The Laplace Transform

The method of Laplace transforms is a system that relies on algebra (rather than calculus) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear equations with discontinuous forcing functions.

Definition: Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathscr{L}\{f(t)\}$, is given by

$$\boxed{\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt\,.}$$

Provided that this (improper) integral exists (i.e. that it is convergent).

Example: Let $f(t) = 1$, then $F(s) = \dfrac{1}{s}$, $\quad s > 0$.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st}\Big|_0^\infty$$

The integral is divergent whenever $s \le 0$. However, when $s > 0$, it converges to

$$\frac{-1}{s}\left(0 - e^0\right) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s).$$

Example: Let $f(t) = t$, then $F(s) = \dfrac{1}{s^2}$, $\quad s > 0$.

[This is left to you as an exercise.]

Example: Let $f(t) = e^{at}$, then $F(s) = \dfrac{1}{s-a}$, $\quad s > a$.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$$

The integral is divergent whenever $s \le a$. However, when $s > a$, it converges to

$$\frac{1}{a-s}\left(0 - e^0\right) = \frac{1}{a-s}(-1) = \frac{1}{s-a} = F(s).$$

Definition: A function $f(t)$ is called *piecewise continuous* if it only has finitely many (including zero) discontinuities on any interval $[a, b]$, and that both one-sided limits exist as t approaches each of those discontinuity from within the interval. (The last part of the definition means that f could have removable and/or jump discontinuities only; it cannot have any infinity discontinuity.)

Theorem: Suppose that

1. f is piecewise continuous on the interval $0 \le t \le A$ for any $A > 0$.

2. $|f(t)| \le K e^{at}$ when $t \ge M$, for any real constant a, and any positive constants K and M. (This means that f is "of exponential order".)

Then the Laplace transform, $F(s) = \mathscr{L}\{f(t)\}$, exists for $s > a$.

Some properties of the Laplace Transform

1. $\mathscr{L}\{0\} = 0$

2. $\mathscr{L}\{f(t) \pm g(t)\} = \mathscr{L}\{f(t)\} \pm \mathscr{L}\{g(t)\}$

3. $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, for any constant c.

Properties 2 and 3 together means that the Laplace transform is *linear*.

4. $\mathscr{L}\{(-t)f(t)\} = F'(s)$ or, equivalently $\mathscr{L}\{t f(t)\} = -F'(s)$

Example: $\mathscr{L}\{t^2\} = -(\mathscr{L}\{t\})' = -\dfrac{d}{ds}\dfrac{1}{s^2} = -\dfrac{-2}{s^3} = \dfrac{2}{s^3}$

In general, the derivatives of Laplace transforms satisfy

$\mathscr{L}\{(-t)^n f(t)\} = F^{(n)}(s)$ or, equivalently $\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

Solution of Initial Value Problems

We now shall meet “the new System”: how the Laplace transforms can be used to solve differential equations algebraically.

Theorem: [**Laplace transform of derivatives**] Suppose f is of exponential order, and that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Then

$$\boxed{\mathscr{L}\{f'(t)\} = s\,\mathscr{L}\{f(t)\} - f(0)}$$

Applying the theorem multiple times gives:

$$\mathscr{L}\{f''(t)\} = s^2\,\mathscr{L}\{f(t)\} - sf(0) - f'(0),$$

$$\mathscr{L}\{f'''(t)\} = s^3\,\mathscr{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0),$$

$$\vdots$$

$$\vdots$$

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\,\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots$$
$$- s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

This is an extremely useful aspect of the Laplace transform: that it changes differentiation with respect to t into multiplication by s (and differentiation with respect to s into multiplication by $-t$, on the other hand). Equally importantly, it says that the Laplace transform, when applied to a differential equation, would change derivatives into algebraic expressions in terms of s and (the transform of) the dependent variable itself. Thus, <u>it could transform a differential equation into an algebraic equation</u>.

We are now ready to see how the Laplace transform can be used to solve differentiation equations.

Solving initial value problems using the method of Laplace transforms

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

1. Take the Laplace transforms of both sides of an equation.

2. Simplify algebraically the result to solve for $\mathscr{L}\{y\} = Y(s)$ in terms of s.

3. Find the inverse transform of $Y(s)$. (Or, rather, find a function $y(t)$ whose Laplace transform matches the expression of $Y(s)$.) This inverse transform, $y(t)$, is the solution of the given differential equation.

The first two steps are rather mechanical. The last step is the heart of the process, and it will take some practice. Let's get started.

Example: $y'' - 6y' + 5y = 0,$ $y(0) = 1,\ \ y'(0) = -3$

[Step 1] Transform both sides

$$\mathscr{L}\{\, y'' - 6y' + 5y \,\} = \mathscr{L}\{0\}$$

$$(s^2 \mathscr{L}\{y\} - s\,y(0) - y'(0)) - 6(s\mathscr{L}\{y\} - y(0)) + 5\,\mathscr{L}\{y\} = 0$$

[Step 2] Simplify to find $Y(s) = \mathscr{L}\{y\}$

$$(s^2 \mathscr{L}\{y\} - s - (-3)) - 6(s\mathscr{L}\{y\} - 1) + 5\mathscr{L}\{y\} = 0$$

$$(s^2 - 6s + 5)\,\mathscr{L}\{y\} - s + 9 = 0$$

$$(s^2 - 6s + 5)\,\mathscr{L}\{y\} = s - 9$$

$$\mathscr{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathscr{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}\,.$$

The last expression corresponding to the Laplace transform of $2e^t - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^t - e^{5t}.$$

Many of the observant students no doubt have noticed an interesting aspect (out of many) of the method of Laplace transform: that it finds the particular solution of an initial value problem directly, without solving for the general solution first.

The Laplace Transform method will also solve a nonhomogeneous linear differential equation directly, using the exact same three basic steps, without having to separately solve for the complementary and particular solutions. This point is illustrated in the next example.

Example: $y'' - 3y' + 2y = e^{3t}, \qquad y(0) = 1, \;\; y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathscr{L}\{y\} - s\,y(0) - y'(0)) - 3(s\mathscr{L}\{y\} - y(0)) + 2\,\mathscr{L}\{y\} = \mathscr{L}\{\, e^{3t} \,\}$$

[Step 2] Simplify to find $Y(s) = \mathscr{L}\{y\}$

$$(s^2 \mathscr{L}\{y\} - s - 0) - 3(s\mathscr{L}\{y\} - 1) + 2\mathscr{L}\{y\} = 1 \,/\, (s - 3)$$

$$(s^2 - 3\,s + 2)\,\mathscr{L}\{y\} - s + 3 = 1 \,/\, (s - 3)$$

$$(s^2 - 3\,s + 2)\,\mathscr{L}\{y\} = s - 3 + \frac{1}{s-3} = \frac{(s-3)^2 + 1}{s-3}$$

$$\mathscr{L}\{y\} = \frac{s^2 - 6s + 10}{(s^2 - 3s + 2)(s-3)} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathscr{L}\{y\} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}\,.$$

Therefore, $y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$.

For the next example, we will need the following Laplace transforms:

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2} \quad , \quad s>0$$

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \quad , \quad s>0$$

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad , \quad s>a$$

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \quad , \quad s>a$$

Note: The values of a and b in the last two expressions' denominators can be determined without completing the squares. Any irreducible quadratic polynomial $s^2 + Bs + C$ can always be written in the required from of $(s-a)^2 + b^2$ by using the quadratic formula to find (necessarily complex-valued) s. The value a is the real part of s, and the value b is just the absolute value of the imaginary part of s. That is, if $s = \lambda \pm \mu i$, then $a = \lambda$ and $b = \mu$.

Example: $y'' - 2y' + 2y = \cos(t), \qquad y(0) = 1, \quad y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathscr{L}\{y\} - s\,y(0) - y'(0)) - 2(s\mathscr{L}\{y\} - y(0)) + 2\,\mathscr{L}\{y\} = \mathscr{L}\{\cos(t)\}$$

[Step 2] Simplify to find $Y(s) = \mathscr{L}\{y\}$

$$(s^2 \mathscr{L}\{y\} - s - 0) - 2(s\mathscr{L}\{y\} - 1) + 2\mathscr{L}\{y\} = s \,/\, (s^2 + 1)$$

$$(s^2 - 2s + 2)\mathscr{L}\{y\} - s + 2 = s \,/\, (s^2 + 1)$$

$$(s^2 - 2s + 2)\mathscr{L}\{y\} = s - 2 + \frac{s}{s^2+1} = \frac{(s-2)(s^2+1)+s}{s^2+1}$$

$$\mathscr{L}\{y\} = \frac{s^3 - 2s^2 + s - 2 + s}{(s^2+1)(s^2-2s+2)} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathscr{L}\{y\} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} + \frac{4s-6}{s^2-2s+2}\right]$$

$$= \frac{1}{5}\left[\frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{4(s-1)}{s^2-2s+2} - \frac{2}{s^2-2s+2}\right]$$

which corresponds to

$$y(t) = \frac{1}{5}\left[\cos(t) - 2\sin(t) + 4e^t\cos(t) - 2e^t\sin(t)\right]$$

Examples: Find the inverse Laplace transform of each

(i) $F(s) = \dfrac{2s-5}{s^2+4s+8}$

Answer: $f(t) = 2e^{-2t}\cos(2t) - \dfrac{9}{2}e^{-2t}\sin(2t)$

(ii) $F(s) = \dfrac{s+4}{(s-2)^3}$

Answer: $f(t) = te^{2t} + 3t^2e^{2t}$

Note: $\mathscr{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$