrak{g}$, $\partial : \wedge^*\mathfrak{g}^* \longrightarrow \wedge^{*-1}\mathfrak{g}^*$ is the Lie algebra homology differential of $\mathfrak{g}^*$, and $\xi_0$, $X_0$ are the modular vectors of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively.

More generally, if $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{s}$ is a Manin pair, and $S(\mathfrak{d}) \cong \wedge\mathfrak{s}$,

$$D = d + \partial + \frac{1}{2}(\xi_0 + X_0) - \frac{1}{24}\phi_{ijk}v^i v^j v^k,$$

where $\phi_{ijk}$ denotes the Drinfeld 3-cocycles [?] $\phi_{ijk} =$.

**Example 2.2.** Consider again the vector bundle $E = TP \oplus T^*P$ with the standard bilinear form (2). Then as in Section 1.1, $\Omega(P)$ becomes a Clifford module of Cliff$E$ with the action defined by Equation (1).

Let $\pi \in \Gamma(\wedge^2 TP)$ be a Poisson tensor. In terms of the graph $L$ of $\pi^\# : T^*P \longrightarrow TP$, one obtains another decomposition $E \cong L \oplus TP$. Thus $\Gamma(\wedge^* L)$ can also be considered as a Clifford module of $\Gamma$Cliff$E$, where the action is essentially the same as described by Equation (1) with $L$ being identified with $T^*P$ via the standard bilinear form (2). On the other hand, there is a natural vector bundle isomorphism between $L$ and $T^*P$:

$$T^*P \longrightarrow L, \quad \xi \longrightarrow \xi + \pi^\#\xi, \quad \forall \xi \in T^*P.$$

Hence, $\Gamma(\wedge^* L) \cong \Omega(P)$ and the Clifford action is given by

$$(X + \xi) \cdot \mu := \iota(X + \pi^\#\xi)\mu + \xi \wedge \mu, \tag{27}$$

where $\xi \in \Omega^1(P)$, $X \in \mathfrak{X}(P)$ and $\mu \in \Omega(P)$. It thus follows that these two Clifford modules ((1) and (27)) are isomorphic and the isomorphism is established by $\iota(\exp\pi) :\ \Omega(P) \longrightarrow \Omega(P)$:

$$\iota(\exp\pi)(\mu) = \mu + \iota(\pi)\mu + \frac{1}{2!}\iota(\pi \wedge \pi)\mu + \cdots, \quad \forall \mu \in \Omega(P).$$

As a consequence, in terms of the standard action (1), we have, for any $e \in \Gamma(E) \cong \mathrm{Cliff}_1(E)$,

$$\iota(\exp\pi)\circ e \circ \iota(\exp\pi)^{-1} = \phi(e), \tag{28}$$

where $\phi(e) = (X + \pi^\#\xi) + \xi$ if $e = X + \xi$.

On the other hand, a simple calculation yields [6] that

$$\iota(\exp\pi)\circ d \circ \iota(\exp\pi)^{-1} = d + \partial, \tag{29}$$

where $\partial = [\iota(\pi),\ d]$ is the Koszul-Brylinski differential.

Let

$$D = d + \partial.$$

Thus, we have

$$[D, f] = \iota(\exp\pi)\circ[d, f]\circ\iota(\exp\pi)^{-1} = \phi(df) = X_f + df,$$

$D^2 = 0$ and

$$\begin{aligned} & \{e_1, e_2\tilde{\}}_D \\ = & [[D, e_1], e_2] \\ = & [[\iota(\exp\pi)\circ d\circ\iota(\exp\pi)^{-1}, e_1], e_2] \\ = & \iota(\exp\pi)\circ[[d, \phi^{-1}e_1], \phi^{-1}e_2]\circ\iota(\exp\pi)^{-1} \\ = & \phi\{\phi^{-1}e_1, \phi^{-1}e_2\tilde{\}}_d \end{aligned}$$

Thus $D = d+\partial$ is also a Dirac generating operator, whose induced Courant algebroid is isomorphic to the standard one as in Section 1.1 with $\phi^{-1}$ being the isomorphism. Note that this Courant algebroid is also the double of the Lie bialgebroid $(TP, T^*P)$ associated to the Poisson manifold $P$ (see Section 2.3).

**Example 2.3.** Let $E = TP \oplus T^*P$ be equipped with the standard bilinear form on the fibers, and Cliff$E$ and $\Gamma(S(E)) \cong \Omega(P)$ as in Section 1.1. Now we will consider another differential operator $D = d + \Omega$ on $\Omega(P)$ for some $\Omega \in \Omega^3(P)$:

$$D\mu = d\mu + \Omega \wedge \mu, \quad \forall \mu \in \Omega(P).$$

It is clear that $D^2 = d\Omega$. In order for $D^2 \in C^\infty(P)$, it is necessary that $d\Omega = 0$, i.e., $\Omega$ is a closed 3-form. It is also trivial to see that $[D, f] = df \in \Gamma(E)$ and $[[D, e_1], e_2] \in \Gamma(E)$. Therefore, $D$ is indeed a Dirac generating operator. The induced bracket on $\Gamma(E)$ is in fact similar to the one described in Section 1.1 with some modification for the bracket between vector fields in $\mathfrak{X}(P)$. More precisely, the bracket takes the following form:

(30)

$$\{X_1+\xi_1, X_2+\xi_2\} = [X_1, X_2]+\{L_{X_1}\xi_2-L_{X_2}\xi_1+\frac{1}{2}d(\langle\xi_1, X_2\rangle-\langle\xi_2, X_1\rangle)-\iota(X_1\wedge X_2)\Omega\},$$

while the anchor $\rho : TP \oplus T^*P \longrightarrow TP$ is the projection map to the first component. This type of Courant algebroids was due to Severa [18], called exact Courant algebroids. More precisely, an *exact Courant algebroid* is a Courant algebroid on $E = TP \oplus T^*P$ with the standard bilinear form (2) and the anchor $\rho(X + \xi) = X$[1].

In fact, it is simple to see, by using the properties of Courant algebroids, that the bracket of any exact Courant algebroid on $E$ is of the form (30), where the three form $\Omega$ can be recovered by the following formula:

$$\Omega(X, Y, Z) = -2(\{X, Y\},\ Z), \quad \forall X, Y, Z \in \mathfrak{X}(P). \tag{32}$$

Hence, one obtains a one-one correspondence between exact Courant algebroids and closed 3-forms on $P$.

[1]This definition can be rephrased in a more intrinsic way. Namely, a Courant algebroid $E$ with anchor $\rho$ is *exact*, if the sequence

$$0 \longrightarrow T^*P \xrightarrow{\rho^*} E \xrightarrow{\rho} TP \longrightarrow 0 \tag{31}$$

is exact, where $\rho^* : T^*P \longrightarrow E$ is defined by $(\rho^*\xi, e) = \langle\xi, \rho e\rangle$, $\forall\xi \in T^*P$ and $e \in \Gamma(E)$. Such a vector bundle $E$ is always isomorphic, as a vector bundle with Riemanian metric in fibers, to $TP \oplus T^*P$ by choosing an isotropic complement of $T^*P \cong \mathrm{Ker}\rho$.

Now assume that $\Omega_1,\ \Omega_2 \in Z^3(P)$ are cohomologous, i.e., $\Omega_1 = \Omega_2 + d\theta$ for $\theta \in \Omega^2(P)$. Consider the spinsor bundle isomorphism $\wedge^* T^*P \longrightarrow \wedge^* T^*P$ induced by the Clifford action of $e^\theta \in \Omega(P)$. Namely, on $\Omega(P)$:

$$e^\theta \cdot \mu = \mu + \theta \wedge \mu + \frac{1}{2!}\theta \wedge \theta \wedge \mu + \cdots, \quad \forall \mu \in \Omega(P). \tag{33}$$

¿From the relation $e^{-\theta} \circ d \circ e^\theta = d + d\theta$, it follows that

$$D_1 = e^{-\theta} \circ D_2 \circ e^\theta.$$

It is also simple to see that

$$e^{-\theta} \circ e \circ e^\theta = \phi(e), \quad \forall e \in \Gamma(E),$$

where

$$\phi(e) = X + (\iota(X)\theta + \xi), \quad \text{if } \ e = X + \xi. \tag{34}$$

Thus it follows that $\{e_1, e_2\}_1 = \phi\{\phi^{-1}e_1, \phi^{-1}e_2\}_2$, $\forall e_1, e_2 \in \Gamma(E)$, where $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are the induced derived brackets on $\Gamma(E)$ corresponding to $D_1$ and $D_2$, respectively. As a consequence, the induced Courant algebroids for $D_1$ and $D_2$ are isomorphic.

Conversely, it is simple to see, using the properties of Courant algebroids, that any two exact isomorphic Courant algebroid structures on $E$ must be related by an isomorphism $\phi : E \longrightarrow E$ of the form (34) for some two form $\theta \in \Omega^2(P)$. Therefore, their corresponding closed three forms on $P$ must be cohomologous. Thus, we recover the following theorem of Severa.

**Theorem 2.4.** [18] *The isomorphic classes of exact Courant algebroids are parametrized by $H^3(P)$.*

The corresponding class in $H^3(P)$ is called the *Severa characteristic class* of the Courant algebroid.

2.3. **Double of Lie bialgebroids.** A large class of Courant algebroids arise as the double of Lie bialgebroids. The notion of Lie bialgebroids is a natural generalization of that of Lie bialgebras. Roughly speaking, a Lie bialgebroid is a pair of Lie algebroids $(A,\ A^*)$ satisfying a certain compatibility condition. We quote here its definition below [16] [10].

**Definition 2.5.** *A* Lie bialgebroid *is a dual pair $(A, A^*)$ of vector bundles equipped with Lie algebroid structures such that the differential $d_*$ on $\Gamma(\wedge^* A)$ coming from the structure on $A^*$ is a derivation of the Schouten-type bracket on $\Gamma(\wedge^* A)$ obtained by extension of the structure on $A$. Equivalently, $d_*$ is a derivation for sections of $A$, i.e.,*

$$d_*[X,Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A). \tag{35}$$

Given a Lie bialgebroid $(A,\ A^*)$ over the base $P$, with anchors $a$ and $a_*$ respectively, let $E$ denote their vector bundle direct sum $E := A \oplus A^*$. On $E$, there exists a natural non-degenerate symmetric bilinear form:

$$(X_1 + \xi_1, X_2 + \xi_2) = \frac{1}{2}(\langle \xi_1 X_2\rangle + \langle \xi_2, X_1\rangle), \quad \forall X_1, X_2 \in \Gamma(A), \quad \xi_1, \xi_2 \in \Gamma(A^*). \tag{36}$$

Introduce a bracket on $\Gamma(E)$ by

$$\begin{aligned}[e_1, e_2] &= \{[X_1, X_2] + L_{\xi_1} X_2 - L_{\xi_2} X_1 - \tfrac{1}{2} d_*(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle)\} \\ &\quad + \{[\xi_1, \xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + \tfrac{1}{2} d(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle)\},\end{aligned} \tag{37}$$

where $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$. Let $\rho : E \longrightarrow TP$ be the bundle map $\rho = a + a_*$. That is,

$$\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A) \text{ and } \xi \in \Gamma(A^*). \tag{38}$$

The following result was proved in [14].

**Theorem 2.6.** *If $(A, A^*)$ is a Lie bialgebroid, then $E = A \oplus A^*$ together with $([\cdot,\cdot], \rho, (\cdot,\cdot))$ is a Courant algebroid. In this case, $Df = d_* f + df$, $\forall f \in C^\infty(P)$.*

Such a Courant algebroid $E$ is called the *double of a Lie bialgebroid* $(A, A^*)$. For a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the bracket (37) reduces to the well known Lie bracket on the double $\mathfrak{g} \oplus \mathfrak{g}^*$. On the other hand, if $A$ is the tangent bundle Lie algebroid $TM$ and $A^* = T^*M$ with zero bracket, then Equation (37) takes the form:

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \{L_{X_1}\xi_2 - L_{X_2}\xi_1 + \frac{1}{2} d(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle)\}. \tag{39}$$

This is exactly the one discussed in Section 1.1.

2.4. **Courant algebroids associated to Manin pairs.** Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair [**?**]. Namely, $\mathfrak{d}$ is a Lie algebra with an invariant symmetric non-degenerate pairing $(\cdot,\cdot)$ of signature $(n, n)$ and $\mathfrak{g}$ is an isotropic subalgebra. Suppose that $D$ is a Lie group with Lie algebra $\mathfrak{d}$ and $G \subset D$ is a Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g}$. Consider the trivial vector bundle

$$E = \mathfrak{d} \times D/G \longrightarrow D/G,$$

where $D/G$ is the standard homogeneous space obtained by taking the orbit space of the right $G$-action on $D$. Equip $E$ with a fiberwise bilinear form $(\cdot,\cdot)$. Now $D$ acts on $D/G$ from the left, and by $\mathfrak{d} \longrightarrow \mathfrak{X}(D/G)$, $\; e \longrightarrow \hat{e}$ we denote its induced infinitesimal action. Define an anchor map $\rho : E \longrightarrow T(D/G)$ by

$$\rho(e, x) = \hat{e}_x, \quad \forall (e, x) \in \mathfrak{d} \times D/G. \tag{40}$$

Sections of $E$ are naturally identified with $\mathfrak{d}$-valued functions on $D/G$. The Lie bracket on $\mathfrak{d}$ thus induces a bracket between constant $\mathfrak{d}$-valued functions. Extend this bracket to non-constant functions using Equation (21).

**Proposition 2.7.** *The above bracket on $\Gamma(E)$ is well-defined. In fact, this bracket together with the anchor map (40) and the bilinear form $(\cdot,\cdot)$ makes $E$ into an exact Courant algebroid. Moreover, the Courant algebroid structure on $E$ is invariant under the group $D$-action:*

$$d \cdot (e, x) = (Ad_d e, d \cdot x) \quad \forall d \in D,\ (e, x) \in (e, x) \in \mathfrak{d} \times D/G. \tag{41}$$

*Proof.* Since $E$ is a trivial bundle, $\mathcal{C}E \cong \mathcal{C}\mathfrak{d} \times D/G$ and $S(E) \cong S\mathfrak{d} \times D/G$. Consider the twisted spinor bundle $\tilde{S} = (S\mathfrak{d} \times D/G) \otimes (\wedge^{top} T^*(D/G))^{\frac{1}{2}}$, and the operator

$$D := -\frac{1}{2} e^i L_{\hat{e}_i} - \frac{1}{24} C_{ijk} e^i e^j e^k. \tag{42}$$

Then, $[D,f] = -\frac{1}{2}e^i[L_{\hat{e}_i}, f] = -\frac{1}{2}(\hat{e}_i f)e^i$. Hence, $[D,f] \in \Gamma(E)$, and the corresponding anchor map $\rho(e) = -\hat{e}$.

For any constant sections $u, v$ of $E$,

$$[[D,u],v] = -\frac{1}{24}[[C_{ijk}e^ie^je^ku],v] = [u,v].$$

This implies that $[[D,e_1],e_2] \in \Gamma(E)$ for any two sections $e_1, e_2$ according to Lemma ?.

Next,

$$\begin{array}{rcl} D^2 & = & (-\frac{1}{2}e^iL_{\hat{e}_i} - \frac{1}{24}C_{ijk}e^ie^je^k)^2 \\ & = & \frac{1}{4}(e^i,e^j)L_{\hat{e}_i}L_{\hat{e}_j} + \frac{1}{8}e^ie^j[L_{\hat{e}_i}, L_{\hat{e}_j}] - \frac{1}{8}C_{ijk}e^ie^jL_{\hat{e}_k} + \frac{1}{576}C_{ijk}C^{ijk}. \end{array}$$

Now

$$[L_{\hat{e}_i}, L_{\hat{e}_j}] = L_{\hat{[e_i,e_j]}} = C_{ijk}L_{\hat{e}_k},$$

and

$$\begin{array}{rcl} (e^i,e^j)\nabla_{e_i}\nabla_{e_j} & = & (e^i,e^j)(\rho_i^a\partial_a + \frac{1}{2}(\partial_a\rho_i^a))(\rho_j^b\partial_b + \frac{1}{2}(\partial_b\rho_j^b)) \\ & = & (e^i,e^j)(\rho_i^a\rho_j^b\partial_a\partial_b + \partial_a(\rho_i^a\rho_j^b)\partial_b + \frac{1}{2}(\rho_i^a\partial_a\partial_b\rho_j^b) + \frac{1}{4}(\partial_a\rho_i^a)^2) = \frac{1}{2}(e^i,e^j)(\rho_i^a\partial_a\partial_b\rho_j^b) + \frac{1}{4} \end{array}$$

Hence $D^2$ is indeed a function on the base.

Finally, note that $D$ interchange with the natural $D$ action on $\tilde{S}$. Hence the induced Courant algebroid is indeed invariant under the $D$-action. ☐

Next we will investigate some special cases of this construction of Courant algebroids.

Assume that $\mathfrak{g}$ admits an isotropic complement $\mathfrak{m}$ which is stable under the adjoint action of $\mathfrak{g}$, i.e., $[\mathfrak{g},\ \mathfrak{m}] \subset \mathfrak{m}$. This decomposition defines a (left)-$D$ invariant connection on the principal bundle $G \longrightarrow D \longrightarrow D/G$ [?]. In other words, one obtains a $D$-equivariant bundle map $\sigma : T(D/G) \longrightarrow (TD)/G$, where $D$ acts on $TD$ by lifting the left translation which commutes with the lifted right $G$ action, hence it descends to an action on the quotient space $(TD)/G$. Trivialize $TD \cong \mathfrak{d} \times D$ by right translation, which induces a vector bundle isomorphism $(TD)/G \longrightarrow \mathfrak{d} \times D/G (\cong E$ given by $[e,x] \longrightarrow (e,[x])$, where $e \in \mathfrak{d}$ and $x \in D$. It is simple to see that, under this identification, the left $D$-action on $(TD)/G$ goes to the $D$-action defined by Equation (41). Thus we obtain an $D$-equivariant bundle map

$$\sigma : T(D/G) \longrightarrow E$$

such that $\rho \circ \sigma = id$. Since $\mathfrak{m}$ is assumed to be isotropic, hence the image of $\sigma$ is isotropic in $E$. In other words, we obtain an $D$-equivariant isotropic spiltting of the exact sequence

$$0 \longrightarrow T^*(D/G) \xrightarrow{\rho^*} E \xrightarrow{\rho} T(D/G) \longrightarrow 0. \tag{43}$$

According to Example (**??**), $E$ can be identified with a standard Courant algebroid $T(D/G) \oplus T^*(D/G)$ with the 3-form $\Omega \in \Omega^3(D/G)$ defined by Equation (32). Since $\sigma$ is $D$-equivariant and the Courant algebroid $E$ is $D$-invariant, it is simple to see that $\Omega$ is indeed $D$-invariant, i.e., $\Omega \in \Omega^3(D/G)^D \cong \wedge^3(\mathfrak{d}/frakg)^*) \cong \wedge^3\mathfrak{g}$. Under this identification, Equation (32) implies that $\Omega = -2\phi$, where $\phi \in \wedge^3\mathfrak{g}$ is the Drinfeld 3-cocycle corresponding to the splitting $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{m}$. Note that $\phi$ is in general a Lie algebra $\mathfrak{g}$-cocycle, but our assumption that $[\mathfrak{g},\mathfrak{m}] \subset \mathfrak{m}$ assurs that it is also a relative 3-cocycle of the Lie algebra pair $(\mathfrak{d},\mathfrak{g})$, which is responsable for $\Omega$ being a

$D$-invrainat closed 3-form on $D/G$. Alternatively, our construction together with Equation (32) shows that $\phi$ is essentially the curvature tensor of the principle bundle $D \longrightarrow D/G$, under the identification $\Omega^2(D/G)^D \otimes \mathfrak{g} \cong \wedge^2(\mathfrak{d}/\mathfrak{g})^* \otimes \mathfrak{g} \cong \wedge^2\mathfrak{g} \otimes \mathfrak{g}$.

**Example 2.4.** Let $(\mathfrak{g}\oplus\mathfrak{g}^*, \mathfrak{g})$ be the standard Manin pair, and let $\mathfrak{m} = \mathfrak{g}^*$. It is clear that all the assumptions are satisfied and $\phi = 0$ since $\mathfrak{m}$ is a Lie subalgebra. In this case $D = G \times \mathfrak{g}^*$ is the semi-direct product, and $D/G \cong \mathfrak{g}^*$. Thus $E$ is isomorphic to the standard Courant algebroid $T\mathfrak{g}^* \oplus T^*\mathfrak{g}^*$, the double of the Lie bialgebroid corresponding to the zero Poisson $\mathfrak{g}^*$.

**Example 2.5.** Assume that $\mathfrak{g}$ admits an invariant, nondegenerate symmetric bilinear form $Q$. Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, the direct sum of Lie algebra $\mathfrak{g}$. Define a bilinear form on $\mathfrak{d}$ by

$$(\xi_1 + \eta_1, \xi_2 + \eta_2) = -\frac{1}{4}(Q(\xi_1, \xi_2) - Q(\eta_1, \eta_2)). \tag{44}$$

Embed $\mathfrak{g} \longrightarrow \mathfrak{d}$ by the diagonal embedding: $\mathfrak{g}_{diag} = \{(\xi, \xi)|\xi \in \mathfrak{g}\}$ and let $\mathfrak{m} = \{(\xi, -xi)|\xi \in \mathfrak{g}\}$. It is clear that $\mathfrak{d} = \mathfrak{g}_{diag} \oplus \mathfrak{m}$, $\mathfrak{g}_{diag}$ is a Lie subalgebra, and $[\mathfrak{g}_{diag}, \mathfrak{m}] \subset \mathfrak{m}$. Therefore we can apply the above construction to our situation so that the Courant algebroid $E$ is isomorphic to an exact Courant algebroid $T(D/G)\oplus T^*(D/G)$. Now $D = G \times G$ and we can identify $D/G$ with $G$ by the following map

$$\phi : [x, y] \longrightarrow xy^{-1}, \quad \forall x,\ y \in G.$$

Under this identification, the left-$D$ action on $D/G$ becomes $d \cdot g = d_1 g d_2^{-1}$, $\forall d = (d_1, d_2) \in G \times G$ and $g \in G$ since $\phi((d_1, d_2) \cdot [x, y]) = \phi([d_1 x, d_2 y]) = d_1\phi([x, y])d_2^{-1}$. In particular, the induced $G$ actions on $G$ by embedding $G$ as a subgroup of $D$ via $G \times \{1\}$, $\{1\} \times G$, and the diagonal, correspond to the left translation, the right translation, and the adjoint action, respectively. A $D$-invariant closed 3-form on $D/G$ corresponds to, via $\phi$, a biinvariant closed 3-form on $G$. A simple calculation shows that $\Omega$ is indeed the Cartan form corresponding to $Q$.

Finally, we have

**Proposition 2.8.** *Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ be the double of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Assume that the corresponding Poisson groups $G$ and $G^*$ are complete so that $D = G \times G^*$ as a manifold. Then $E$ is isomorphic to the Courant algebroid $TG^* \oplus T^*G^*$ assocaited to the Poisson manifold $G^*$.*

*Proof.* Identifying $TG^* \oplus T^*G^*$ with the trivial vector bundle $(\mathfrak{g} \oplus \mathfrak{g}^*) \times G^*$ by left translations, one obtains a vector bundle isomorphism $\phi : TG^* \oplus T^*G^* \cong E$. It is clear that this identification preserves the bilinear forms on the vector bundles. By the definition of dressing action [18], one also sees that $\phi$ interchanges the anchor maps.

To show that $\phi$ is a Courant algebroid isomorphism, it suffices to verify that $\phi$ preserves the Courant bracket for invariant sections. To this purpose, for any $e_1,\ e_2 \in \mathfrak{g} \oplus \mathfrak{g}^*$, let $\bar{e_1}$ and $\bar{e_2}$ be their corresponding left invariant sections in $TG \oplus T^*G$. Assume that $e_1 = X + \xi$, $e_2 = Y + \eta \in \mathfrak{g} \oplus \mathfrak{g}^*$, and $\bar{e_1} = \bar{X} + \bar{\xi}$, $\bar{e_2} = \bar{Y} + \bar{\eta}$. Here $\bar{X}$, $\bar{Y}$ and $\bar{\xi}$, $\bar{\eta}$ denote the corresponding left invariant vector fields and 1-forms on $G$ for $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ respectively.From the fact that

$[\bar{X}, \bar{Y}] = [X, Y]^-$ and $[\bar{\xi}, \bar{\eta}] = [\xi, \eta]^-$ (see [?]), it follows that

$$L_{\bar{X}}\bar{\xi} = (ad^*_X\xi)^-, L_{\bar{\xi}}\bar{X} = (ad^*_\xi X)^-.$$

This means that

$$\begin{aligned}[\bar{X},\ \bar{\xi}] &= L_{\bar{X}}\bar{\xi} - L_{\bar{\xi}}\bar{X} + \frac{1}{2}(d^* - d) < \bar{X}, \bar{\xi} > \\ &= (ad^*_X\xi)^- - (ad^*_\xi X)^- \\ &= [X, \xi]^-.\end{aligned}$$

Therefore,

$$\begin{aligned}[\bar{e}_1,\ \bar{e}_2] &= [\bar{X}, \bar{Y}] + [\bar{X}, \bar{\eta}] + [\bar{\xi}, \bar{Y}] + [\bar{\xi}, \bar{\eta}] \\ &= [X, Y]^- + [X, \eta]^- + [\xi, Y]^- + [\xi, \eta]^- \\ &= [e_1, e_2]^-.\end{aligned}$$

This concludes the proof. □

## 3. Connections, divergence and torsion

The purpose of this Section is to extend the notions of connection, divegence and torsion to Courant algebroids.

### 3.1. $E$-connections.

Let $E$ be a Courant algebroid over $P$, $\rho$ be its anchor map, and $V \to P$ be a vector bundle. In analogy to ordinary connections we define a notion of an $E$-connection.

**Definition 3.1.** A linear map $\nabla : \Gamma(E \otimes V) \to \Gamma(V)$ is called an $E$-connection on $V$ if it satisfies the following porperties,

$$\nabla_{fe}s = f\nabla_e s, \tag{45}$$

$$\nabla_e(fs) = f\nabla_e s + (\rho(e)f)s. \tag{46}$$

Here $e \in \Gamma(E), s \in \Gamma(V)$ and $f \in C^\infty(P)$.

Given an ordinary connection $\nabla$ on $V$ one can define an $E$-connection by formula

$$\nabla_e s := \nabla_{\rho(e)} s. \tag{47}$$

Let $\nabla^E$ be a linear connection on the Courant algebroid $E$ preserving the bilinear form $(\cdot, \cdot)$. That is,

$$X(e_1, e_2) = (\nabla^E_X e_1, e_2) + (e_1, \nabla^E_X e_2).$$

By formula (47) an ordinary connection on $E$ preserving the bilinear form on fibers defines an $E$-connection which will be denoted by the same letter $\nabla^E$. For any $e \in \Gamma(E)$, we define a function $div_\nabla e \in C^\infty(P)$ called the divegence of $e$,

$$div_\nabla e = \sum_i (\nabla^E_{e_i} e,\ e^i), \tag{48}$$

where $\{e_1, \cdots, e_n\}$ is a local basis of $\Gamma(E)$, and $\{e^1, \cdots, e^n\}$ its dual basis.

**Lemma 3.1.** *For any $f \in C^\infty(P)$ and $e \in \Gamma(E)$,*

$$div_\nabla(fe) = f div_\nabla e + \rho(e)f.$$

*Proof.* The proof is by direct computation,

$$div_{\nabla}(fe) = \sum_i (\nabla^E_{e_i}(fe),\ e^i) = f div_{\nabla} e + \sum_i (e, e^i)\rho(e_i) f = f div_{\nabla} e + \rho(e) f.$$

□

In the following we shall need an $E$-connection on the line bundle $\Lambda := \wedge^{top} T^*P$.

**Lemma 3.2.** *Formula*

$$\nabla^{\Lambda}_e s := L_{\rho(e)} s - (div_{\nabla} e) s \tag{49}$$

*defines an $E$-connection on $\Lambda$.*

*Proof.* Using Lemma 3.1 we obtain,

$$\nabla^{\Lambda}_{fe} s = L_{f\rho(e)} s - (div_{\nabla}(fe))s = f L_{\rho(e)} + \rho(e) f - f(div_{\nabla} e)s - \rho(e) f = f\nabla^{\Lambda}_e s.$$

The following computation completes the proof,

$$\nabla^{\Lambda}_e (fs) = L_{\rho(e)}(fs) - (div_{\nabla} e)(fs) = f\nabla^{\Lambda}_e s + (\rho(e) f)s.$$

□

Note that in general the connection $\nabla^{\Lambda}$ is not induced by an ordinary connection on $\Lambda$ since $\rho(e) = 0$ does not necessarily imply $div_{\nabla} e = 0$.

3.2. **Courant algebroid torsion.** In this section we develop a notion of torsion for Courant algebroids. Given an $E$-connection $\nabla^E$ on a Courant algebroid $E$, for any three sections $e_1, e_2, e_3 \in \Gamma(E)$ we define a skew-symmetric function $C(e_1, e_2, e_3) \in C^{\infty}(P)$ by formula,

$$C(e_1, e_2, e_3) := \frac{1}{3}\mathrm{Cycl}_{123}(\{u, v\}, w) - \frac{1}{2}\mathrm{Cycl}_{123}(\nabla^E_{e_1} e_2 - \nabla^E_{e_2} e_1, e_3).$$

**Lemma 3.2.** *The function $C(e_1, e_2, e_3)$ defines a 3-form on $E$, i.e., $C \in \Gamma(\wedge^3 E^*)$.*

*Proof.* The first term in the expression for $C$ transforms as follows,

$$\begin{aligned} \mathrm{Cycl}_{123}(\{fe_1, e_2\}, e_3) &= f\mathrm{Cycl}_{123}(\{e_1, e_2\}, e_3) \\ &+ ((e_1, e_2)Df - (\rho(e_2)f)e_1, e_3) - ((e_1, e_3)Df - (\rho(e_3)f)e_1, e_2) \\ &= f\mathrm{Cycl}_{123}(\{e_1, e_2\}, e_3) + \tfrac{3}{2}((e_1, e_2)\rho(e_3)f - (e_1, e_3)\rho(e_2)f). \end{aligned}$$

For the second term, we obtain,

$$\begin{aligned} \mathrm{Cycl}_{123}(\nabla^E_{fe_1} e_2 - \nabla^E_{e_2}(fe_1), e_3) &= f\mathrm{Cycl}_{123}(\nabla^E_{e_1} e_2 - \nabla^E_{e_2} e_1, e_3) \\ &+ ((e_1, e_2)\rho(e_3)f - (e_1, e_3)\rho(e_2)f). \end{aligned}$$

The statement of the Lemma follows by comparison of these two formulas. □

The form $C \in \Gamma(\wedge^3 E^*) \cong \Gamma(\wedge^3 E)$ is (up to a sign) a direct analog of torsion when the Lie bracket of vector fields is replaced by the Courant bracket on sections of the Courant algebroid.

3.3. **Flat bases.** It is often convenient to choose a local basis in the space of section of a Courant algebroid. Let $U \subset P$ be an open neighborhood of some point $p \in P$ and let $E_U$ be the restriction of the Courant algebroid $E$ to $U$.

**Lemma 3.3.** *As a vector bundle with scalar product $E_U$ has a pair of complementary Lagrangian subbundles. That is, $E_U \cong L_1 \oplus L_2$, where both $L_1$ and $L_2$ are of dimension $n$ and the restriction of the scalar product to each of them vanishes.*

*Proof.* As a vector bundle, $E$ is a direct sum of two $n$-dimensional subbundles $E_+$ and $E_-$, where the metric is positive definite on $E_+$ and negative definite on $E_-$. Lagrangian subbundles $L \subset E$ are in one-to-one correspondence with isomorphisms between Riemannian vector bundles $E_+$ and $E_-$, with metric on $E_-$ the inverse of the scalar product. More precisely, given such an isomorphism $\lambda : E_+ \to E_-$, the corresponding Lagrangian subbundle $L_\lambda$ is spanned by vectors of the form $(x, \lambda(x)) \in (E_+ \oplus E_-)_p$. Over $U$, both $E_+$ and $E_-$ are trivial as vector bundles. Since all metrics on the vector space $\mathbb{R}^n$ are isomorphic, $E_+$ and $E_-$ are isomorphic as Riemannian vector bundles. Choose a particular isomorphism $\lambda$ and the corresponding Lagrangian subbundle $L_\lambda$. The Lagrangian subbundle $L_{-\lambda}$ is complementary to $L_\lambda$. This copmletes the proof. □

The scalar product on $E$ gives rise to a duality $L_1 \cong L_2^*$ between two complementary Lagrangian subbundles. Over $U$, $L_1$ can be trivilized, and we can choose a basis $\{l_\alpha\} \subset \Gamma(U, L_1)$ and a dual basis $\{l^\alpha\} \subset \Gamma(U, L_2)$. Together, they form a flat basis $\{e_i\} \subset \Gamma(P, E_U)$ such that $(e_i, e_j)$ is constant for all $i$ and $j$. Define a flat connection $\nabla^E$ with property $\nabla^E_e e_i = 0$ for all $i$.

Using a basis $\{e_i\}$ and the corresponding flat connection $\nabla^E$ simplifies many formulas. In particular, all elements of the basis have vanishing divergence $div_{\nabla^E} e_i = 0$ since $\nabla^E_{e_j} e_i = 0$ for all $i$ and $j$. Also, the expression for the torsion $C$ simplifies,

$$C_{ijk} := C(e_i, e_j, e_k) = \frac{1}{3}\mathrm{Cycl}_{ijk}(\{e_i, e_j\}, e_k).$$

**Proposition 3.4.** *Let $\nabla^E$ be a flat connection on $E$ and $\{e_i\}$ be a basis in the space of section of $E$ such that $\nabla^E_e e_i = 0$ for all $e \in \Gamma(E)$ and for all $i$, and $(e_i, e_j)$ are constant for all $i$ and $j$. Then, the Courant bracket can be expressed by formula,*

$$\{e_i, e_j\} = C_{ijk} e^k, \tag{50}$$

*and the tensor $C_{ijk} := C(e_i, e_j, e_k)$ satisfies the following differential equation,*

$$\mathrm{Cycl}_{jkl}\left(\rho(e_i) C_{jkl} + 6 C_{ijq}(e^q, e^p) C_{pkl}\right) = 0. \tag{51}$$

*Proof.* By definition, the expression $(\{e_i, e_j\}, e_k)$ is skew-symmetric with respect to the first two indices. And formula (**??**) shows,

$$\begin{aligned} 0 = \rho(e_i)(e_j, e_k) &= (\{e_i, e_j\} + D(e_i, e_j), e_k) + (e_j, \{e_i, e_k\} + D(e_i, e_k)) \\ &= (\{e_i, e_j\} e_k) + (\{e_i, e_k\}, e_j), \end{aligned}$$

that it is in fact completely skew-symmetric. Hence, it coincides with $C_{ijk}$ which implies (50).

To prove the differential equation we use formula (??)

$$\begin{aligned}\rho(e_i)C_{jkl} &= 2(DT(e_j,e_k,e_l),e_i)\\ &= 2\mathrm{Cycl}_{jkl}(\{\{e_j,e_k\},e_l\},e_i)\\ &= 2\mathrm{Cycl}_{jkl}C_{jkp}(e^p,e^q)\{\{e_q,e_l\},e_i\}\\ &= -2\mathrm{Cycl}_{jkl}C_{ilq}(e^q,e^p)C_{pjk}.\end{aligned}$$

□

3.4. **Clifford bundles and Clifford modules.** Since a Courant algebroid $E$ is equipped with a bilinear form $(\cdot,\cdot)$ on fibers, one can associate to it a bundle of Clifford algebras $\mathcal{C}(E)$. The Clifford (super-) commutator on sections of $E$ is given by:

$$[e_1,e_2]=2(e_1,e_2).$$

Each $E$-connection $\nabla^E$ on $E$ preserving the bilinear form gives rise to an $E$-connection $\nabla^{\mathcal{C}(E)}$ on $\mathcal{C}(E)$ such that

$$\nabla_e^{\mathcal{C}(E)}(uv)=(\nabla_e^{\mathcal{C}(E)}u)v+u(\nabla_e^{\mathcal{C}(E)}v)$$

for any $e\in\Gamma(E)$ and $u,v\in\Gamma(\mathcal{C}(E))$.

The difference $A_e=\tilde{\nabla}_e-\nabla_e$ of two $E$-connections $\nabla^E$ and $\tilde{\nabla}^E$ preserving the scalar product on $E$ is a linear operator on the fibers of $E$. It is skew-symmetric with respect to the scalar product on $E$. On a local basis $\{e_i\}\subset\Gamma(E)$ the action of $A_e$ is given by $A_e(e_i)=A_{ij}e^j$ with $A_{ij}=-A_{ji}$. The corresponding Clifford connections $\nabla^{\mathcal{C}(E)}$ and $\tilde{\nabla}^{\mathcal{C}(E)}$ are related by formula,

$$\tilde{\nabla}_e^{\mathcal{C}(E)}u=\nabla_e^{\mathcal{C}(E)}u+[a_e,u],$$

where $a_e=-\frac{1}{2}A_{ij}e^ie^j$ is a section of the Clifford bundle.

Locally, the Clifford bundle $\mathcal{C}(E)$ always possesses an irreducible Clifford module. That is, a bundle $S\to P$ such that a fiber $S_p$ is an irreducible representation of the Clifford algebra $\mathcal{C}(E)_p$. Indeed, by Lemma **??**, over an open neighborhood $U$ of $p$ the bundle $E$ is isomorphic to a direct sum of two Lagrangian subbundles $L_1\oplus L_2$. Then, by choosing $S=\wedge L_1$ we define the Clifford action by formulas,

$$e_1\cdot s=e_1\wedge s\,,\; e_2\cdot s=\iota(e_2)s,$$

where $s\in\Gamma(S)=\Gamma(\wedge L_1)$, $e_1\in\Gamma(L_1)$, and $e_2\in\Gamma(L_2)\cong\Gamma(L_1^*)$.

Globally, if $E$ admits a splitting into a sum of two Lagrangian subbundles, it possesses an irreducible Clifford module $S$ constructed as above. Then, any other irreducible Clifford module $S'$ is obatined as tensor product of $S$ with a line bundle, $S'\cong S\otimes\mathcal{L}$.

Connections $\nabla^S$ and $\nabla^{\mathcal{C}(E)}$ are called compatible if the following condition is satisfied,

$$\nabla_e^S(u\cdot s)=(\nabla_e^{\mathcal{C}(E)}u)\cdot s+u\cdot(\nabla_e^S s).$$

Different connections on $S$ compatible with a given Clifford connection differ by a function on the base,

$$\tilde{\nabla}_e^S-\nabla_e^S=\langle\gamma,e\rangle,$$

where $\gamma\in\Gamma(E^*)\cong\Gamma(E)$.

In the case of $E$ split into a direct sum of two Lagrangian subbundles $E\cong L\oplus L^*$, and $S\cong\wedge L$, any connection $\nabla^L$ induces compatible connections on $\mathcal{C}(E)$ and on

$S$. In particular, by Lemma **??**, locally one can always construct such compatible connections.

## 4. Courant brackets as derived brackets

In this Section we show that, at least locally, every Courant bracket can be obtained as a derived bracket for some choice of the Dirac generating operator.

4.1. **Construction of the Dirac generating operator.** Let $E$ be a Courant algebroid, $\mathcal{C}(E)$ be the corresponding Clifford bundle and $S$ be an irreducible Clifford module over $\mathcal{C}(E)$. The vector bundle $S$ has fibers of dimension $N = 2^n$. Let $\tilde{S} = S \otimes \mathcal{L}$ be the twisted Clifford module with line bundle $\mathcal{L} := \mathcal{S} \otimes \Lambda$, where $\mathcal{S} = (\mathrm{det} S^*)^{1/N}$ and $\Lambda = (\wedge^{top} T^*P)^{1/2}$. In general, the Clifford modules $S$ and $\tilde{S}$ can only be constructed locally, over some open neighborhood $U \subset E$.

**Example 4.1.** Assume that $E$ admits a global splitting into a direct sum of two Lagrangian subbundles, $E \cong L \oplus L^*$, and choose $S = \wedge L$. In this case, $\mathrm{det} S^* \cong \mathrm{det}(\wedge L^*) \cong (\wedge^{top} L^*)^{\frac{N}{2}}$, and formula for $\tilde{S}$ simplifies,

$$\tilde{S} \cong \wedge L \otimes (\wedge^{top} L^* \otimes \wedge^{top} T^*P)^{\frac{1}{2}}.$$

In particular, if $L \cong T^*P$ we obtain $\tilde{S} \cong \wedge T^*P$ since the bundle $(\wedge^{top} TP \otimes \wedge^{top} T^*P)$ has a canonical trivialization. If $L \cong TP$ we obtain, $\tilde{S} \cong \wedge TP \otimes \wedge^{top} T^*P$.

Let $\nabla^E$ be an $E$-connection on $E$, $\nabla^{\mathcal{C}(E)}$ be the induced connection on the Clifford bundle, and $\nabla^S$ be a compatible connection on $S$. The connection $\nabla^S$ induces a connection on the line bundle $\mathcal{S}$, and by Lemma **??** a connection $\nabla^E$ defines an $E$-connection on $\Lambda$. Together, they define a compatible connection on the Clifford module $\tilde{S}$ that we denote by $\nabla$,

$$\nabla = \nabla^S \otimes \nabla^{\mathcal{S}} \otimes \nabla^{\Lambda}. \tag{52}$$

**Theorem 4.1.** *For any compatible connection* $(\nabla^E, \nabla^S)$*, the operator*

$$D := \frac{1}{2} e^i \nabla_{e_i} - \frac{1}{24} C_{ijk} e^i e^j e^k \tag{53}$$

*acting on sections of the Clifford module* $\tilde{S}$ *is a Dirac generating operator for the Courant bracket on* $E$.

This Theorem is the main result of the paper. We postpone the proof until the end of this Section. In the formulation of the Theorem we used a pair of dual bases $\{e_i\}$ and $\{e^i\}$ in the space of sections of $E$. The definition of $D$ is independent of this choice since the torsion $C$ is a 3-form on $E$ (see Lemma **??**). The definition of the generating operator $D$ is reminiscent of the standard construction of a Dirac operator (plus a cubic term correction) on a Riemannian manifold with a spin-structure. This operator is also ultimately related to the cubic Dirac operator considered by Konstant [13]. Recall that the generating operator $D$ squares to a function on the base, $D^2 \in C^\infty(P)$, whereas the square of the usual Dirac operator is a generalized Laplacian.

The following Theorem shows that the Dirac generating operator on a given irreducible Clifford module is essentially unique.

**Theorem 4.2.** *Let $E$ be a Courant algebroid, $S$ be an irreducible Clifford module of $\mathcal{C}(E)$ and let $D$ and $\tilde{D}$ be two Dirac generating operators acting on $\Gamma(S)$ and generating the anchor map and the Clifford bracket on $E$. Then, the difference of $\tilde{D}$ and $D$ is a section of $E$, $\delta := (\tilde{D} - D) \in \Gamma(E)$, with property $[D, \delta] \in C^\infty(P)$.*

*Proof.* Since the anchor maps generated by $D$ and $\tilde{D}$ coincide, we obtain,

$$[\delta, f] = \tilde{D}f - Df = 0 \tag{54}$$

for any $f \in C^\infty(P)$. By assumption, FIND the Clifford module $E$ is irreducible, and formula (54) imples $[\delta, f] = 0$ and $\delta \in \Gamma(\mathcal{C}(E))$.

The derived brackets generated by $\tilde{D}$ and $D$ coincide which implies,

$$[[\tilde{D}, e_1], e_2] - [[D, e_1], e_2] = [[\delta, e_1], e_2] = 0$$

for all $e_1, e_2 \in \Gamma(E)$. This shows that $\delta$ is a section of $E$, $\delta \in \Gamma(E)$.

Finally,

$$[D, \delta] = \tilde{D}^2 - D^2 - \delta^2,$$

and since all the terms on the right hand side a smooth functions on the base, so is the left hand side. □

4.2. $D$ **is independent of the conenction.** In this Section we show that the generating operator is independent of the choice of the connection $\nabla^E$ on $E$ and of the compatible connection $\nabla^S$ on $S$.

**Theorem 4.3.** *The Dirac generating operator $D$ defined by formula (**??**) on sections of $\tilde{S}$ is independent of the choice of a connection $\nabla^E$ on $E$ and of a compatible connection $\nabla^S$ on $S$.*

*Proof.* Choose two connections $\nabla^E$ and $\tilde{\nabla}^E$ on $E$ and compatible connections $\nabla^S$ and $\tilde{\nabla}^S$ on $S$. Then, as operators on $S$,

$$\tilde{\nabla}^S_{e_i} - \nabla^S_{e_i} = -\frac{1}{4} a_{i,jk} e^j e^k + f_i,$$

where $a_{i,jk}, f_i \in C^\infty(P)$ and $a_{i,jk} = -a_{i,kj}$. A straightforward calculation shows that the torsion forms induced by $\nabla^E$ and by $\tilde{\nabla}^E$ compare as follows,

$$\tilde{C}_{ijk} - C_{ijk} = -\frac{1}{2} \mathrm{Asym}_{ijk}(a_{i,jk}),$$

where $\mathrm{Asym}_{ijk}$ stands for complete anti-symmetrisation in indices $i, j$ and $k$. The connections induced by $\nabla^S_{e_i}$ and $\tilde{\nabla}^S_{e_i}$ on the linde bundle $\mathcal{S} := (\mathrm{det} S^*)^{1/N}$ differ by a function $(-f_i)$,

$$\tilde{\nabla}^{\mathcal{S}}_{e_i} - \nabla^{\mathcal{S}}_{e_i} = -f_i.$$

Finally, the difference of induced $E$-connections on the bundle $\Lambda = (\wedge^{top} T^* P)^{1/2}$ is of the form,

$$\tilde{\nabla}^\Lambda_{e_i} - \nabla^\Lambda_{e_i} = \frac{1}{2}(div_\nabla e_i - div_{\tilde{\nabla}} e_i) = \frac{1}{2}(e^j, e^k) a_{j,ki}.$$

Putting all the pieces together we compare the two generating operators,

$$\tilde{D} - D = \frac{1}{2} e^i (-\frac{1}{4} a_{i,jk} e^j e^k + f_i - f_i + \frac{1}{2}(e^j, e^k) a_{j,ki}) + \frac{1}{48} \mathrm{Asym}_{ijk}(a_{i,jk}) e^i e^j e^k = 0.$$

Here we used that

$$a_{i,jk}e^ie^je^k = \frac{1}{6}\mathrm{Asym}_{ijk}(a_{i,jk})e^ie^je^k + 2(e^i,e^j)a_{i,jk}e^k.$$

□

4.3. **Proof of Theorem ??.** By the property (45) of $E$-connections, we obtain,

$$[D,f] = \frac{1}{2}e^i[\nabla_{e_i},f] = \frac{1}{2}(\rho(e_i)f)e^i.$$

Hence, $[D,f]\in\Gamma(E)$, and the corresponding anchor map $\rho(e)f := 2([D,f],e) = (e,e^i)\rho(e_i)f$ coincides with the anchor map of $E$.

Since the operator $D$ is independent of the choice of compatible connections on $E$ and $S$, we can locally choose a slitting $E\cong L\oplus L^*$, dual bases $\{l_\alpha\}\subset\Gamma(L)$ and $\{l^\alpha\}\subset\Gamma(L^*)$ and an isomorphism $S\cong\wedge L$ as in Section **??**. The flat connection $\nabla^E_{e_i}e_j=0$, where $\{e_i\}=\{l_\alpha\}\cup\{l^\alpha\}$ induces a flat connection of S with property,

$$\nabla^S_{e_i}(l_{\alpha_1}\wedge\cdots\wedge l_{\alpha_k})=0,$$

and a flat connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}\cong(\wedge^{top}L^*)^{1/2}$.

Choose local coordinates $x^a$ on the base $P$ and trivialize the bundle $\wedge^{top}T^*P$ by the volume form $\mu := dx^1\wedge\dots dx^p$. In local coordinates, the anchor map takes the form,

$$\rho(e_i)=\rho_i^a\,\partial_a,$$

where $\rho_i^a\in C^\infty(P)$. The condition $\rho\circ D=0$ reads,

$$(e^i,e^j)\rho_i^a\rho_j^b=0$$

for any $a,b$. The connection on $\Lambda=(\wedge^{top}T^*P)^{1/2}$ can be written as,

$$\nabla^\Lambda_{e_i}f\mu^{1/2}=(\rho(e_i)f+\frac{1}{2}(\partial_a\rho_i^a)f)\mu^{1/2}.$$

We compute the commutator of $D$ with two elements of the basis $e_i,e_j$ of $E$,

$$[[D,e_i],e_j]=[\nabla_{e_i}-\frac{1}{4}C_{ikl}e^ke^l,e_j]=C_{ikl}e^l.$$

By Lemma **??** +++ADD LEMMA+++ it implies that $[[D,e_1],e_2]\in\Gamma(E)$ for any two sections $e_1,e_2$. Moreover, by Lemma **??** we obtain,

$$[[D,e_i],e_j]=\{e_i,e_j\}.$$

Next, we verify that the operator $D$ sqaures to a function on $P$.

$$\begin{aligned} D^2 &= (\tfrac{1}{2}e^i\nabla_{e_i}-\tfrac{1}{24}C_{ijk}e^ie^je^k)^2\\ &= \tfrac{1}{4}(e^i,e^j)\nabla_{e_i}\nabla_{e_j}+\tfrac{1}{8}e^ie^j[\nabla_{e_i},\nabla_{e_j}]\\ &- \tfrac{1}{8}C_{ijk}e^ie^j\nabla_{e_k}-\tfrac{1}{48}(\rho(e_i)C_{jkl}++\tfrac{1}{6}\mathrm{Cycl}_{jkl}(C_{ijp}(e^p,e^q)C_{qkl}))e^ie^je^ke^l+\tfrac{1}{576}C_{ijk}C^{ijk}.\end{aligned}$$

Note that,

$$(\rho(e_i)C_{jkl})e^ie^je^ke^l=\mathrm{Asym}_{ijkl}(\rho(e_i)C_{jkl})e^ie^je^ke^l+(\rho(e_i)C_{ikl})e^ke^l.$$

Lemma **??** implies that the fourth term vanishes, and the first term is canceled by the next term in the expression for $D^2$. Next, we notice that for our choice of $\nabla$,

$$[\nabla_{e_i},\nabla_{e_j}]=\nabla_{\{e_i,e_j\}}=C_{ijk}\nabla_{e_k},$$

and the second and the third terms in the expression for $D^2$ cancel each other. Finally,

$$\begin{aligned}(e^i, e^j)\nabla_{e_i}\nabla_{e_j} &= (e^i, e^j)(\rho_i^a\partial_a + \tfrac{1}{2}(\partial_a\rho_i^a))(\rho_j^b\partial_b + \tfrac{1}{2}(\partial_b\rho_j^b)) \\ &= (e^i, e^j)(\rho_i^a\rho_j^b\partial_a\partial_b + \partial_a(\rho_i^a\rho_j^b)\partial_b + \tfrac{1}{2}(\rho_i^a\partial_a\partial_b\rho_j^b) + \tfrac{1}{4}(\partial_a\rho_i^a)^2).\end{aligned}$$

Since $(e^i, e^j)\rho_i^a\rho_j^b = 0$, the first two terms on the right hand side vanish, and the other two terms are functions on the base. In conclusion, we obatin,

$$D^2 = \frac{1}{576}C_{ijk}C^{ijk} + \frac{1}{2}(\rho_i^a\partial_a\partial_b\rho_j^b) + \frac{1}{4}(\partial_a\rho_i^a)^2) \in C^\infty(P).$$

We have shown that the operator $D$ is a Dirac generating operator, and that the corresponding anchor map coincides with the one of $E$. It is sufficient to verify that the derived bracket generated by $D$ coincides with the one of $E$ on any basis in the space of sections of $E$. But we already checked it in equation (**??**). This observation completes the proof of Theorem **??**.

## 5. Splittable case

In this section, we discuss the case when the Courant algebroid is splittable, i.e., it admits a transversal pair of Dirac structures. In this case, the Courant algebroid $E$ corresponds to the double of a Lie bialgebroid $(A, A^*)$, i.e., $E = A \oplus A^*$. Thus we can take $S = \wedge A$ as the spinor bundle, and the line bundle $\mathcal{L} = (\wedge^{top}A^* \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$. Hence the twisted spinor bundle $\tilde{S} = S \otimes \mathcal{L} \cong \wedge A \otimes (\wedge^{top}A^* \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$. We shall show, by a suitable choice of connections, the Dirac generating operator $D$ can be described by the differential operators defining the differential BV-algebra structure on $\Gamma(\wedge A)$ [19].

**5.1. Statement of the theorem.** Recall that given a Lie algebroid $A$ over $P$, there exists a canonical Lie algebroid representation of $A$ on the line bundle $\wedge^{top}A \otimes \wedge^{top}T^*P$ [7], which is simply taking the Lie derivative. Hence $A$ has a representation on the square root of this line bundle: $(\wedge^{top}A \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$. The corresponding Lie algebroid cohomology differential gives ries to an operator

$$\Gamma(\wedge^l A^* \otimes (\wedge^{top}A \otimes \wedge^{top}T^*P)^{\frac{1}{2}}) \longrightarrow \Gamma(\wedge^{l+1}A^* \otimes (\wedge^{top}A \otimes \wedge^{top}T^*P)^{\frac{1}{2}}).$$

It is clear that as a vector bundle $\wedge^l A^* \otimes (\wedge^{top}A \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$ is canonically isomorphic to $\wedge^{top-l}A \otimes \mathcal{L}$, where $\mathcal{L} = (\wedge^{top}A^* \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$, Therefore, one obtains an operator

$$\tilde{\partial} : \Gamma(\wedge^k A \otimes \mathcal{L}) \longrightarrow \Gamma(\wedge^{k-1}A \otimes \mathcal{L}).$$

On the other hand, by considering the Lie algebroid $A^*$ and its canonical representation on $\mathcal{L}$, we obtain a differential operator

$$\tilde{d}_* : \wedge^k A \otimes \mathcal{L} \longrightarrow \wedge^{k+1}A \otimes \mathcal{L}.$$

The main theorem of the Section is

**Theorem 5.1.** *If a Courant algebroid $E$ is the double of a Lie bialgebroid $(A, A^*)$, by identifying $\tilde{S}$ with $\wedge A \otimes (\wedge^{top}A^* \otimes \wedge^{top}T^*P)^{\frac{1}{2}}$, the Dirac generating opeartor $D$ equals to $\tilde{d}_* + \tilde{\partial}$.*

5.2. **Compatible connections and an alternative formula for $D$.** Let us fix a non-zere section $V \in \Gamma(\wedge^{top} A)$ (which always exists locally), and let $\Omega \in \Gamma(\wedge^{top} A^*)$ be its dual section. Choose a (local) volume form $s \in \Gamma(\wedge^{top} T^*P)$. Using the section $\Omega \otimes s \in \Gamma(\wedge^{top} A^* \otimes \wedge^{top} T^*P)$, we can trivialize the line bundle $\mathcal{L}$, and write the operators $\tilde{d}_*$ and $\tilde{\partial}$ as follows:

$$\tilde{d}_* = d_* + \frac{1}{2} X_0 \quad \text{and} \tag{55}$$

$$\tilde{\partial} = -\partial + \frac{1}{2}\xi_0, \tag{56}$$

where $d_* : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*+1} A)$ is the Lie algebroid cohomology differential of the Lie algebroid $A^*$, and $\partial : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*-1} A)$ is the BV-generating operator [19] for the Gerstenhaber algebra $\oplus\Gamma(\wedge^* A)$ corresponding to the non-zero section $V \in \Gamma(\wedge^{top} A)$:

$$\partial U = -(-1)^{|\omega|}(d\omega \lrcorner V),$$

when $U = \omega \lrcorner V$. Here $X_0$ and $\xi_0$ are, respectively, the modular vector fields of the Lie algebroids $A^*$ and $A$ corresponding to the non-zero sections $\Omega \otimes s$ and $V \otimes s$:

$$L_Y V \otimes s + V \otimes L_{aY} s = < \xi_0, Y > V \otimes s, \quad \forall Y \in \Gamma(A), \tag{57}$$

and

$$L_\theta \Omega \otimes s + \Omega \otimes L_{a_*\theta} s = < \theta, X_0 > \Omega \otimes s \quad \forall \theta \in \Gamma(A^*). \tag{58}$$

In this case, we may choose a pair of compatible connections on $E$ and on $S$ as follows. Choose a linear connection $\nabla$ on the vector bundle $A$, and denote by the same symbol its induced connection on the dual bundle $A^*$. In this way, we obtain a connection on $E = A \oplus A^*$ preserving the bilinear form $(\cdot, \cdot)$. By abuse of notations, we use the same symbol $\nabla$ to denote all kinds of different connections.

The connection $\nabla$ also induces naturally a connection on the spinor bundle $S = \oplus \wedge^* A$. It is trivial to check that these two connections are indeed compatible. Note that a linear connection on $A$ induces a Lie algebroid $A$-connection on $A$ by $\nabla_X Y = \nabla_{a(X)} Y$, $\forall X, Y \in \Gamma(A)$. Then the torsion tensor is a section of $\wedge^2 A^*$ defined by

$$\mathrm{Tor}(X, Y) = [X, Y] - (\nabla_X Y - \nabla_Y X).$$

Similarly, one obtains an $A^*$-connection on $A^*$ with the torsion being a section of $\wedge^2 A$ defined by $\mathrm{Tor}(\xi, \eta) = [\xi, \eta] - (\nabla_\xi \eta - \nabla_\eta \xi)$, $\forall \xi, \eta \in \Gamma(A^*)$.

**Lemma 5.2.** *For any $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$,*

$$C(X, Y, \xi) = \frac{1}{2} < Tor(X, Y),\ \xi > \tag{59}$$

$$C(\xi, \eta, X) = \frac{1}{2} < Tor(\xi, \eta),\ X > . \tag{60}$$

*Proof.*

$$
\begin{aligned}
& \mathrm{Cycl}(\nabla_X Y - \nabla_Y X,\ \xi) \\
= \ & (\nabla_X Y - \nabla_Y X,\ \xi) + (\nabla_Y \xi,\ X) - (\nabla_X \xi,\ Y) \\
= \ & \frac{1}{2}\langle \nabla_X Y - \nabla_Y X,\ \xi\rangle + \frac{1}{2}\langle \nabla_Y \xi,\ X\rangle - \frac{1}{2}\langle \nabla_X \xi,\ Y\rangle \\
= \ & \frac{1}{2}\langle \nabla_X Y - \nabla_Y X,\ \xi\rangle + \frac{1}{2}(a(Y)\langle \xi, X\rangle - \langle \xi, \nabla_Y X\rangle) - \frac{1}{2}(a(X)\langle \xi, Y\rangle - \langle \xi, \nabla_X Y\rangle) \\
= \ & \langle \nabla_X Y - \nabla_Y X,\ \xi\rangle + \frac{1}{2}a(Y)\langle \xi, X\rangle - \frac{1}{2}a(X)\langle \xi, Y\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& C(X, Y, \xi) \\
= \ & T(X, Y, \xi) - \frac{1}{2}\,\mathrm{Cycl}(\nabla_X Y - \nabla_Y X, \xi) \\
= \ & T(X, Y, \xi) - \frac{1}{2}[\langle \nabla_X Y - \nabla_Y X, \xi\rangle + \frac{1}{2}a(Y)\langle \xi, X\rangle - \frac{1}{2}a(X)\langle \xi, Y\rangle] \\
= \ & T(X, Y, \xi) - \frac{1}{4}a(Y)\langle \xi, X\rangle + \frac{1}{4}a(X)\langle \xi, Y\rangle - \frac{1}{2}\langle \nabla_X Y - \nabla_Y X, \xi\rangle (\text{using Equation (26)}) \\
= \ & ([X, Y], \xi) - \frac{1}{2}\langle \nabla_X Y - \nabla_Y X, \xi\rangle \\
= \ & \frac{1}{2}\langle [X, Y] - (\nabla_X Y - \nabla_Y X),\ \xi\rangle \\
= \ & \frac{1}{2}\langle Tor(X, Y),\ \xi\rangle.
\end{aligned}
$$

Equation (60) can be proved similarly. □

Let us choose a (local) basis $\{X_1, \cdots, X_n\}$ of $\Gamma(A)$, and let $\{\alpha^1, \cdots, \alpha^n\}$ be its dual basis in $\Gamma(A^*)$, i.e., $\langle \alpha^i,\ X_j\rangle = \delta^i_j$. Then we may choose $\{X_1, \cdots, X_n, \alpha^1, \cdots, \alpha^n\}$ as a basis of $\Gamma(E)$, and $\{2\alpha^1, \cdots, 2\alpha^n, 2X_1, \cdots, 2X_n\}$ will be its dual basis.

Under such a choice of basis, the operator $D$ can be written as

**Proposition 5.3.**

$$
D = \alpha^i \nabla_{X_i} + X_i \nabla_{\alpha^i} + \frac{1}{2}(\langle\, Tor(X_i, X_k), \alpha^j\rangle \alpha^i X_j \alpha^k + \langle\, Tor(\alpha^i, \alpha^k), X_j\rangle X_i \alpha^j X_k) \tag{61}
$$

*Proof.* It is clear that $C_{ijk}e^i e^j e^k$ can be decomposed into four parts $I_1 + I_2 + I_3 + I_4$, where

$$
\begin{aligned}
I_1 &= 8C(X_i, X_j, X_k)\alpha^i \alpha^j \alpha^k = 0 \\
I_2 &= 8C(\alpha^i, \alpha^j, \alpha^k)X_i X_j X_k = 0 \\
I_3 &= 8C(X_i, X_j, \alpha^k)\alpha^i \alpha^j X_k + 8C(X_i, \alpha^j, X_k)\alpha^i X_j \alpha^k + 8C(\alpha^i, X_j, X_k)X_i \alpha^j \alpha^k \\
I_4 &= 8C(\alpha^i, \alpha^j, X_k)X_i X_j \alpha^k + 8C(\alpha^i, X_j, \alpha^k)X_i \alpha^j X_k + 8C(X_i, \alpha^j, \alpha^k)\alpha^i X_j X_k
\end{aligned}
$$

Now

$$\begin{aligned} & C(X_i, X_j, \alpha^k)\alpha^i\alpha^j X_k \\ = \; & C(X_i, X_j, \alpha^k)\alpha^i(-X_k\alpha^j + \delta_k^j) \\ = \; & -C(X_i, X_k, \alpha^j)\alpha^i X_j\alpha^k + C(X_i, X_j, \alpha^j)\alpha^i. \end{aligned}$$

Similarly,

$$\begin{aligned} & C(\alpha^i, X_j, X_k)X_i\alpha^j\alpha^k \\ = \; & -C(X_i, X_k, \alpha^j)\alpha^i X_j\alpha^k - C(\alpha^j, X_i, X_j)\alpha^i. \end{aligned}$$

Hence,

$$\begin{aligned} I_3 &= -24C(X_i, X_k, \alpha^j)\alpha^i X_j\alpha^k (\text{by Equation (59)}) \\ &= -12\langle \mathrm{Tor}(X_i, X_k), \alpha^j\rangle\alpha^i X_j\alpha^k. \end{aligned}$$

Similarly, we have

$$I_4 = -12\langle \mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle X_i\alpha^j X_k.$$

Therefore,

$$C_{ijk}e^ie^je^k = -12(\langle \mathrm{Tor}(X_i, X_k), \alpha^j\rangle\alpha^i X_j\alpha^k + \langle \mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle X_i\alpha^j X_k).$$

The conclusion thus follows immediately. □

Next we divide the computation of $D$ into two parts, namely on the part of $\wedge A$, and on the part of the line bundle $\mathcal{L}$.

5.3. **Contribution of the $\wedge A$-part.** Let $\tilde{\nabla}$ denote the torsion free $A$-connection on $A$:

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\mathrm{Tor}(X, Y), \quad \forall X, Y \in \Gamma(A).$$

Then

$$\partial' = -\alpha^i\tilde{\nabla}_{X_i} \tag{62}$$

is the corresponding BV-generating operator of the Schouten bracket on $\oplus\Gamma(\wedge^* A)$ [19].

Similarly, we denote by $\tilde{\nabla}_\xi\eta = \nabla_\xi\eta + \frac{1}{2}\mathrm{Tor}(\xi, \eta)$, $\forall \xi, \eta \in \Gamma(A^*)$, the torsion free $A^*$-connection on $A^*$.

Note that, for any fixed $\xi \in \Gamma(A^*)$, $\mathrm{Tor}(\xi, \cdot) : A^* \longrightarrow A^*$ is a bundle map. Hence $\mathrm{trace}\mathrm{Tor}(\xi, \cdot)$ is a smooth function on the base space $P$, which is $C^\infty(P)$-linear on $\xi$. Therefore, there exists a $\tilde{X}_0 \in \Gamma(A)$ such that

$$\langle \xi, \tilde{X}_0\rangle = \mathrm{trace}\mathrm{Tor}(\xi, \cdot) = \langle \mathrm{Tor}(\xi, \alpha^i), X_i\rangle. \tag{63}$$

More explicitly,

$$\tilde{X}_0 = \langle \mathrm{Tor}(\alpha^i, \alpha^k), X_k\rangle X_i \tag{64}$$

Similarly, let

$$\tilde{\xi}_0 = \langle \mathrm{Tor}(X_i, X_k), \alpha^k\rangle\alpha^i. \tag{65}$$

Set

$$\begin{aligned} D_1 &= \alpha^i\nabla_{X_i} + \frac{1}{2}\langle \mathrm{Tor}(X_i, X_k), \alpha^j\rangle \alpha^i X_j \alpha^k \\ D_2 &= X_i\nabla_{\alpha^i} + \frac{1}{2}\langle \mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle X_i \alpha^j X_k \end{aligned}$$

The following lemma is obvious.

**Lemma 5.4.** *For any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$, $X\xi$, as an operator on $\oplus\Gamma(\wedge^* A)$, is a derivation with respect to the wedge product, i.e.,*

$$(X\xi)(U \wedge V) = ((X\xi)U) \wedge V + U \wedge (X\xi)V, \quad \forall U,\ V \in \Gamma(\wedge^* A).$$

**Proposition 5.5.** *For any $U \in \Gamma(\wedge^* A)$, $D_1 U = -\partial' U$, and $D_2 U = d_* U + \frac{1}{2}\tilde{X}_0 \wedge U$.*

*Proof.*

$$\begin{aligned} D_1 &= \alpha^i(\nabla_{X_i} + \frac{1}{2}\langle \mathrm{Tor}(X_i, X_k), \alpha^j\rangle X_j \alpha^k) \\ &= \alpha^i(\nabla_{X_i} + \frac{1}{2}\mathrm{Tor}(X_i, X_k)\alpha^k) \text{(by Lemma 5.4)} \\ &= \alpha^i\tilde{\nabla}_{X_i} \end{aligned}$$

Hence,

$$D_1 U = \alpha^i \lrcorner \tilde{\nabla}_{X_i} U = -\partial' U.$$

On the other hand,

$$\begin{aligned} D_2 &= X_i\nabla_{\alpha^i} + \frac{1}{2}\langle \mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle X_i \alpha^j X_k \\ &= X_i(\nabla_{\alpha^i} + \frac{1}{2}\langle \mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle \alpha^j X_k) \\ &= X_i(\nabla_{\alpha^i} + \frac{1}{2}\mathrm{Tor}(\alpha^i, \alpha^k) X_k) \\ &= X_i(\nabla_{\alpha^i} - \frac{1}{2}X_k\mathrm{Tor}(\alpha^i, \alpha^k) + \frac{1}{2}\langle \mathrm{Tor}(\alpha^i, \alpha^k),\ X_k\rangle) \\ &= X_i(\nabla_{\alpha^i} - \frac{1}{2}X_k\mathrm{Tor}(\alpha^i, \alpha^k)) + \frac{1}{2}\tilde{X}_0. \end{aligned}$$

It is simple to check that for any $X \in \Gamma(A)$

$$(\nabla_{\alpha^i} - \frac{1}{2}X_k\mathrm{Tor}(\alpha^i, \alpha^k))X = \tilde{\nabla}_{\alpha^i} X.$$

Using the derivation property: Lemma 5.4, we thus have

$$(\nabla_{\alpha^i} - \frac{1}{2}X_k\mathrm{Tor}(\alpha^i, \alpha^k))U = \tilde{\nabla}_{\alpha^i} U, \quad \forall U \in \Gamma(\wedge^* A).$$

Thus

$$\begin{aligned} D_2 U &= X_i \wedge \tilde{\nabla}_{\alpha^i} U + \frac{1}{2}\tilde{X}_0 \wedge U \text{(according to Lemma 3.10 in [19])} \\ &= d_* U + \frac{1}{2}\tilde{X}_0 \wedge U. \end{aligned}$$

This concludes the proof of the proposition. □

5.4. **Contribution of the line bundle $\mathcal{L}$-part.** Next we need to compute the portion, of which $D$ acts on the line bundle $\mathcal{L}$. The following lemma can be verified directly (see [**?**]).

**Lemma 5.6.**

$$\begin{aligned} div_\nabla X &= \langle \nabla_{X_i} X,\ \alpha^i \rangle, \\ div_\nabla \xi &= \langle \nabla_{\alpha^i} \xi,\ X_i \rangle. \end{aligned}$$

Take $\Omega = \alpha^1 \wedge \cdots \wedge \alpha^n \in \Gamma(\wedge^{top} A^*)$, and $V = X_1 \wedge \cdots \wedge X_n \in \Gamma(\wedge^{top} A)$. Then clearly $\langle \Omega,\ V \rangle = 1$.

**Lemma 5.7.** *For any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$,*

$$\begin{aligned} \nabla_\xi (\Omega \otimes s) &= \langle \xi, X_0 - \tilde{X}_0 \rangle (\Omega \otimes s); \\ \nabla_X (V \otimes s) &= \langle X, \xi_0 - \tilde{\xi}_0 \rangle (V \otimes s). \end{aligned}$$

*Proof.* For any $s \in \Gamma(\wedge^{top} T^* P)$,

$$\begin{aligned} (L_\xi - \nabla_\xi) s &= L_{(a_* \xi)} s - \nabla_\xi s \\ &= L_{(a_* \xi)} s - (L_{(a_* \xi)} s - (div_\nabla \xi) s) \\ &= (div_\nabla \xi) s \qquad \text{(by Lemma 5.6)} \\ &= \langle \nabla_{\alpha^i} \xi, X_i \rangle s. \end{aligned}$$

On the other hand,

$$\langle (L_\xi - \nabla_\xi) \Omega, V \rangle = \langle (L_\xi - \nabla_\xi) \alpha^i,\ X_i \rangle.$$

Therefore,

$$(L_\xi - \nabla_\xi) \Omega = \langle (L_\xi - \nabla_\xi) \alpha^i,\ X_i \rangle \Omega.$$

It thus follows that

$$\begin{aligned} & L_\xi (\Omega \otimes s) - \nabla_\xi (\Omega \otimes s) \\ = \ & (L_\xi - \nabla_\xi) \Omega \otimes s + \Omega \otimes (L_\xi - \nabla_\xi) s \\ = \ & (\langle [\xi, \alpha^i] - (\nabla_\xi \alpha^i - \nabla_{\alpha^i} \xi),\ X_i \rangle) \Omega \otimes s \\ = \ & \langle \mathrm{Tor}(\xi, \alpha^i), X_i \rangle \Omega \otimes s \\ = \ & \langle \xi, \tilde{X}_0 \rangle \Omega \otimes s. \end{aligned}$$

Hence

$$\nabla_\xi (\Omega \otimes s) = L_\xi (\Omega \otimes s) - \langle \xi, \tilde{X}_0 \rangle \Omega \otimes s = \langle \xi,\ X_0 - \tilde{X}_0 \rangle \Omega \otimes s.$$

The second equation can be proved similarly. □

By $\theta$, we denote the section of $A^*$ defined by

$$\nabla_X (\Omega \otimes s) = \langle X, \theta \rangle (\Omega \otimes s), \quad \forall X \in \Gamma(A). \tag{66}$$

**Lemma 5.8.**

$$\partial - \partial' = \frac{1}{2} (\xi_0 - \theta).$$

According to Lemma 5.7, we have

$$\nabla_X V \otimes s + V \otimes \nabla_X s = \langle X,\ \xi_0 - \tilde{\xi}_0 \rangle (V \otimes s).$$

It thus follows that

$$\nabla_X s = \langle X,\ \xi_0 - \tilde{\xi}_0 \rangle s - \langle \Omega, \nabla_X V \rangle s. \tag{67}$$

From the definition of $\theta$, we have

$$\langle X, \theta \rangle (\Omega \otimes s) = \nabla_X \Omega \otimes s + \Omega \otimes \nabla_X s.$$

Thus,

$$\begin{aligned}
& \langle X, \theta \rangle s \\
= \ & \langle \nabla_X \Omega, V \rangle s + \nabla_X s \\
= \ & -\langle \Omega, \nabla_X V \rangle s + \nabla_X s \quad \text{(by Equation (67))} \\
= \ & -2\langle \Omega, \nabla_X V \rangle s + \langle X, \xi_0 - \tilde{\xi}_0 \rangle s \\
= \ & 2\langle \Omega,\ (\tilde{\nabla}_X - \nabla_X) V \rangle s + \langle X,\ \xi_0 - \tilde{\xi}_0 \rangle s - 2\langle \Omega, \tilde{\nabla}_X V \rangle s \\
= \ & \langle X, \tilde{\xi}_0 \rangle s + \langle X, \xi_0 - \tilde{\xi}_0 \rangle s - 2\langle \Omega, \tilde{\nabla}_X V \rangle s \\
= \ & \langle X,\ \xi_0 \rangle s - 2\langle \Omega, \tilde{\nabla}_X V \rangle s \\
= \ & \langle X, \xi_0 \rangle s - 2\langle \Omega, \tilde{\nabla}_X V \rangle s.
\end{aligned}$$

It thus follows that

$$\langle X,\ \xi_0 - \theta \rangle = 2\langle \Omega, \tilde{\nabla}_X V \rangle.$$

Hence

$$\tilde{\nabla}_X V = \frac{1}{2} \langle X,\ \xi_0 - \theta \rangle V.$$

Let $\nabla^0$ be the flat $A$-connection on $\wedge^{top} A$ corresponding to the operator $\partial$, i.e., $\nabla^0_X V = 0$, $\forall X \in \Gamma(A)$. Thus, $\tilde{\nabla}_X V - \nabla^0_X V = \langle X,\ \frac{1}{2}(\xi_0 - \theta) \rangle V$. According to Proposition 4.2 in [19], we thus have

$$\partial - \partial' = \frac{1}{2}(\xi_0 - \theta).$$

□

5.5. **Proof of Theorem 5.1.** It is easy to see that for any $U \in \Gamma(\wedge^* A)$,

$$\begin{aligned}
& D(U\otimes(\Omega\otimes s)^{\frac{1}{2}}) \\
= \; & DU\otimes(\Omega\otimes s)^{\frac{1}{2}} + (\alpha^i \lrcorner U)\otimes\nabla_{X_i}(\Omega\otimes s)^{\frac{1}{2}} \\
& \quad +(X_i\wedge U)\otimes\nabla_{\alpha^i}(\Omega\otimes s)^{\frac{1}{2}} \quad \text{(by Lemma 5.7 and Equation (66))} \\
= \; & DU\otimes(\Omega\otimes s)^{\frac{1}{2}} + \frac{1}{2}\langle X_i,\theta\rangle(\alpha^i \lrcorner U)\otimes(\Omega\otimes s)^{\frac{1}{2}} \\
& \quad +\frac{1}{2}\langle\alpha^i,\ X_0-\tilde{X}_0\rangle(X_i\wedge U)\otimes(\Omega\otimes s)^{\frac{1}{2}} \\
= \; & DU\otimes(\Omega\otimes s)^{\frac{1}{2}} + \frac{1}{2}(\theta\lrcorner U)\otimes(\Omega\otimes s)^{\frac{1}{2}} + \frac{1}{2}(X_0-\tilde{X}_0)\wedge U\otimes(\Omega\otimes s)^{\frac{1}{2}} \text{(by Proposition 5.5)} \\
= \; & (-\partial' U + d_* U + \frac{1}{2}\tilde{X}_0\wedge U)\otimes(\Omega\otimes s)^{\frac{1}{2}} \text{(by Lemma 5.8)} \\
= \; & (-\partial U + \frac{1}{2}\xi_0\lrcorner U + d_* U + \frac{1}{2}X_0\wedge U)\otimes(\Omega\otimes s)^{\frac{1}{2}}.
\end{aligned}$$

This concludes the proof of the theorem. □

5.6. **Laplacian operator.** Define the Laplacian operator of a Lie bialgebroid $(A, A^*)$ by

$$\mathcal{L} = \partial d_* + d_*\partial : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^* A), \tag{68}$$

which is clear a derivation of the Gerstenhaber algebra on $\Gamma(\wedge A)$. As an immediate consequence of Theorem 5.1, we see that $\mathcal{L}$ is indeed an inner derivation. This gives an affirmative answer to an open question in [19].

**Corollary 5.9.**
- $\mathcal{L} = \frac{1}{2}(L_{X_0} + L_{\xi_0}) : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^* A)$;
- $D^2 = -\frac{1}{2}(\partial X_0) + \frac{1}{4} < \xi_0, X_0 >$.

*Proof.* We now have

$$D = -\partial + d_* + \frac{1}{2}(\xi_0 + X_0) : \oplus\Gamma(\wedge^* A) \longrightarrow \oplus\Gamma(\wedge^* A).$$

Thus,

$$\begin{aligned}
D^2 &= \frac{1}{2}[D, D] \\
&= -\mathcal{L} + \frac{1}{2}L_{\xi_0} - \frac{1}{2}[\partial, X_0] + \frac{1}{4}[\xi_0, X_0].
\end{aligned}$$

Now for any $U \in \Gamma(\wedge^* A)$,

$$\begin{aligned}
& [\partial, X_0](U) \\
= \; & \partial(X_0\wedge U) + X_0\wedge\partial U \\
= \; & (-[X_0, U] + \partial X_0\wedge U - X_0\wedge\partial U) + X_0\wedge\partial U \\
= \; & -L_{X_0}U + (\partial X_0)U,
\end{aligned}$$

Hence $[\partial, X_0] = -L_{X_0} + \partial X_0$. On the other hand, $[\xi_0, X_0] = \langle\xi_0, X_0\rangle$.

Therefore,

$$\begin{aligned} D^2 &= -\mathcal{L} + \frac{1}{2}L_{\xi_0} - \frac{1}{2}(-L_{X_0} + \partial X_0) + \frac{1}{4} < \xi_0, X_0 > \\ &= (\frac{1}{2}L_{\xi_0} + \frac{1}{2}L_{X_0} - \mathcal{L}) - \frac{1}{2}(\partial X_0) + \frac{1}{4}\langle \xi_0, X_0\rangle. \end{aligned}$$

It is simple to see that $\mathcal{L}$, $L_{\xi_0}$ and $L_{X_0}$ are all derivations of degree 0 with respect to the wedge product on $\Gamma(\wedge A)$. The conclusion thus follows immediately from Theorem **??**. □

## 6. Dirac equation and Dirac structures

In this section we explain the relation between Dirac generating operators and Dirac structures.

6.1. **Dirac structures.** Let us first recall the definition of Dirac structures below.

**Definition 6.1.** Let $E \to P$ be a Courant algebroid. A Lagrangian subbundle $F \subset E$ is called a Dirac structure if the space of sections $\Gamma(F) \subset \Gamma(E)$ is closed with respect to the Courant bracket.

It is clear that the restriction of the anchor map and the bracket on $\Gamma(E)$ to $F$ makes $F$ into a Lie algebroid.

If the Courant bracket on $E$ is a derived bracket corresponding to a Dirac generating operator $D$, there is a simple way to construct Dirac structures.

**Definition 6.2.** A section $s \in \Gamma(S(E))$ is called projectively closed if there is a section $e \in \Gamma(E) \cong \mathcal{C}_1(E)$ such that

$$Ds = e \cdot s. \tag{69}$$

Assume that $s \in \Gamma(S(E))$ is a nonvanishing section. We denote by $\mathbf{c} : \mathcal{C}(E) \to S(E)$, $\mathbf{c}(a) = a \cdot s$, $\forall a \in \mathcal{C}(E)$ the corresponding Clifford map.

Given a nonvanishing section $s$, consider a subspace of $E_x$, $x \in P$ as follows:

$$F_x = E_x \cap \mathrm{Ker}(\mathbf{c}_x).$$

It is clear that $F_x$ is automatically isotropic.

**Definition 6.3.** A nonvanishing section $s$ is called *of maximum rank* at a point $x \in P$ if $F_x$ is maximal isotropic (i.e., $F_x$ is of dimension $n$, or is a Lagrangian subspace).

Let $s \in \Gamma(S(E))$ be a nonvanishing section of maximum rank everywhere on $P$. Then, the subspaces $F_x \subset E_x$ form a smooth subbundle $F$ of rank $n$, which is indeed a Lagrangian subbundle of $E$, called the Lagrangian subbundle generated by $s$. More generally, a line subbundle $\mathcal{L}$ of $S(E)$ is said to be *of maximum rank* , if any local nonvanishing section of $\mathcal{L}$ is of maximum rank . It is clear that any of maximum rank line subbundle of $S(E)$ generates a Lagrangian subbundle of $E$ in a similar fashion.

Conversely, given any Lagrangian subspace $F_x \subset E_x$, elements in $S(E)_x$ which are annihilated by $F_x$ under the Clifford action consist of a one-dimensional subspace $\mathcal{L}_x$ of $S(E)_x$. Hence for a Lagrangian subbundle $F$, $\cup_{x\in P}\mathcal{L}_x$ is a line subbundle of $S(E)$, denoted by $\mathcal{L}_F$.

In summary, there is a bijection between Lagrangian subbundles of $E$ and of maximum rank line subbundles of $S(E)$. Next Theorem describes what a Dirac structure corresponds under this bijection.

**Theorem 6.4.** *Let $E$ be a Courant algebroid corresponding to a Dirac generating operator $D$ on the spinor bundle $S(E)$. Assume that $s \in \Gamma(S(E))$ is a nonvanishing section of maximum rank everywhere on $P$. If $s$ is projectively closed, then its generated Lagrangian subbundle $F$ is a Dirac stucture on $E$.*

*Conversely, given any Dirac stucture $F \subset E$, if $s \in \Gamma(\mathcal{L}_F)$ is a nonvanishing (local) section, then $s$ must be projectively closed.*

*Proof.* Let $e_1, e_2$ be two sections of $F$ so that $e_1 \cdot s = e_2 \cdot s = 0$. Then,

$$(e_1 \triangleleft e_2)s = [De_1, e_2]s = -(e_2e_1)Ds = -e_2e_1e \cdot s = (-e_2(e, e_1) + e_2ee_1) \cdot s = 0,$$

because both $e_1$ and $e_2$ annihilate $s$. Hence, $e_1 \triangleleft e_2 \in \Gamma(F)$, and the space of sections $\Gamma(F)$ is closed with respect to the Courant bracket.

Conversely, assume that $s \in \Gamma(\mathcal{L}_F)$ is a nonvanishing (local) section. Then for any $e_1, e_2 \in \Gamma(F)$, since $(e_1 \triangleleft e_2) \in \Gamma(F)$, we have $[[D, e_1], e_2]s = (e_1 \triangleleft e_2) \cdot s = 0$. It thus follows that $e_2[D, e_1]s = 0$ since $e_2$ annihilates $s$. This implies that $[D, e_1]s \in \Gamma(\mathcal{L}_F)$ by the definition of $\mathcal{L}_F$. Since $s$ is assumed to be nonvanishing, there exists a function $f_{e_1} \in C^\infty(P)$ such that

$$[D, e_1]s = f_{e_1}s. \tag{70}$$

Now $[D, e_1]s = (De_1 + e_1D)s = e_1 \cdot Ds$, which implies that $f_{e_1}$ depends on $e_1$ $C^\infty(P)$-linearly. Thus there is a section $e \in \Gamma(E)$ determined uniquely up to a section of $F$ such that $f_{e_1} = 2(e, e_1)$. And therefore $f_{e_1}s = (ee_1 + e_1e) \cdot s = e_1e \cdot s$. According to Equation (70), we obtain that $e_1 \cdot (Ds - e \cdot s) = 0$. Hence $Ds - e \cdot s = fs$ for some function $f$. Note that both $D$ and $e$ are odd operators on $\Gamma(S(E))$, that is, they change the parity with respect to a fixed grading on $\Gamma(S(E))$ (WE SHOULD SAY SOMETHING REGARDING $Z_2$ GRADDING OF $\Gamma(S(E))$. IT IS NOT CANONICAL, BUT ONE MAY ALWAYS CHOOSE ONE). On the other hand, it is easy to see that a nonvanishing section $s \in \Gamma(\mathcal{L}_F)$ must be homogeneous with respect to this grading. It thus follows that $Ds - e \cdot s = 0$. This completes the proof of the theorem. □

**Definition 6.5.** A of maximum rank line subbundle $\mathcal{L}$ of $S(E)$ is said to be projectively flat if any of its nonvanishing local section is projectively closed.

Geometrically, one may rephrase this definition as follows. A of maximum rank line subbundle $\mathcal{L}$ of $S(E)$ generates another subbundle: $E \cdot \mathcal{L}$, which is of rank $n$. Then $\mathcal{L}$ is projectively closed if for any $s \in \Gamma(\mathcal{L})$, $Ds \in \Gamma(E \cdot \mathcal{L})$. According to Theorem 7.1, Dirac generating is essentially unique up to some section of $E$, therefore the properity being projectively flat is independent of the choice of $D$. Equivalently, Theorem 6.4 can be stated as follows.

**Proposition 6.1.** *There is a bijection between projectively flat of maximum rank line subbundles of $S(E)$ and Dirac structures of $E$.*

6.2. **Properties of $e$.** We note that in Theorem 6.4, when $s$ is fixed $e$ is uniquely determined up to a section of $F$, hence it corresponds to a unique section $\tilde{e}$ of $E/F \cong F^*$. As we see below, $\tilde{e}$ is a 1-cocycle of the Lie algebroid $F$.

**Proposition 6.2.**

$$d_F\tilde{e} = 0,$$

*where* $d_F : \Gamma(\wedge^* F^*) \longrightarrow \Gamma(\wedge^{*+1}F^*)$ *is the differential of the Lie algebroid cohomology of* $F$.

*Proof.* We have, $\forall X, Y \in \Gamma(F)$,

$$\begin{aligned} & d_F\tilde{e}(X,Y) \\ = \;& \rho(X)\langle\tilde{e}, Y\rangle - \rho(Y)\langle\tilde{e}, X\rangle - \langle\tilde{e}, \{X,Y\}\rangle \\ = \;& \frac{1}{2}[\rho(X)(e,Y) - \rho(Y)(e,X) - (e, X \triangleleft Y)] \\ = \;& \frac{1}{2}[(X \triangleleft e, Y) - \rho(Y)(e,X)]. \end{aligned}$$

Here the second from the last equality follows from Equation (17).

On the other hand,

$$\begin{aligned} & D(X,e)\cdot s \\ = \;& \frac{1}{2}D[X,e]\cdot s \\ = \;& \frac{1}{2}([DX,e] - [X,De])s \\ = \;& \frac{1}{2}(X \triangleleft e - [X,De])s. \end{aligned}$$

Now $[X,De]s = (XDe-(De)X)s = X[D,e]s = X\cdot(D(e\cdot s)+e\cdot Ds) = X\cdot(D^2s+ee\cdot s) = (D^2+ee)X\cdot s = 0$. Here we used Dirac Equation (69), and the fact that both $D^2$ and $ee$ are functions on $P$. It thus follows that $D(X,e)\cdot s = \frac{1}{2}(X \triangleleft e)\cdot s$, and $\rho(Y)(e,X)s = [D(X,e),Y]s = YD(X,e)\cdot s = \frac{1}{2}[Y(X\triangleleft e)]\cdot s = \frac{1}{2}[(X\triangleleft e)Y + Y(X\triangleleft e)]\cdot s = (X\triangleleft e, Y)s$. Therefore, we obtain that $d_F\tilde{e}(X,Y)s = 0$, which implies that $d_F\tilde{e}(X,Y) = 0$ since $s$ is a nonvanishing section. This completes the proof. □

We note that $e$, hence $\tilde{e}$, depends on a particular choice of $s$. When $s$ is replaced by $fs$ for a nowhere vanishing function $f$, $e$ becomes $e + Df/f$. On the other hand, it is simple to see that $\widetilde{Df} = d_F f$. Thus the class of $\tilde{e}$ is a well-defined element in $H^1(F)$. We will show below that this is indeed the modular class of the Lie algebroid $F$ when $D$ is self-adjoint.

**Theorem 6.6.** *If $D$ is a self-adjoint Dirac generating operator and $F$ is a Dirac structure, the section $e$ in Dirac Equation* (69) *corresponds to (up to a scalar of $\frac{1}{2}$) the modular class of $F$.*

*Proof.* Let us choose an isotropic complement $F^*$ of $F$ so that $E = F \oplus F^*$. Note that $\Gamma(F^*)$ may not be closed under the Courant bracket. As in Example (4.1), we take $S = \wedge F^*$ and

$$\tilde{S} \cong \wedge F^* \otimes (\wedge^{top} F \otimes \wedge^{top} T^*P)^{\frac{1}{2}}.$$

It is clear that in this case $\mathcal{L}_F \cong 1\otimes(\wedge^{top}F\otimes\wedge^{top}T^*P)^{\frac{1}{2}}$. As in Section 5.1, we may choose a pair of compatible connections on $E$ and on $S$ as follows. Choose a linear connection $\nabla$ on the vector bundle $F$, and denote by the same symbol its induced connection on the dual bundle $F^*$. This induces compatible connections on $E = F\oplus F^*$ and $S = \wedge F^*$. Let us choose a (local) basis $\{\alpha^1, \cdots, \alpha^n\}$ of $\Gamma(F)$, and denote by $\{X_1, \cdots, X_n\}$ its dual basis in $\Gamma(F^*)$, i.e., $\langle\alpha^i,\ X_j\rangle = \delta^i_j$. Then we may choose $\{X_1, \cdots, X_n, \alpha^1, \cdots, \alpha^n\}$ as a basis of $\Gamma(E)$, and $\{2\alpha^1, \cdots, 2\alpha^n, 2X_1, \cdots, 2X_n\}$ will be its dual basis.

As in Proposition 5.3, under such a choice of a basis, $-\frac{1}{24}C_{ijk}e^ie^je^k$ can be decomposed into four parts $I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned}
I_1 &= -\frac{1}{3}C(X_i, X_j, X_k)\alpha^i\alpha^j\alpha^k\\
I_2 &= -\frac{1}{3}C(\alpha^i, \alpha^j, \alpha^k)X_iX_jX_k\\
I_3 &= -\frac{1}{3}C(X_i, X_j, \alpha^k)\alpha^i\alpha^jX_k + 8C(X_i, \alpha^j, X_k)\alpha^iX_j\alpha^k + 8C(\alpha^i, X_j, X_k)X_i\alpha^j\alpha^k\\
I_4 &= -\frac{1}{3}C(\alpha^i, \alpha^j, X_k)X_iX_j\alpha^k + 8C(\alpha^i, X_j, \alpha^k)X_i\alpha^jX_k + 8C(X_i, \alpha^j, \alpha^k)\alpha^iX_jX_k
\end{aligned}$$

It is clear that both $I_1$ and $I_3$ annilate $\mathcal{L}_F$ under the Clifford action. On the other hand, $I_2$ is identically zero since $F$ is a Dirac structure. And as in the proof of Proposition 5.3, $I_4 = -\frac{1}{2}\langle\mathrm{Tor}(\alpha^i, \alpha^k), X_j\rangle X_i\alpha^jX_k = -\frac{1}{2}X_iX_k\mathrm{Tor}(\alpha^i, \alpha^k) + \frac{1}{2}\tilde{X}_0$, where $\tilde{X}_0 \in \Gamma(F^*)$ is defined by Equation (64). In conclusion, when acting on $\mathcal{L}_F$ under the Clifford action, $-\frac{1}{24}C_{ijk}e^ie^je^k$ equals to $\frac{1}{2}\tilde{X}_0$.

Now $\frac{1}{2}e^i\nabla_{e_i} = \alpha^i\nabla_{X_i} + X_i\nabla_{\alpha^i}$. When acting on $\Gamma(\mathcal{L}_F)$, the first part $\alpha^i\nabla_{X_i}$ clearly vanishes, according to Lemma 5.7 the second part equals to $X_i\nabla_{\alpha^i}(1\otimes(\Omega\otimes s)^{\frac{1}{2}}) = \frac{1}{2}(X_0 - \tilde{X}_0)\cdot(1\otimes(\Omega\otimes s)^{\frac{1}{2}})$. Here $\Omega = \alpha^1\wedge\cdots\wedge\alpha^n \in \Gamma(\wedge^{top}F)$, $s \in \Gamma(\wedge^{top}T^*P)$ is a (loca) volume volume, and $X_0 \in \Gamma(F^*)$ is the modular section of the Lie algebroid with respect to the section $\Omega\otimes s$:

$$L_\theta\Omega\otimes s + \Omega\otimes L_{\rho(\theta)}s = \langle\theta, X_0\rangle\Omega\otimes s \quad \forall\theta \in \Gamma(F). \tag{71}$$

Let $\tilde{s} = (1\otimes(\Omega\otimes s)^{\frac{1}{2}})$, which is a nonvanishing (local) section of $\mathcal{L}_F$. Then we obtain that

$$D\tilde{s} = \frac{1}{2}X_0\cdot\tilde{s}.$$

This completes the proof. □

**Remark 6.1.** In the case above, $(F, F^*, E)$ indeed consists of a quasi-Manin triple. The restriction of the Courant bracket to $\Gamma(F^*)$ is decomposed into two components: a cobracket $\Gamma(F) \longrightarrow \Gamma(\wedge^2F)$ and a section $\phi \in \Gamma(\wedge^3F)$. Thus one may still have operators $\tilde{d}_*$ and $\tilde{\partial}$ as in Section 5.1. And then $D = \tilde{d}_* + \tilde{\partial} - \frac{1}{24}\phi$.

As an immediate consequence, we have

**Corollary 6.3.** *If $E$ is a Courant algebroid which admits a unimodular Dirac structure and $D$ its self-adjoint Dirac generating opeartor, then $D^2 = 0$.*

Note that Corollary 6.3 also follows from Corollary 5.9 when the unimodular Dirac structure admits a Dirac complement.

**Example 6.1.** Consider the standard Courant algebroid $E = TP \oplus T^*P$ as in Section 1.1. Take $S(E) = \wedge TP$ and $D = \partial$, the BV-generating operator of the Schouten bracket, i.e., $D = *^{-1} \circ d \circ *$, where $* : \Gamma(\wedge^* TP) \longrightarrow \Omega^{top-*}(P)$ is the linear isomorphism induced by a volume form on $P$. Let $\pi$ be a bivector field, and $F = \{\pi^\# \xi + \xi | \xi \in T^*P\}$ be the graph of $\pi$. Then $F$ is clearly a Lagrangian subbundle of $E$. It is simple to see that $\mathcal{L}_F$ is spanned by the section $e^{-\pi} \in \Gamma(\wedge TP)$. Now $De^{-\pi} = (-\partial\pi + [\pi, \pi]) \cdot e^{-\pi}$. Hence we see that $F$ is a Dirac structure iff $\pi$ is a Poisson tensor, and the modular vector field $\partial\pi$ of the Poisson tensor (which is, up to a sign, one-half of the modular of the corresponding Lie algebroid $T^*P$) naturally appears in the Dirac equation.

**Example 6.2.** Let $E = TP \oplus T^*P$ be an exact Courant algebroid defined by a three form $\Omega \in \Omega^3(P)$ as in Example 2.3. Take $\Gamma(S(E)) \cong \Omega(P)$ and $D = d + \Omega$. For a given two form $\omega \in \Omega^2(P)$, the graph $F = \{v + \omega^b v | v \in TP$ is a Lagrangian subbundle. Then $\mathcal{L}_F$ is linearly spanned by the section $e^{-\omega} \in \Omega(P)$. It is simple to see that $De^{-\omega} = (-d\omega + \Omega) \cdot e^{-\omega}$. Thus it follows that $F$ is a Dirac structure iff $d\omega = \Omega$, i.e., $\omega$ is a quasi-presymplectic structure in the sense of [**?**].

On the other hand, consider the Lagrangian subbundle $F = \{\pi^\# \xi + \xi) | \xi \in T^*P\}$ corresponding to the graph of a bivector field $\pi$. Just like in the previous example, it is more convenient in this case to use $\Gamma(\wedge TP)$ as a model of $\Gamma(S(E))$. Then $\mathcal{L}_F$ is spanned by the section $e^{-\pi} \in \Gamma(\wedge TP)$, and $D = \partial + \Omega$. Now a simple calculation yields that

$$De^{-\pi} = (-\partial\pi + [\pi, \pi] - \pi^\#\Omega + \pi^\#(\pi \lrcorner \Omega)) \cdot e^{-\pi}.$$

Thus it follows that $F$ is a Dirac structure iff $[\pi, \pi] = \pi^\#\Omega$, i.e., $\pi$ is quasi-Poisson [**?**]. In this case, the vector field $-\partial\pi + \pi^\#(\pi \lrcorner \Omega)$ is one-half of the modular of the Lie algebroid $F \cong T^*P$ defined by the quasi-Poisson structure. For this reason, we may call $\partial\pi - \pi^\#(\pi \lrcorner \Omega)$ the *modular vector field* of the quasi-Poisson structure $\pi$.

6.3. **Generalized Dirac structures.** Let $E \to P$ be a Courant algebroid, $F \to Q$ a Lagrangian subbundle of $E$ over a submanifold $Q$ of $P$. We say $F$ is *compatible with the anchor* if the image of the anchor map $\rho$, when restricting to $F$, is tangent to $Q$, i.e., $\rho(F) \subset TQ$. The following is easy to check.

**Lemma 6.7.** *Assume that $F \to Q$ is a Lagrangian subbundle compatible with the anchor. For any sections $e_1, e_2$ of $E$, if $e_1|_Q, e_2|_Q \in \Gamma(F)$, then the value of $\{e_1, e_2\}$ over $Q$ only depends on $e_1|_Q$ and $e_2|_Q$.*

*Proof.* Let $f \in C^\infty(P)$ be any function such that $f|_Q = 0$, then,

$$\{e_1, fe_2\}|_Q = f\{e_1, e_2\} + (\rho(e_1)f)e_2 - (e_1, e_2)Df = 0.$$

Similarly, $\{fe_1, e_2\}|_Q = 0$. This means that $\{e_1, e_2\}_Q$ only depends on $e_1|_Q$ and $e_2|_Q$. □

We are now ready to introduce the definition of generalized Dirac structures.

**Definition 6.8.** Given a Courant algebroid $E \to P$, a Lagrangian subbundle $F \to Q$ over a submanifold $Q$ of $P$ is a *Dirac structure* if

- $F$ is compatible with the anchor;
- for any sections $e_1, e_2$ of $E$ such that $e_1|_Q, e_2|_Q \in \Gamma(F)$, $\{e_1, e_2\}|_Q \in \Gamma(F)$;

- for any sections $e_1, e_2, e_3 \in \Gamma(F)$, there exist extensions $\tilde{e_1}, \tilde{e_2}, \tilde{e_3} \in \Gamma(E)$ such that for any $e \in E|_q$, we have $\rho(e)T(\tilde{e_1}, \tilde{e_2}, \tilde{e_3}) = 0$.

Immediately, we have

**Proposition 6.4.** *A generalized Dirac structure $F \to Q$ is a Lie algebroid under the restriction of the anchor map and the Courant bracket.*

For a given Lagrangian subbundle $F \to Q$ compatible with the anchor, consider $T(e_1, e_2, e_3) = \frac{1}{3}\operatorname{Cycl}_{123}(\{e_1, e_2\}, e_3)$ as in Equation (24). If $e_1|_Q,\ e_2|_Q,\ e_3|_Q \in \Gamma(F)$, by Equation (26), we have $T(e_1, e_2, e_3)|_Q = (\{e_1, e_2\}, e_3)|_Q$. Thus it follows that $T$ defines a three form on $\Gamma(\wedge^3 F^*)$, which is denoted by $T_F$. It is clear that $\{e_1, e_2\}_Q \in \Gamma(F)$ iff $T_F = 0$. Thus we obtain the following equivalent definition of generalized Dirac structures.

**Proposition 6.5.** *A Lagrangian subbundle $F \to Q$ is a* generalized Dirac structure *if*

- *$F$ is compatible with the anchor;*
- *$T_F = 0$.*

Let $D : \Gamma(S(E)) \longrightarrow \Gamma(S(E))$ be a Dirac generating operator of the Courant algebroid. Assume that $\mathcal{L} \to Q$ is a line subbundle of $S(E)$ which is of maximum rank at each point of $Q$. Thus $\mathcal{L}$ defines a Lagrangian subbundle $F \to Q$ such that $F$ annilates $\mathcal{L}$ under the Clifford action, i.e., $\mathcal{L} = \mathcal{L}_F$.

**Proposition 6.6.** *The following are all equivalent*

- *(i) $F$ is compatiable with the anchor;*
- *(ii) $Df|_Q \in \Gamma(F)$ for any $f \in C^\infty(P)$ such that $f|_Q = 0$;*
- *(iii) $Ds|_Q$ is a well-defined section of $S(E)|_Q$ for any $s \in \Gamma(\mathcal{L}_F)$.*

*Proof.* $(i) \to (ii)$ For any $f \in C^\infty(P)$ such that $f|_Q = 0$ and $e \in \Gamma(F)$, we have $(e, Df)|_Q = 2\rho(e)f = 0$. Thus $Df|_Q \in \Gamma(F)$.

$(ii) \to (iii)$ Let $f \in C^\infty(P)$ be any function such that $f|_Q = 0$. Then $D(fs) = fDs + Df \cdot s$. Thus $D(fs)|_Q = 0$, since $Df|_Q \in \Gamma(F)$ which annialtes $s$. It thus follows that $Ds|_Q$ is well-defined.

$(iii) \to (i)$ By assumption, for any nonvanishing local section $s \in \Gamma(\mathcal{L}_F)$ and $f \in C^\infty(P)$ such that $f|_Q = 0$, we have $D(fs)|_Q = 0$. Thus $Df|_Q \cdot s = 0$, which implies that $Df|_Q \in \Gamma(F)$. If $e \in \Gamma(F)$ is any section, then $\rho(e)(f)|_Q = \frac{1}{2}(e, Df|_Q) = 0$. Hence $\rho(e)$ must be tangent to $Q$.

This concludes the proof. □

Similar to ?, we say that a section $s \in \Gamma(\mathcal{L}_F)$ is *projectively closed over $Q$* if Equation (69) holds over $Q$ for some $e \in \Gamma(E|_Q)$, and we say that $\mathcal{L}_F$ is *projectively flat* if any of its nonvanishing local sections is projectively closed, or $Ds \in \Gamma(E \cdot \mathcal{L}_F)$ for any $s \in \Gamma(\mathcal{L}_F)$.

The following can be proved similarly as in Theorem 6.4.

**Theorem 6.9.** *Let $E$ be a Courant algebroid corresponding to a Dirac generating operator $D$ on the spinor bundle $S(E)$. Assume that $s \in \Gamma(S(E))|_Q$ is a nonvanishing section of maximum rank everywhere on $Q$. Then the Lagrangian subbundle $F$ generated by $s$ is a generalized Dirac stucture, if $F$ is compatible with the anchor and $s$ is projectively closed over $Q$.*

*Conversely, given any generalized Dirac stucture $F \to Q$, if $s \in \Gamma(\mathcal{L}_F)$ is a nonvanishing (local) section, then $s$ must be projectively closed over $Q$.*

*In other words, there is a bijection between projectively flat of maximum rank line subbundles of $S(E)$ over $Q$ and Dirac structures of $E$ over $Q$.*

**Theorem 6.10.** *Under the same hypthesis as in Theorem 6.9, if $F \to Q$ is a generalized Dirac stucture defined by the Dirac equation $Ds = e\cdot s$, then the projection $\tilde{e}$ of $e$ in $\Gamma((E/F)|_Q) \cong \Gamma(F^*)$ is a Lie algebroid 1-cocycle. If moreover $D$ is selfadjoint, the class of $\tilde{e}$ is the modular class of the Lie algebroid $F$.*

6.4. **Examples.** In this section, we will discuss some examples of generalized Dirac stuctures.

**Proposition 6.7.** *Let $P$ be a Poisson manifold with Poisson tensor $\pi$, and $E = TP \oplus T^*P$ the Courant algebroid as in Example 2.2. For a submanifold $Q$, $F = TQ \oplus N^*Q$ is a generalized Dirac stucture iff $Q$ is a coisotropic submanifold.*

*Proof.* If $F$ is a generalized Dirac stucture, it follows from the compatibility between $F$ and the anchor that $Q$ must be a coisotropic submanifold. Now we will prove its inverse.

Assume that $Q$ is coisotropic. Then $F$ is compatible with the anchor. Take $S(E) \cong \wedge T^*P$ and then $D = d + \partial$. It is clear that $\mathcal{L}_F \cong \wedge^{top} N^*Q$. Let us choose a local coordinates $(x_1, \cdots, x_q, x_{q+1}, \cdots, x_n)$ of $P$ such that $Q$ is the slice defined by $x_{q+1} = \cdots = x_n = 0$. Thus $s = dx_{q+1} \wedge \cdots \wedge dx_n|_Q$ is a (local) nonvanishing section of $\mathcal{L}_F$. It is simple to see that

$$Ds = \partial(dx_{q+1} \wedge \cdots dx_n) = \sum_{i\neq j} (-1)^{i+j} d\{x_{q+i}, x_{q+j}\} \wedge dx_{q+1} \wedge \cdots \hat{\cdots} \cdots \hat{\cdots} \cdots \wedge dx_n = e \cdot s,$$

where $e = \sum_{i\neq j}(\frac{\partial\{x_{q+i},x_{q+j}\}}{\partial x_{q+i}} \frac{\partial}{\partial x_{q+j}} + \frac{\partial\{x_{q+i},x_{q+j}\}}{\partial x_{q+j}} \frac{\partial}{\partial x_{q+i}})$.

This concludes the proof. □

The proposition above is a special case of the following more general theorem.

**Theorem 6.11.** *Let $E = A \oplus A^*$ be the double of a Lie bialgebroid $(A, A^*)$ over a base $P$, $L \to Q$ a vector subbundle of $A$ over a submanifold $Q \subset P$. Let $F = L \oplus L^\perp$. Then $F$ is a generalized Dirac structure iff both $L$ and $L^\perp$ are subalgebroids of $A$ and $A^*$ respectively.*

*Proof.* It is clear that $F$ is automatically a Lagrangian subbundle. If $F$ is a generalized Dirac structure, by definition $L$ and $L^\perp$ must be Lie subalgebroids of $A$ and $A^*$ respectively.

Conversely, assume that $L$ and $L^\perp$ are Lie subalgebroids of $A$ and $A^*$ respectively. Then it is simple to see that $F$ is compatible with anchor. Take $S(E) \cong \wedge A^*$, then according to Theorem 5.1, $D = d - \partial_* + \frac{1}{2}(X_0 + \xi_0)$ is a Dirac generating operator. We also see that $\mathcal{L}_F \cong \wedge^{top} L^\perp$, and $E \cdot \mathcal{L}_F \cong \wedge^{top-1} L^\perp \oplus (\wedge^{top} L^\perp \wedge A^*)$. Now given any section $\xi \in \Gamma(A^*)$ such that $\xi|_Q \in \Gamma(L^\perp)$, we have $d\xi|_Q \in \Gamma(L^\perp \wedge A^*)$. This is because, for any $X, Y \in \Gamma(L)$,

$$d\xi|_Q(X, Y) = \rho(X)\langle \xi, Y\rangle - \rho(Y)\langle \xi, X\rangle - \langle \xi, [X, Y]\rangle = 0.$$

It thus follows that $d\Gamma(\mathcal{L}_F) \subseteq \Gamma(\wedge^{top} L^{\perp} \wedge A^*)$. On the other hand, since $L^{\perp}$ is a Lie subalgebroid of $A^*$, $\Gamma(\wedge L^{\perp})$ is indeed stable under the BV-generating operator $\partial$. Hence, in particular, $\partial\Gamma(\wedge^{top} L^{\perp}) \subseteq \Gamma(\wedge^{top-1} L^{\perp})$. It thus follows that $\mathcal{L}_F$ is indeed projectively flat under $D$. This completes our proof. □

**Proposition 6.8.** *Let $E = TP \oplus T^*P$ be an exact Courant algebroid corresponding to the three form $\Omega \in \Omega^3(P)$. For any smooth submanifold $Q \subset P$, $F = TQ \oplus N^*Q$ is a generalized Dirac structure iff $i^*\Omega = 0$, where $i : Q \longrightarrow P$ is the inclusion.*

*Proof.* Take $S(E) = \wedge T^*P$ and then $D = d + \Omega$. We have $\mathcal{L}_F \cong \wedge^{top} N^*Q$, and $E \cdot \mathcal{L}_F \cong \wedge^{top-1} N^*Q \oplus (\wedge^{top} N^*Q \wedge T^*P|_Q)$. It is simple to see that $d\Gamma(\mathcal{L}_F) \subset \Gamma(\wedge^{top} N^*Q \wedge T^*P|_Q) \subset E \cdot \mathcal{L}_F$. Hence, $\mathcal{L}_F$ is projectively flat iff $\Omega|_Q \cdot \mathcal{L}_F \subset E \cdot \mathcal{L}_F$, which is equivalent to $i^*\Omega = 0$. □

6.5. **Courant algebroid morphisms.** We are now ready to introduce Courant algebroid morphisms.

**Definition 6.12.** Let $E_1 \to P_1$ and $E_2 \to P_2$ be Courant algebroids. A morphism between these Courant algebroids is a generalized Dirac structure $F \subset E_1 \times \bar{E}_2$ over a graph of some smooth map $\phi : P_1 \longrightarrow P_2$, where $\bar{E}_2$ is the Courant algebroid obtained from $E_2$ by changing the sign of the metric, and $E_1 \times \bar{E}_2$ denotes the product Courant algebroid over $P_1 \times P_2$.

If $P_1 = P_2 = P$ and $F$ is the graph of a bundle map $\Phi : E_1 \longrightarrow E_2$ over the identity $P \longrightarrow P$, it is simple to see that $F$ defines a Courant algebroid morphism iff $\Phi$ is a Courant algebroid isomorphism. In general, we have

**Proposition 6.9.** *The graph of a bundle map $\Phi : E_1 \longrightarrow E_2$ over $\phi : P_1 \longrightarrow P_2$ defines a morphism of Courant algebroids iff*

- *$\Phi$ commutes with the anchor maps, i,e., $\rho_2 \circ \Phi = \phi_* \circ \rho_1$;*
- *$\Phi$ preserves the bilinear forms on the vector bundles;*
- *for any $e_1, e_2 \in \Gamma(E_2)$, $\Phi[\hat{e_1}, \hat{e_2}] = [e_1, e_2]$, where for any $e \in \Gamma(E_2)$, $\hat{e} \in \Gamma(E_1)$ denotes the pull back of $e$ by $\Phi$ with $E_1$ and $E_2$ being identified with their duals.*

*Proof.* It follows from definition. We leave it for the reader. □

As we see above, Courant algebroid morphisms are quite rigid when the defining relation $F$ comes from a bundle map. In many interesting cases, however $F$ does not come from a bundle map.

**Theorem 6.10.** *Let $(A, A^*)$ and $(B, B^*)$ be Lie bialgebroids over bases $M$ and $N$ respectively, and $E_1 = A \oplus A^*$ and $E_2 = B \oplus B^*$ be their doubles. Assume that $\Phi : A \longrightarrow B$ is a Lie bialgebroid morphism. Then*

$$F = \{(a + \Phi^* b_*, \Phi a + b_*) | \forall a \in A \text{ and } b_* \in B^* \text{ over compatible fibers}\} \subset E_1 \times \bar{E}_2$$

*is a morphism of Courant algebroids.*

*Proof.* Let $L = \{(a, \Phi a) | \forall a \in A\}$ be the graph of $\Phi$, which is a subbundle of $A \times B$. By assumption, $L$ is a Lie subalgebroid of the product Lie algebroid $A \times B$, and $L^{\perp} = \{(\Phi^* b_*, -b_*) | \forall b_* \in B^*\}$ is a Lie subalgebroid of $A^* \times \bar{B}^*$, where $\bar{B}^*$ is the opposite Lie algebroid of $B^*$. According to Theorem 6.11, $L \oplus L^*$ is a generalized

Dirac structure of the double $A \times B \oplus A^* \times \bar{B}^*$. It is clear that $A \times B \oplus A^* \times \bar{B}^*$ is the Courant algebroid product of $A \oplus A^*$ with $B \oplus \bar{B}^*$. Finally, note that the bundle map $b + b_* \longrightarrow b - b_*$, $\forall b + b_* \in B \oplus B^*$ is a Courant algebroid isomorphism between $B \oplus \bar{B}^*$ and $\bar{E}_2$, and $L \oplus L^*$ becomes $F$ by applying this isomorphism on $B$'s-components. This concludes the proof. □

As a special case, we have

**Corollary 6.11.** *Let $P_1$ and $P_2$ be Poisson manifolds, and $\phi : P_1 \longrightarrow P_2$ a Poisson map. And let $E_1 = TP_1 \oplus T^*P_1$ and $E_1 = TP_2 \oplus T^*P_2$ the standard Courant algebroids as in Example 2.2. Then*

$$F = \{(v + \phi^*\xi, \phi_*v + \xi)|v \in T_xP_1,\ \xi \in T^*_mP_2 \ \ \textit{with}\ \phi(x) = m\} \subset E_1 \times \bar{E}_2$$

*is a Courant algebroid morphism.*

*Proof.* We note that $\phi_* : TP_1 \longrightarrow TP_2$ is a Lie bialgebroid morphism since $\phi$ is a Poisson map. □

**Remark 6.2.** *Note that $\mathcal{L}_F \cong N^*Q$, where $Q$ is the graph of $\phi$. It might be interesting to write down an explict formula of $s$ when $P_1$ is symplectic.*

**Example 6.3.** If $P$ is a Hamiltonian Poisson group $G$ space with a momentum map $J : P \longrightarrow G^*$, then

$$F = \{(v+\phi^*\xi, \phi_*v+\xi)|v \in T_xP,\ \xi \in T^*_mG^* \ \ \text{with}\ \phi(x) = m\} \subset (TP \oplus T^*P) \times (TG^* \oplus T^*G^*)^-$$

is a morphism between $TP \oplus T^*P$ and the Courant algebroid $(\mathfrak{g} \oplus \mathfrak{g}^*) \times G^*$ as in Proposition 2.8.

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We end this Section with the following last example.

**Proposition 6.12.** *Let $E = TP_1 \oplus T^*P_1$ and $E_1 = TP_2 \oplus T^*P_2$ be exact Courant algebroids with $\Omega_1 \in \Omega^3(P_1)$ and $\Omega_2 \in \Omega^3(P_2)$ respectively. Let $\phi : P_1 \longrightarrow P_2$ be a smooth map. Then*

$$F = \{(v + \phi^*\xi, \phi_*v + \xi)|v \in T_xP_1,\ \xi \in T^*_mP_2 \ \ \textit{with}\ \phi(x) = m\} \subset E_1 \times \bar{E}_2$$

*is a Courant algebroid morphism iff $\phi^*\Omega_2 = \Omega_1$.*

*Proof.* This is a direct consequence of Proposition 6.8. □

## 7. SELF-ADJOINT

**Proposition 7.1.** *Let $E \to P$ be a Courant algebroid, $D$ a self-adjoint first order odd differential operator $D$ on $\Gamma(S)$ satisfing the following properties:*

- *For any function, $f \in C^\infty(P)$,*

$$[D, f] = Df \in \Gamma(E). \tag{72}$$

- *For any two sections $e_1, e_2 \in \Gamma(E)$,*

$$[[D, e_1], e_2] = e_1 \triangleleft e_2 \in \Gamma(E). \tag{73}$$

*Then $D$ is a Dirac generating operator of the Courant algebroid, i.e., $D^2 \in C^\infty(P)$.*

*Proof.* $\forall g \in C^\infty(P)$, we have $[[D, Df], g] = [Df, Dg] = 2(Df, Dg) = 0$. It thus follows that $[D, Df] \in \Gamma(\mathcal{C}(E))$. On the other hand, $\forall e \in \Gamma(E)$, by assumption, $[[D, Df], e] = Df \triangleleft e = 0$. Hence $[D, Df] \in C^\infty(P)$. By graded Jacobi identity, $\forall f \in C^\infty(P)$, we have $[D^2, f] = [D, Df]$. Hence, we obtain that $[D^2, f] \in C^\infty(P)$.

Now since $D$ is self-adjoint, $(D^2)^t = D^2$, and therefore $[D^2, f]^t = -[D^2, f]$, which implies that $[D^2, f] = 0$. Hence $D^2 \in \Gamma(\mathcal{C}(E))$. Finally, again by assumption, $D$ only involves $\Gamma(\mathcal{C}_3(E))$, and therefore $D^2 \in \Gamma(\mathcal{C}_2(E))$. Now the condition $(D^2)^t = D^2$ implies that $D^2 \in C^\infty(P)$. □

THEOREM 4.2

The following Theorem shows that the Dirac generating operator is essentially unique.

**Theorem 7.1.** *Let $E$ be a Courant algebroid, $S$ be an irreducible Clifford module of $\mathcal{C}(E)$ and let $D$ a Dirac generating operator acting on $\Gamma(S)$ and generating the anchor map and the Clifford bracket on $E$.*

- *If $S' = S \otimes \mathcal{L}$ is another irreducible Clifford module of $\mathcal{C}(E)$, where $\mathcal{L}$ is a line bundle, then $D' = D \otimes 1 + e^i \otimes \nabla_{\rho(e_i)}$ is a Dirac generating operator acting on $\Gamma(S')$. Here $\nabla$ is a flat linear connection on the line bundle $\mathcal{L}$.*
- *If $\tilde{D}$ is also a Dirac generating operators acting on $\Gamma(S)$ and generating the anchor map and the Clifford bracket on $E$, then, the difference of $\tilde{D}$ and $D$ is a section of $E$, $\delta := (\tilde{D} - D) \in \Gamma(E)$, with property, $\rho(\delta) = 0$ and $\delta \triangleleft e = 0$, $\forall e \in\in \Gamma(E)$.*

## References


[1] Alekseev, A, Malkin, A, and Meinrenken, E., Lie group valued moment maps, *J. Diff. Geom.* **48** (1998), 445-495.

[2] Cabras, A., Vinogradov, A. M., Extensions of the Poisson bracket to differential forms and multi-vector fields, *J. Geom. Phys.* **9** (1992), 75-100.

[3] Courant, T.J., Dirac manifolds, *Trans. A.M.S.* **319** (1990), 631-661.

[4] Drinfel'd, V. G., Quasi-Hopf algebras, *Leningrad Math. J.* **2** (1991), 829-860.

[5] Drinfel'd, V.G., On Poisson homogeneous spaces of Poisson-Lie groups, *Theor. Math. Phys.* **95** (1993), 524-525.

[6] Evens, S,, and Lu, J.-H., Poisson harmonic forms, Kostant harmonic forms, and the $S^1$-equivariant cohomology of $K/T$, *Adv. Math.* **142** (1999), 171–220.

[7] Evens, S., Lu, J.-H., and Weinstein, A., Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, *Quart. J. Math. Oxford*, to appear.

[8] Kosmann-Schwarzbach, Y., Modular vector fields and Batalin-Vilkovisky algebras, *Banach Center Publ.* **51** (2000), 109-129.

[9] Kosmann-Schwarzbach, Y., and Monterde, J., Divergence operators and odd Poisson brackets, math.QA/0002209.

[10] Kosmann-Schwarzbach, Y., Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.* **41** (1995), 153-165.

[11]

[12] Y. Kosmann-Schwarzbach, From Poisson algebras to Gerstenhaber algebras, *Ann. Inst. Fourier* **46** (1996), 1243-1274.

[13] Kostant, B., A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups, *Duke Math. J.* **100** (1999), 447-501.

[14] Liu, Z.-J., Weinstein, A. and Xu, P., Manin triples for Lie bialgebroids, *J. Diff. Geom.*, **45** (1997), 547-574.

[15] Lu, J. H., Weinstein, A., Poisson Lie groups, dressing transformations, and Bruhat decompositions, *J. Diff. Geom.* **31** (1990), 501 - 526.
[16] Mackenzie, K. and Xu, P., Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **18** (1994), 415-452.
[17] Courant algebroids, derived brackets and even symplectic supermanifolds, math.DG/9910078.
[18] Severa, P., letter to Alan Weinstein.
[19] Xu, P., Gerstenhaber algebras and BV-algebras in Poisson geometry, *Commun. Math. Phys.* **200** (1999), 545-560.