Bi-quantization and application for Symmetric spaces

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Charles Torossian

CNRS

ENS (now Univ. Paris 7)

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(joint work with A. Cattaneo, Univ. Zurich)

I- Introduction - Motivations

Let G be a Lie group (connec.), $\mathfrak{g}$ his Lie algebra, $\mathfrak{g}^*$ the dual space. Let's consider H a Lie subgroup (connec.). Consider the homogeneous space G/H.

Let's denote by $\mathbb{D}(G/H)$ the algebra of invariant differential operators.

Let $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$ and $S(\mathfrak{g})$ the symmetric algebra. Then you have the following description :

$$\mathbb{D}(G/H) = \Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}\Big)^{\mathfrak{h}}$$

(H-invariant elements in $U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}$).

Remarks :

1- In general $U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}$ is not an algebra. The associated graded space is

$$\mathrm{gr}\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}\Big) \quad = \quad S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}$$

In general it's not a Poisson algebra.

2- The algebra

$$\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right)^{\mathfrak{h}}$$

is a Poisson algebra (H-invariant polynomial functions on $\mathfrak{h}^{\perp}$).

You have got the symbol map

$$\underbrace{\mathbb{D}(G/H)}_{\substack{Asso.\\ algebra}} \underset{\substack{Vector\\ spaces}}{\approx} \underbrace{\mathrm{gr}\Big(\mathbb{D}(G/H)\Big)}_{\substack{Poisson\\ algebra}} \hookrightarrow \underbrace{\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right)^{\mathfrak{h}}}_{\substack{Poisson\\ algebra}}$$

3- Example of the group case : $G_1 = G \times G$ and $H =$ diagonal, then

$$\mathbb{D}(G_1/H) \approx U(\mathfrak{g})^{\mathfrak{g}} \quad \textit{center of } U(\mathfrak{g})$$

and

$$\left(S(\mathfrak{g}_1)/S(\mathfrak{g}_1)\cdot\mathfrak{h}\right)^{\mathfrak{h}} \approx S(\mathfrak{g})^{\mathfrak{g}}$$

Poisson center of $S(\mathfrak{g})$.

In that case, it's well known that both algebras are isomorphic (Duflo's Isomorphism).

Hope : there is a similar connection between centers of

$$\mathbb{D}(G/H) \quad \text{and} \quad \left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right)^{\mathfrak{h}}$$

Question of Duflo 86' :

Is the **center** of $\mathbb{D}(G/H)$ isomorphic (as algebra) to the Poisson **center** of $\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right)^{\mathfrak{h}}$?

In particular, both algebras should be simultaneously commutative.

This is the case if G/H is a nilpotent homogeneous spaces.

Corwin-Greenleaf (88'-92'), Baklouti, Fujiwara, Ludwig, Lion, Magneron, Mehdi (01'-04')

Theorem : G nilpotent (simp. conn.)

$\mathbb{D}(G/H)$ is commutative

$\Leftrightarrow L^2(G/H)$ is finite multiplicity

$\Leftrightarrow$ generic $f \in \mathfrak{h}^\perp$ $\quad H \cdot f$ lagrangian in $G \cdot f$

$\Leftrightarrow \left(\text{Frac}(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h})\right)^{\mathfrak{h}}$ Poisson com.

$\Leftrightarrow \left(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\right)^{\mathfrak{h}}$ Poisson com.

Moreover if $\mathbb{D}(G/H)$ is commutative

$$\mathbb{D}(G/H) \underset{algebras}{\hookrightarrow} \left(\text{Frac}(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h})\right)^{\mathfrak{h}}.$$

Remarks :

1- In general there are no obvious map (of vectors spaces) from

$$\left(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\right)^{\mathfrak{h}} \quad \to \quad \mathbb{D}(G/H).$$

If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with a H-invariant complement $\mathfrak{q}$ then the symmetrization β do the job. It's the case for symmetric spaces.

2- If G/K is a general symmetric space, id. K is the fixed points of a group involution

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

then $\mathbb{D}(G/K)$ and $\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{k}\right)^{\mathfrak{k}}$ are commutative algebras and I proved

$$\mathbb{D}(G/K) \underset{algebras}{\hookrightarrow} \left(\mathrm{Frac}(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{k})\right)^{\mathfrak{k}}.$$

3- All these problems are connected with the orbit's method in Lie theory and the works of Harish-Chandra, Helgason, Dixmier, Kirillov, Duflo etc...

II- Kontsevich's construction

In 97' M. Kontsevich proved that *every Poisson variety admits a formal quantization.*

The proof uses a explicit construction for a (associative) star-product in $\mathbb{R}^d$ in terms of diagrams, configuration's spaces and bi-differential operators. Key argument for the associativity is the Stokes formula.

Theorem (Kontsevich '97) :

There exists a L_∞-quasi-isomorphism $(U_n)_{n\geq 1}$ between the dg-Lie algebras

$$\mathfrak{g}_1 = T_{poly}(\mathbb{R}^d) \quad \text{et} \quad \mathfrak{g}_2 = D_{poly}(\mathbb{R}^d).$$

In particular it induces a bijection between (formal) solutions of Maurer-Cartan equations (modulo gauge).

If α is a Poisson bi-vector $[\alpha, \alpha] = 0$ then

$$\underset{t}{\star} := m + \sum_{n\geq 1} \frac{t^n}{n!} U_n(\underbrace{\alpha, \ldots, \alpha}_{n \text{ times}})$$

is a star-product on $\mathcal{C}^\infty(\mathbb{R}^d)[[t]]$.

For $\mathfrak{g}^*$ consider $\alpha = \frac{1}{2}\sum_{i,j}[e_i, e_j]\partial_{e_i^*} \wedge \partial_{e_j^*}$
Then the Kontsevich's formula

$$f \underset{Kont}{\star} g = fg + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_{n,2}} w_\Gamma B_\Gamma(f, g).$$

defines a star product on $\mathcal{C}_{poly}(\mathfrak{g}^*)[t]$.

Take now $X, Y \in \mathfrak{g}$ and apply this formula for $f = e^X$ and $g = e^Y$. The Kontsevich's formula gives a new expression for the BCH's formula written with all brackets (not only iterated brackets as is Dynkin formula !).

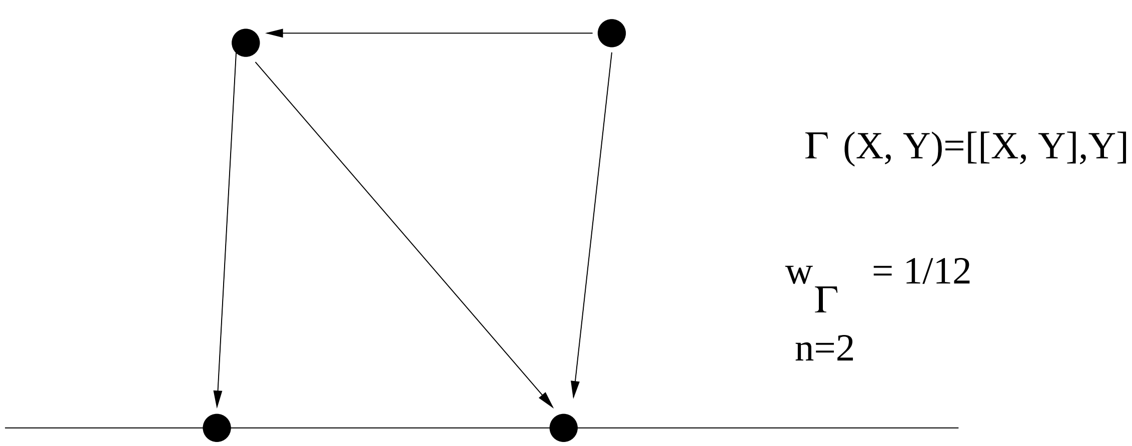
New BCH formula (Khatotia 99') :

$$Z(X,Y) = X + Y + \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,2)}} w_\Gamma \Gamma(X,Y).$$

Definition of each terms in the formula.

$\Gamma :=$ simple graph with $2n$ arrows.

$\Gamma(X,Y) :=$ is the Lie word associated to the graph Γ



$$Z(X,Y) = X + Y + \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type (n,2)}}} w_\Gamma \Gamma(X,Y).$$

To each arrow $e \in \Gamma$ you associate a 1-form $d\phi_e$ by using the Kontsevich's angle map on the upper half plane

$$\phi(p,q) = \arg(p-q) + \arg(p-\overline{q}).$$

Define the $2n$-form on $C_{n,2}$

$$\Omega_\Gamma := \bigwedge_{e \in \text{arrows}} \frac{d\phi_e}{2\pi}$$

Coefficients w_Γ are the real numbers defined by

$$w_\Gamma := \int_{C_{n,2}} \Omega_\Gamma.$$

$C_{n,2} :=$ compactification of the space of configurations of n points in the Poincaré half plane and 2 points on the real line (modulo the group $az + b$).

Application of the BCH formula : Proof of the Kashiwara-Vergne conjecture (00'-06' Alekseev-Meinrenken, To., Alek.-To., Dvorsky-Andler-Sahi, An.-Sa.-To.).

III- Cattaeno-Felder's construction for co-isotropic subspaces

Let's consider G/H a homogeneous. Fix a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then $\mathfrak{h}^{\perp}$ is a coisotropic subspace of $\mathfrak{g}^*$. Consider the super space

$$M := \mathfrak{h}^{\perp} \oplus \Pi\mathfrak{h},$$

avec Π the parity functor. Take π the Poisson bi-vector sur $\mathfrak{g}^*$ and apply a odd Fourier transform in the normal direction $\mathfrak{q}^{\perp}$. You get $\widehat{\pi}$ a polyvector on M solution of MC's equation

$$[\widehat{\pi}, \widehat{\pi}]_S = 0.$$

Explicitly in coordinates for symmetric pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$\pi = [K_i, K_j]\partial_{K_i^*} \wedge \partial_{K_j^*} + [P_i, P_j]\partial_{P_i^*} \wedge \partial_{P_j^*} + 2[K_i, P_j]\partial_{K_i^*} \wedge \partial_{P_j^*}$$

and you get ($\theta_i := \Pi K_i^*$ function on $\Pi\mathfrak{k}$)

$$\overset{\wedge}{\pi} = \theta_i\theta_j\partial_{\Pi[K_i,K_j]} + 2[K_i, P_j]\theta_i\partial_{P_j^*} + \partial_{P_i^*} \wedge \partial_{P_j^*} \wedge \partial_{\Pi[P_i,P_j]}.$$

Apply the super-formality theorem

$$\widehat{\pi} \xrightarrow{\text{SuperFormality}} \star$$

You have $[\star,\star]_{GH} = 0$ but $\star$ is not homogeneous for the Hochschild graduation on $\mathcal{C}^\infty(M)$ (because of the odd degree).

Indeed you get $(m_n)_{n\geq 1}$ a flat A_∞-structure on

$$\big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\big)\otimes\bigwedge\mathfrak{h}^*$$

m_1 is a differential and m_2 is a associative structure up to m_1.

Expansion in terms of Kontsevich's graphs :

Change of parity corresponds to reverse the direction of arrows in the graph expansion or equivalently to color the arrows.

Some explanations : Take the same kind of ingredients as in Kontsevich's formula, but put 2 colors on the arrows of the graph Γ.

That means that derivation the $\partial_{e_i^*}$ in B_Γ and the angle map $d\phi_e$ depend of the color.

If the color is $+$ then the angle map is

$$\phi_+(p,q) = \arg(p-q) + \arg(p-\overline{q})$$

and deriv. $\partial_{e_i^*}$ is horizontal : $e_i^* \in \mathfrak{h}^\perp$.

If the color is $-$ then the angle map is

$$\phi_-(p,q) = \arg(p-q) - \arg(p-\overline{q})$$

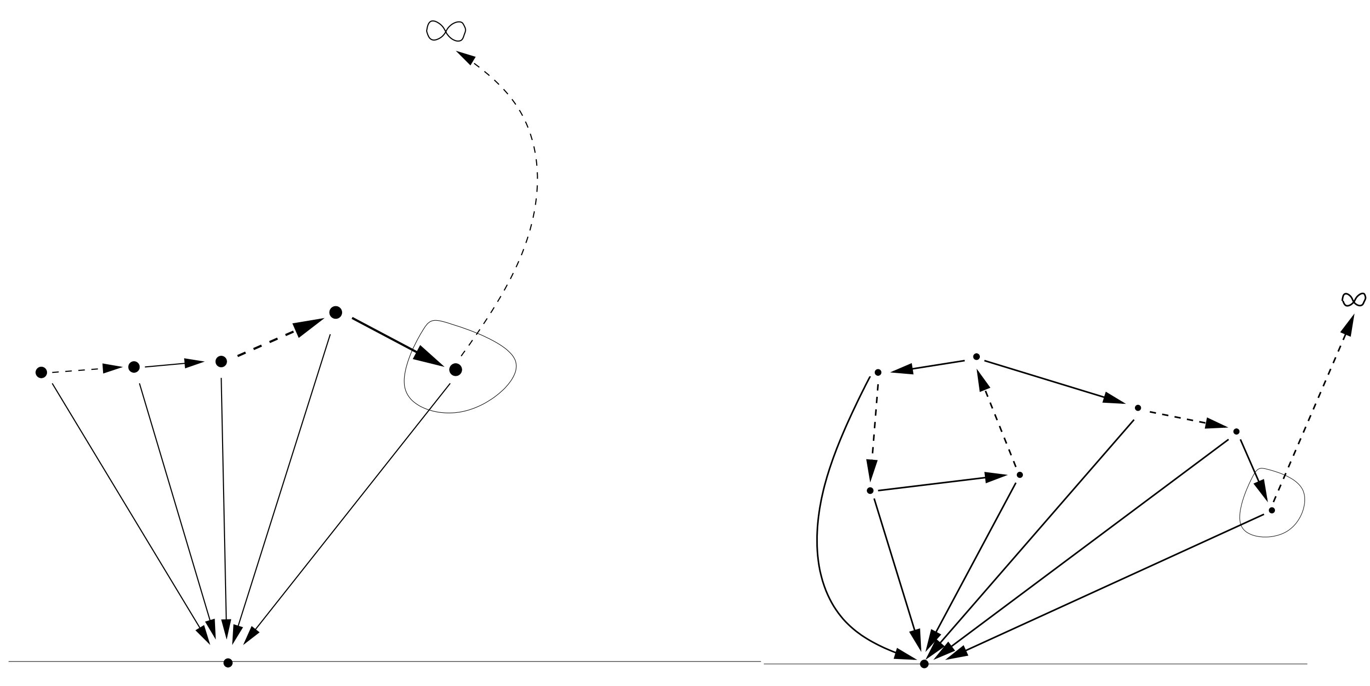
and deriv. $\partial_{e_i^*}$ is transversal : $e_i^* \in \mathfrak{q}^\perp$.

Those A_∞-structure depend of the choice of the complement $\mathfrak{q}$. However they are A_∞-quasi-isomorphe.

1- The first term m_1 is a explicit differential (contributions of all diagrams with 1 outside arrow colored by $-$)

$$\mathsf{d}_{\mathfrak{h}} := m_1 = t\,\mathsf{d}_{CE} + o(t),$$

id. a deformation of the Cartan-Eilenberg differential for the adjoint action of $\mathfrak{h}$ on $S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}$.



2- m_2 defines a star-product on the cohomology space $H(\mathsf{d}_{\mathfrak{h}})$.

$\left(H(\mathsf{d}_{\mathfrak{h}}), m_2\right)$ is called the reduced algebra. If you change the complement $\mathfrak{q}$, you get isomorphic algebras. So the algebra $H^0(\mathsf{d}_{\mathfrak{h}})$ is a serious candidate for a deformation of $\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right)^{\mathfrak{h}}$.

Applications to symmetric pairs :

There are two natural decompositions for symmetric pairs

- Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
- Iwasawa type decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_o + \mathfrak{n}_+$$

with $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$ the little symmetric pair (a kind of MA space) and $\mathfrak{n}_+$ is a nilpotent sub-algebra.

Prop (Ca.-To. 06', Ca.-Felder 04') :

1 - Suppose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with $\mathfrak{q}$ a H-invariant complement. Then

$$\mathrm{d}_{\mathfrak{h}} = t\mathrm{d}_{CE}.$$

and $H^0(d_{\mathfrak{h}}) = \Big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\Big)^{\mathfrak{h}}$.

2 - Moreover you get a explicit isomorphism (wheel's contributions)

$$\Big(H^0(d_{\mathfrak{h}})\ ,\ \underset{CF}{\star}\Big) \underset{\text{algebras}}{\sim} \Big(\Big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}\Big)^{\mathfrak{h}}\ ,\ \cdot\Big)$$

3 - If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a symmetric pair, then the product $\underset{CF}{\star}$ coincide with the Rouviere's product defined on $S(\mathfrak{p})^{\mathfrak{k}}$:

$$\beta\Big(J^{1/2}(\partial)(P \underset{CF}{\star} Q)\Big) = \beta\Big(J^{1/2}(\partial)P\Big) \cdot \beta\Big(J^{1/2}(\partial)Q\Big)$$

with $J(X) = \det_{\mathfrak{p}} \left(\frac{\sinh \mathrm{ad} X}{\mathrm{ad} X}\right)$.

4 - For $X, Y \in \mathfrak{p}$ we have

$$m_2(e^X, e^Y) = E(X, Y) e^{X+Y}.$$

This $E(X, Y)$ function is symmetric and corresponds to the contributions of wheel type diagrams.

For solvable symmetric pairs this function is equal identically to 1. Same thing for quadratic Lie algebras (view as symmetric pairs).

5 - New formula for the Harish-Chandra homomorphism.

Consider the two standard decompositions (Iwasawa and Cartan). The reduced algebras are isomorphic. You get a explicit expression for the isomorphism by solving a gauge differential equation.

IV - Bi-quantization

Let's consider now $\mathfrak{h}$ and $\mathfrak{b}$ to subalgebras. You have 2 coisotropic subspaces $C_1 = \mathfrak{h}^\perp$ and $C_2 = \mathfrak{b}^\perp$ or more generally if λ is a character for $\mathfrak{h}$ and $\mathfrak{b}$ (id. $\lambda[\mathfrak{h},\mathfrak{h}] = \lambda[\mathfrak{b},\mathfrak{b}] = 0$)

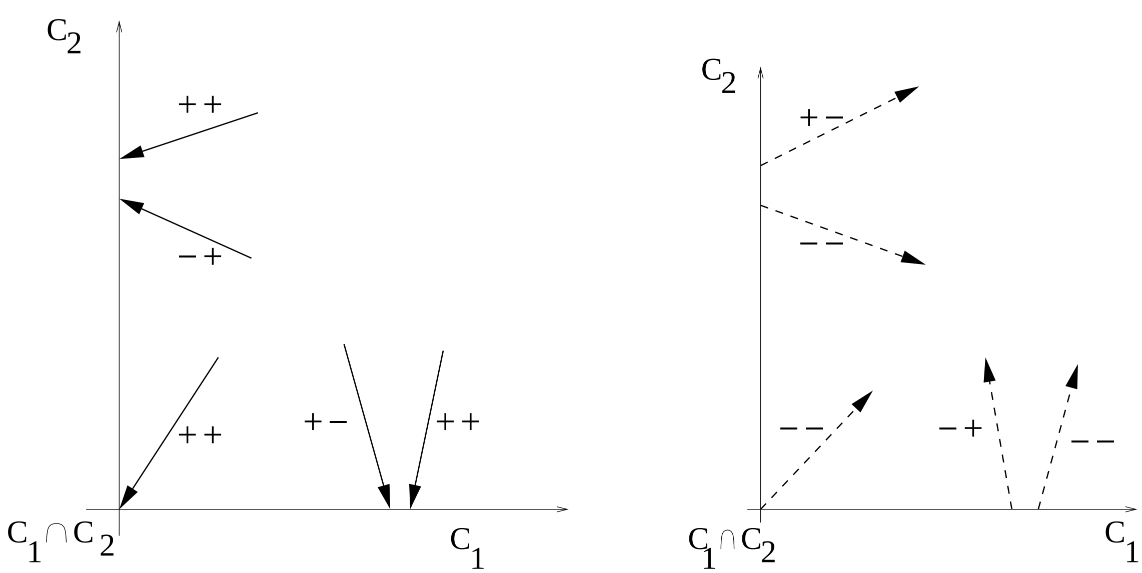
$$C_1 = \lambda + \mathfrak{h}^\perp \quad \text{and} \quad C_2 = \lambda + \mathfrak{b}^\perp.$$

Instead of considering configurations in the half plane, consider now configurations in the first quadrant modulo the dilation group and graphs with 4 colors on the arrows.

Let $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. These are the four colors.

Cattaneo-Felder defined a 4 colors angle map :

$$\phi_{\epsilon_1, \epsilon_2}(p,q) = \arg(p-q) + \epsilon_1 \arg(p-\overline{q}) + \epsilon_2 \arg(p+\overline{q}) + \epsilon_1\epsilon_2 \arg(p+q)$$



As previously, in the operator B_Γ the variable $\partial_{e_i^*}$ depends of the color.

Color $(+,+)$ means e_i^* tangent to $\mathfrak{h}^\perp$ and tangent to $\mathfrak{b}^\perp$.

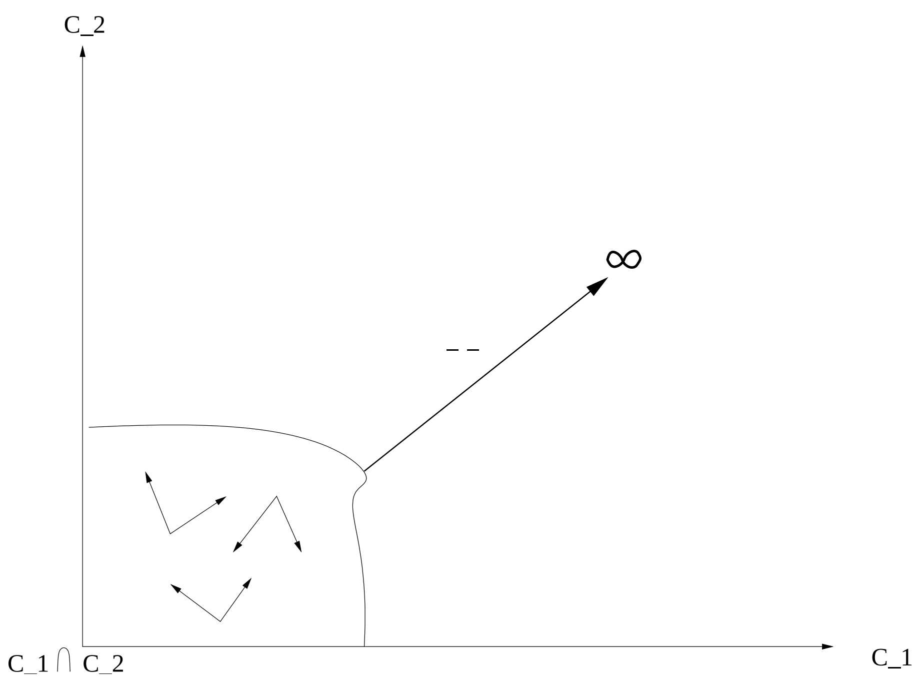
Colors $(+,-)$ means e_i^* tangent to $\mathfrak{h}^\perp$ and transversal to $\mathfrak{b}^\perp$.

Color $(-,+)$ means e_i^* transversal to $\mathfrak{h}^\perp$ and tangent to $\mathfrak{b}^\perp$.

Colors $(-,-)$ means e_i^* transversal to $\mathfrak{h}^\perp$ and transversal to $\mathfrak{b}^\perp$.

As previously contributions of all graphs with one point at the corner and one outside arrow ($(-,-)$ color) define a differential $\mathsf{d}_{\mathfrak{b},\mathfrak{h}}$ on polynomial functions on

$$\lambda + \mathfrak{h}^{\perp} \cap \mathfrak{b}^{\perp} = \lambda + (\mathfrak{h} + \mathfrak{b})^{\perp}.$$



Proposition (C-F 04') Stokes formula proves that there is a two side action

$$\left(H(\mathsf{d}_{\mathfrak{b}}), m_{2,\mathfrak{b}}\right) \underset{\star_2}{\longrightarrow} H(\mathsf{d}_{\mathfrak{b},\mathfrak{h}}) \underset{\star_1}{\longleftarrow} \left(H(\mathsf{d}_{\mathfrak{h}}), m_{2,\mathfrak{h}}\right)$$

Applications to symmetric spaces and homogeneous spaces (Ca.- To.'06) :

1 - Explicit formula for invariant differential operators.

Consider the co-isotropic subspace $C_1 = \mathfrak{k}^\perp$ et $C_2 = 0^\perp = \mathfrak{g}^*$. Take $X \in \mathfrak{p}$ and $R \in S(\mathfrak{p})^{\mathfrak{k}}$. The formula

$$e^X \underset{1}{\star} R$$

give, in terms of diagrams, the invariant differential operator $D_{\beta(J^{1/2}(\partial)R)}$ on G/K in exponentiate coordinates (up $J^{1/2}$-conjugacy).

2 - Characters for $\mathbb{D}(G/H)$

Suppose now $f \in \mathfrak{h}^{\perp}$ and $\mathfrak{b}$ a subalgebra such that

$$f[\mathfrak{b}, \mathfrak{b}] = 0.$$

Suppose the following geometric condition satisfied

$$(\mathfrak{h} \cap \mathfrak{b}) \underset{\substack{coadjoint \\ action}}{\cdot} f = (\mathfrak{h} + \mathfrak{b})^{\perp}$$

This condition is satisfied for example if $\mathfrak{b}$ is a polarization and $H \cdot f$ is lagrangian in $G \cdot f$.

(Remember there are similar conditions for G/H nilpotent).

Then $H^0(\mathrm{d}_{\mathfrak{b},\mathfrak{h}}) = \mathbb{C}$ and you get a character of the reduced algebra

$$(H^0(\mathrm{d}_{\mathfrak{h}}), m_{2,\mathfrak{h}}).$$

That means you are able to construct many characters for the reduced algebra in terms of diagrams under some expected hypothesis.

3 - Dependance of $\mathfrak{b}$.

Suppose $\mathfrak{b}_1$, $\mathfrak{b}_2$ are two polarizations for $f \in \mathfrak{k}^\perp$. If $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{k}$ are in normal crossing then by using a homotopic argument and a **8-colors** angle map on a strip, we can show that the character is the same.