Finiteness and duality on complex symplectic manifolds

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Abstract

For a complex compact manifold $X$, denote by $T$ the category $D^b_{coh}(\mathcal{O}_X)$.

• This category is a $\mathbb{C}$-triangulated category,
• this category is Ext-finite, that is,

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_T(F, G[n])$$

is finite dimensional for any $F, G \in T$,
• this category admits a Serre functor $S(\bullet)$ (see [1]), that is,

$$(\text{Hom}_T(F, G))^* \simeq \text{Hom}_T(G, S(F))$$

where $^*$ is the duality functor for $\mathbb{C}$-vector spaces, and $S(F) = F \otimes \Omega_X[d_X]$.

By analogy with this situation, for a field $k$, a $k$-triangulated category $T$ is said to be a Calabi-Yau category of dimension $d$ if $T$ is Ext-finite, admits a Serre functor and this Serre functor is a shift by $d$. 
Here, we shall consider a complex symplectic manifold $\mathcal{X}$. The natural base field is now $\hat{k} = \mathbb{C}[[\tau^{-1}, \tau]]$ or a subfield $k$ of $\hat{k}$. (Note that $\hat{k}$ and $k$ may be considered as deformation-quantizations of $\mathbb{C}$.)

The manifold $\mathcal{X}$ is endowed with the $k$-algebroid stack $\mathcal{W}_X$ of deformation quantization, a variant of the sheaf of microdifferential operators on a cotangent bundle.

We shall consider the triangulated category $\mathbf{D}^{\text{b}}_{\text{gd}}(\mathcal{W}_X)$ consisting of objects with good cohomology (roughly speaking, coherent modules endowed with a good filtration on compact subsets) and its subcategory $\mathbf{D}^{\text{b}}_{\text{gd},c}(\mathcal{W}_X)$ of objects with compact support.

We shall show that, under a natural properness condition, the composition $K_2 \circ K_1$ of two good kernels $K_i \in \mathbf{D}^{\text{b}}_{\text{gd}}(\mathcal{W}_{\mathcal{X}_{i+1} \times \mathcal{X}_0})$ ($i = 1, 2$) is a good kernel and that this composition commutes with duality.

As a particular case, we obtain that the triangulated category $\mathbf{D}^{\text{b}}_{\text{gd},c}(\mathcal{W}_X)$ is Calabi-Yau of dimension $[d_\mathcal{X}]$ over the field $k$, where $d_\mathcal{X}$ is the complex dimension of $\mathcal{X}$.

Finally, we shall discuss a kind of Riemann-Roch theorem in this framework.

This paper summarizes various joint works with A. D’Agnolo [6], M. Kashiwara [13], P. Polesello [15] and J-P. Schneiders [19].
1 Review on deformation-quantization

The field $k$.

Let $\hat{k} := \mathbb{C}[[\tau^{-1}, \tau]]$ be the field of formal Laurent series in $\tau^{-1}$. We consider the filtered subfield $k$ of $\hat{k}$ consisting of series $a = \sum_{-\infty < j \leq m} a_j \tau^j$ ($a_j \in \mathbb{C}$, $m \in \mathbb{Z}$) satisfying:

(1.1) there exist $C > 0$ with $|a_j| \leq C^{-j}(-j)!$ for all $j \leq 0$.

We denote by $k_0$ the subring of $k$ consisting of elements of order $\leq 0$.

Affine case

When $X$ is affine, one defines the filtered sheaf of $k$-algebras $\mathcal{W}_{T^*X}$ as follows, a variant of the sheaf of microdifferential operators of Sato-Kashiwara-Kawai [16] (see also [10, 17] for an exposition).

- A section $P \in \mathcal{W}_{T^*X}$ of order $m \in \mathbb{Z}$ on $U \subset T^*X$ is given by its total symbol
  \begin{equation}
  \sigma_{\text{tot}}(P)(x; u, \tau) = \sum_{-\infty < j \leq m} p_j(x; u) \tau^j, \quad p_j \in \mathcal{O}_{T^*X}(U),
  \end{equation}
  whith the condition:
  \begin{equation}
  \begin{cases}
  \text{for any compact subset $K$ of $U$ there exists a positive constant $C_K$ such that}
  \sup_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j \leq 0.
  \end{cases}
  \end{equation}

- The total symbol of the product is given by the Leibniz rule:
  \begin{equation}
  \sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!} \partial_u^\alpha (\sigma_{\text{tot}} P) \partial_x^\alpha (\sigma_{\text{tot}} Q).
  \end{equation}

Note that

- $k = \mathcal{W}_{pt}$,

- There is an embedding $\pi^{-1}D_X \rightarrow \mathcal{W}_{T^*X}$ given by $x_i \mapsto x_i$ and $\partial_{x_i} \mapsto \tau u_i$.

- There is an $\mathbb{C}$-linear isomorphism of rings $^t : \mathcal{W}_{T^*X} \simeq (\mathcal{W}_{T^*X})^{\text{op}}$ which satisfies: $^tx_i = x_i, ^tu_i = u_i, ^t\tau = -\tau$.

Note that many authors use the parameter $\hbar$ instead of $\tau^{-1}$. 
The ring $\mathcal{W}_{T^*X}$

Let $\mathfrak{X}$ be a complex symplectic manifold. We denote by $\mathfrak{X}^a$ the manifold $\mathfrak{X}$ endowed with the opposite symplectic form.

**Definition 1.1.** A $W$-ring on $\mathfrak{X}$ is a filtered sheaf of $k$-algebras $\mathcal{W}_{\mathfrak{X}}$ such that for any $x \in \mathfrak{X}$ there exists an open neighborhood $U$ of $x$ and a symplectic isomorphism $\varphi: U \cong V$ with $V$ open in $T^*\mathbb{C}^n$ and an isomorphism of filtered sheaves of $k$-algebras $\varphi^*\mathcal{W}_{\mathfrak{X}} \cong \mathcal{W}_{T^*\mathbb{C}^n}$.

Note that for a $W$-ring $\mathcal{W}_{\mathfrak{X}}$:

- the sheaf of rings $\mathcal{W}_{\mathfrak{X}}$ is right and left coherent and Noetherian,
- $\text{gr} \, \mathcal{W}_{\mathfrak{X}} \cong \mathcal{O}_{\mathfrak{X}}[\tau^{-1}, \tau]$, in particular $\mathcal{W}_{\mathfrak{X}}(0)/\mathcal{W}_{\mathfrak{X}}(-1) \cong \mathcal{O}_{\mathfrak{X}}$,
- denoting by $\sigma_0: \mathcal{W}_{\mathfrak{X}}(0) \to \mathcal{O}_{\mathfrak{X}}$ the natural map, we have for $P, Q \in \mathcal{W}_{\mathfrak{X}}(0)$, $\sigma_0(\tau[P, Q]) = \{\sigma_0(P), \sigma_0(Q)\}$,
- for any $k$-algebra automorphism $\Phi$ of $\mathcal{W}_{\mathfrak{X}}$, there locally exists an invertible section $P$ of $\mathcal{W}_{\mathfrak{X}}(0)$ such that $\Phi = \text{Ad}(P)$. Moreover, $P$ is unique up to a unique scalar multiple. Hence (denoting as usual by $A^\times$ the subgroup of invertible elements of a ring $A$):

\[
\begin{array}{ccc}
\mathcal{W}_{\mathfrak{X}}^a(0)/k^\times & \xrightarrow{\text{Ad}(\cdot)} & \text{Aut}(\mathcal{W}_{\mathfrak{X}}(0)) \\
\downsim & & \downsim \\
\mathcal{W}_{\mathfrak{X}}^a/k^\times & \xrightarrow{\text{Ad}(\cdot)} & \text{Aut}(\mathcal{W}_{\mathfrak{X}}),
\end{array}
\]

- $(\mathcal{W}_{\mathfrak{X}})^{op}$ is a $W$-ring on $\mathfrak{X}^a$.

On a cotangent bundle $T^*X$ one can construct a $W$-ring $\mathcal{W}_{T^*X}$ endowed with an anti-$k$-linear anti-automorphism $P \mapsto {}^tP$. The section ${}^tP$ is called the adjoint of $P$.

However, on a complex symplectic manifold $\mathfrak{X}$ there do not exist $W$-rings in general.
2 The algebroid $\mathcal{W}_X$

Stacks and algebroids

See Giraud [7], Breen [2], Kashiwara [9], Kontsevitch [14], D’Agnolo-Polesello [5]. See also [12] for an exposition on stacks.

Consider

- a commutative unital ring $\mathbb{K}$,
- a topological space $X$,
- an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$,
- for $i \in I$, a sheaf of $\mathbb{K}$-algebras $\mathcal{A}_i$ on $U_i$,
- for $i, j \in I$, an isomorphism $f_{ij}: \mathcal{A}_j|_{U_{ij}} \simeq \mathcal{A}_i|_{U_{ij}}$.

The existence of a sheaf of $\mathbb{K}$-algebras $\mathcal{A}$ locally isomorphic to $\mathcal{A}_i$ requires the condition $f_{ij}f_{jk} = f_{ik}$ on triple intersections.

Let us weaken this last condition by assuming that there exist invertible sections $a_{ijk} \in \mathcal{A}_i^*(U_{ijk})$ satisfying

\[
\begin{cases}
    f_{ij}f_{jk} = \text{Ad}(a_{ijk})f_{ik} \text{ on } U_{ijk}, \\
    a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl} \text{ on } U_{ijkl}.
\end{cases}
\]

We call

\[
\{\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I}\}
\]

a $\mathbb{K}$-algebroid descent data on $\mathcal{U}$.

In this case, there exists a $\mathbb{K}$-algebroid stack $\mathcal{A}$ locally equivalent to the algebroids associated with the $\mathcal{A}_i$’s. More precisely, if $A$ is an algebra, denote by $A^+$ the category with one object and having $A$ as morphisms of this object. Consider the prestack on $X$ given by $U_i \supset U \mapsto (\mathcal{A}_i(U))^+$. Then the the algebroid $\mathcal{A}$ is the stack associated with this prestack.

Although $\mathcal{A}$ is not a sheaf of algebras, modules over $\mathcal{A}$ are well-defined: they are described by pairs $\mathcal{M} = (\{M_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I})$, where $\mathcal{M}_i$ are $\mathcal{A}_i$-modules and $\xi_{ij}: f_{ji}M_j|_{U_{ij}} \rightarrow M_i|_{U_{ij}}$ are isomorphisms of $\mathcal{A}_i$-modules satisfying

\[\xi_{ij} \circ \xi_{jk} = \xi_{ik} \circ a_{kji}^{-1} \text{.}\]

Here, $f_{ji}M_j$ is the $\mathcal{A}_i$-module deduced from the $\mathcal{A}_j$-module $M_j|_{U_{ij}}$ by the isomorphism $f_{ji}$. 
Twisted modules on complex manifolds

Let $X$ be a topological space and let $c \in H^2(X; \mathbb{C}_X^\times)$. By choosing an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ and a 2-cocycle $\{c_{ijk}\}_{i,j,k \in I}$ representing $c$, one gets a descent data, hence an algebroid stack:

$$\mathbb{C}_{X,c} := (\{\mathbb{C}_X|_{U_i}\}_{i}, \{f_{ij} = 1\}_{ij}, \{a_{ijk} = c_{ijk}\}_{ijk}).$$

A twisted sheaf with twist $c$ is an object of $\text{Mod}(\mathbb{C}_{X,c})$.

Assume now that $X$ is a complex manifold. Consider the short exact sequence

$$1 \rightarrow \mathbb{C}_X^\times \rightarrow \mathcal{O}_X^\times \xrightarrow{d\log} d\mathcal{O}_X \rightarrow 0$$

which gives rise to the long exact sequence

$$H^1(X; \mathbb{C}_X^\times) \xrightarrow{\alpha} H^1(X; \mathcal{O}_X^\times) \xrightarrow{\beta} H^1(X; d\mathcal{O}_X) \xrightarrow{\gamma} H^2(X; \mathbb{C}_X^\times)$$

If $\mathcal{L}$ is a line bundle, it defines a class $[\mathcal{L}] \in H^1(X; \mathcal{O}_X^\times)$. For $\lambda \in \mathbb{C}$, one sets

$$c_\lambda^\mathcal{L} = \gamma(\lambda \cdot \beta([\mathcal{L}])) \in H^2(X; \mathbb{C}_X^\times).$$

We shall apply this construction when $\mathcal{L} = \Omega_X$ and $\lambda = \frac{1}{2}$ and set for short:

$$\text{Mod}(\mathbb{C}_{X,\frac{1}{2}}) = \text{Mod}(\mathbb{C}_{X,c_{\Omega_X}^{\frac{1}{2}}}).$$
Quantization of symplectic manifolds

In 2000, M. Kontsevich [14] has proved the existence of a $\hat{k}$-algebroid stack $\mathcal{W}_X$ on any complex Poisson manifold $X$ in the formal case. The analytic case for symplectic manifolds has been treated with a different proof in [15]. Indeed, the story began in 1996 when M. Kashiwara [9] proved the existence of a canonical $\mathbb{C}$-algebroid stack $\mathcal{E}_Y$ over any complex contact manifold $Y$ locally equivalent to the stack associated with the sheaf of microdifferential operators of Sato-Kashiwara-Kawai [16] on the projective cotangent bundle $P^*X$ to a complex manifold $X$.

In particular, for a complex symplectic manifold $\mathfrak{X}$, there is a canonical $k$-algebroid stack $\mathcal{W}_\mathfrak{X}$ locally equivalent to the algebroid stack associated with the sheaf of algebras $\mathcal{W}_{T^*X}$. (The same result holds with $\mathcal{W}_X$ replaced with $\mathcal{W}_X(0)$ and $k$ with $k_0$.)

**Notation 2.1.** For short, as far as there is no risk of confusion, we shall write $\mathcal{W}_X$ instead of $\mathcal{W}_\mathfrak{X}^+$. Let $\mathfrak{X}$ be a complex symplectic manifold. Then Mod($\mathcal{W}_X$) is a well-defined Grothendieck abelian category and admits a bounded derived category $\mathcal{D}^b(\mathcal{W}_X)$.

One proves as usual that the sheaf of algebras $\mathcal{W}_{T^*X}$ is coherent and the support of a coherent $\mathcal{W}_{T^*X}$-module is a closed complex analytic subvariety of $T^*X$. This support is involutive in view of Gabber’s theorem. Hence, the (local) notions of a coherent or a holonomic $\mathcal{W}_X$-module make sense. We denote by $\mathcal{D}^b_{\text{coh}}(\mathcal{W}_X)$ the full triangulated subcategory of $\mathcal{D}^b(\mathcal{W}_X)$ consisting of objects with coherent cohomologies, and by $\mathcal{D}^b_{\text{hol}}(\mathcal{W}_X)$ the full triangulated subcategory of $\mathcal{D}^b_{\text{coh}}(\mathcal{W}_X)$ consisting of objects with Lagrangian supports in $\mathfrak{X}$.
Simple holonomic $\mathcal{W}_\mathcal{X}$-modules

Definition 2.2. Let $\mathcal{L}$ be a coherent $\mathcal{W}_\mathcal{X}$-module supported by a closed analytic Lagrangian variety $\Lambda$ of $\mathcal{X}$.

(a) Assume $\Lambda$ is smooth. One says that $\mathcal{L}$ is simple along $\Lambda$ if there exists locally a coherent $\mathcal{W}_\mathcal{X}(0)$-submodule $\mathcal{L}(0)$ of $\mathcal{L}$ such that $\mathcal{L}(0)$ generates $\mathcal{L}$ over $\mathcal{W}_\mathcal{X}$ and $\mathcal{L}(0)/\mathcal{L}(-1)$ is an invertible $\mathcal{O}_\Lambda$-module.

(b) A coherent $\mathcal{W}_\mathcal{X}$-module supported by $\Lambda$ is regular if it is locally a finite direct sum of simple modules at generic points of $\Lambda$.

It follows from Gabber’s theorem (see [10, Th. 7.34]) that when $\Lambda$ is smooth, a regular $\mathcal{W}_\mathcal{X}$-module is locally a finite direct sum of simple modules.

Example 2.3. Let $X$ be a complex manifold. We denote by $\mathcal{O}_X^\tau$ the $\mathcal{W}_{T^*X}$-module supported by the zero-section $T^*_X X$ defined by $\mathcal{O}_X^\tau = \mathcal{W}_{T^*X}/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by the vector fields which annihilate the section $1 \in \mathcal{O}_X$. A section $f(x, \tau)$ of this module may be written as a series:

\[
f(x, \tau) = \sum_{-\infty < j \leq m} f_j(x) \tau^j, \quad m \in \mathbb{Z},
\]

(2.1)

the $f_j$’s satisfying Condition (1.3). Then $\mathcal{O}_X^\tau$ is a simple $\mathcal{W}_{T^*X}$-module along $T^*_X X$.

One proves easily that any two $\mathcal{W}_\mathcal{X}$-modules simple along $\Lambda$ are locally isomorphic. Moreover, a smooth Lagrangian submanifold is locally isomorphic to the zero section $X$ of $T^*X$. Hence, any simple $\mathcal{W}_\mathcal{X}$-module is locally isomorphic to $\mathcal{O}_X^\tau$. 

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**Theorem 2.4.** [6] Let $\Lambda$ be a smooth Lagrangian submanifold of $\mathfrak{X}$. There is a $k$-equivalence of stacks

\begin{equation}
\text{Mod}(W_{\mathfrak{X}}|_{\Lambda})_{\text{reg-}\Lambda} \simeq \text{Mod}(k_{\Lambda} \otimes_{C} C_{\Lambda,1/2})_{\text{loc-sys}}.
\end{equation}

Here, the left-hand side is the subcategory of $\text{Mod}(W_{\mathfrak{X}}|_{\Lambda})$ consisting of regular holonomic modules along $\Lambda$ and the right-hand side is the subcategory of the category of twisted sheaves of $k_{\Lambda}$-modules with twist $C_{\Lambda,1/2}$ consisting of objects locally isomorphic to local systems over $k$. The proof uses the corresponding theorem for contact manifolds due to Kashiwara.

**Theorem 2.5.** [13] Let $\Lambda_i$ ($i = 0, 1$) be two smooth Lagrangian submanifolds of $\mathfrak{X}$ and let $L_i$ be a regular holonomic module along $\Lambda_i$. Then $\mathcal{R}\text{Hom}_{W_{\mathfrak{X}}}(L_0, L_1)$ is $k$-perverse.

We conjecture that the hypothesis that the $\Lambda_i$ are smooth can be deleted.
Operations on $\mathcal{W}_\mathcal{X}$-modules

- There is a natural equivalence of algebroid stacks $\mathcal{W}_\mathcal{X}^{op} \simeq \mathcal{W}_\mathcal{X}$.

- Let $\mathcal{X}$ and $\mathcal{Y}$ be two complex symplectic manifolds and let $\mathcal{M} \in \text{D}^b(\mathcal{W}_\mathcal{X})$, $\mathcal{N} \in \text{D}^b(\mathcal{W}_\mathcal{Y})$. Their exterior product

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{W}_{\mathcal{X} \times \mathcal{Y}} \boxtimes \mathcal{W}_\mathcal{X} \boxtimes \mathcal{W}_\mathcal{Y} (\mathcal{M} \boxtimes \mathcal{N})$$

is well defined in $\text{D}^b(\mathcal{W}_{\mathcal{X} \times \mathcal{Y}})$.

- Denote by $\Delta_\mathcal{X}$ the diagonal of $\mathcal{X} \times \mathcal{X}^a$. There is a canonical simple $\mathcal{W}_\mathcal{X} \times \mathcal{X}^a$-module $C_{\Delta_\mathcal{X}}$ along $\Delta_\mathcal{X}$.

- Let $\mathcal{M} \in \text{D}^b(\mathcal{W}_\mathcal{X})$. One sets

$$D'_w \mathcal{M} := R\text{Hom}_{\mathcal{W}_\mathcal{X}}(\mathcal{M}, C_{\Delta_\mathcal{X}}), \quad D_w \mathcal{M} := D'_w \mathcal{M} \left[ \frac{1}{2} d_\mathcal{X} \right].$$

($d_\mathcal{X}$ is the complex dimension of $\mathcal{X}$.)

- Let $\mathcal{M}, \mathcal{N} \in \text{D}^b_{\text{coh}}(\mathcal{W}_\mathcal{X})$. There is a natural isomorphism

$$R\text{Hom}_{\mathcal{W}_\mathcal{X}}(\mathcal{M}, \mathcal{N}) \simeq R\text{Hom}_{\mathcal{W}_{\mathcal{X} \times \mathcal{X}^a}}(\mathcal{M} \boxtimes D'_w \mathcal{N}, C_{\Delta_\mathcal{X}}).$$
3 Finiteness

**Good \(W\)-modules**

The following definition adapt to \(W\)-modules a definition of Kashiwara [10] for \(D\)-modules.

**Definition 3.1.** (i) A coherent \(W_X\)-module \(M\) is good if, for any open relatively compact subset \(U\) of \(X\), there exists a coherent \(W_X(0)|_U\)-module \(M_0\) contained in \(M|_U\) which generates \(M|_U\).

(ii) One denotes by \(\text{Mod}_{gd}(W_X)\) the full subcategory of \(\text{Mod}_{coh}(W_X)\) consisting of good modules.

(iii) One denotes by \(\text{D}_{gd}^b(W_X)\) the full subcategory of \(\text{D}_{coh}^b(W_X)\) consisting of objects \(M\) such that \(H^j(M)\) is good for all \(j \in \mathbb{Z}\).

(iv) One denotes by \(\text{D}_{gd,c}^b(W_X)\) the full subcategory of \(\text{D}_{gd}^b(W_X)\) consisting of objects with compact supports.

One proves that the category \(\text{Mod}_{gd}(W_X)\) is thick in \(\text{Mod}_{coh}(W_X)\) and one deduces that the full subcategory \(\text{D}_{gd}^b(W_X)\) of \(\text{D}_{coh}^b(W_X)\) is triangulated.
Main theorem

Consider three complex symplectic manifolds $\mathcal{X}_i$ ($i = 1, 2, 3$) and denote as usual by $p_i$ and $p_{ji}$ the projections defined on $\mathcal{X}_3 \times \mathcal{X}_2 \times \mathcal{X}_1$.

For $\Lambda_i$ a closed subset of $\mathcal{X}_{i+1} \times \mathcal{X}_i$ ($i = 1, 2$), we set

$$\Lambda_2 \circ \Lambda_1 := p_{31}(p_{32}^{-1}\Lambda_2 \cap p_{21}^{-1}\Lambda_1).$$

For $\mathcal{K}_i \in D_{gd}^b(W_{\mathcal{X}_{i+1} \times \mathcal{X}_i})$ ($1 \leq i \leq 2$), we set

$$\mathcal{K}_2 \circ \mathcal{K}_1 := Rp_{31}(p_{32}^{-1}\mathcal{K}_2 \otimes_{p_2^{-1}W_{\mathcal{X}_3}} p_{21}^{-1}\mathcal{K}_1).$$

**Theorem 3.2.** [19] Assume that $p_{31}$ is proper on $p_{32}^{-1}\text{supp}(\mathcal{K}_2) \cap p_{21}^{-1}\text{supp}(\mathcal{K}_1)$.

Then

(i) the object $\mathcal{K}_2 \circ \mathcal{K}_1$ belongs to $D_{gd}^b(W_{\mathcal{X}_3 \times \mathcal{X}_1})$,

(ii) there is a natural isomorphism in $D_{gd}^b(W_{\mathcal{X}_3 \times \mathcal{X}_1})$:

$$(3.1) \quad D_w\mathcal{K}_2 \circ D_w\mathcal{K}_1 \cong D_w(\mathcal{K}_2 \circ \mathcal{K}_1).$$
Application: Calabi-Yau category

Choosing $\mathfrak{X}_3 = \mathfrak{X}_1 = \{\text{pt}\}$ in the main theorem, and using

$$\text{RHom}_{W_{x}}(\mathcal{M}, \mathcal{N}) \simeq D'_{w} M \otimes_{W_{x}} N$$

we get:

**Theorem 3.3.** Let $\mathfrak{X}$ be a complex symplectic manifold and let $\mathcal{M}$ and $\mathcal{N}$ be two objects of $D_{\text{gd}}^b(W_{x})$. Assume that $\text{supp} (\mathcal{M}) \cap \text{supp} (\mathcal{N})$ is compact. Then

(i) the object $\text{RHom}_{W_{x}}(\mathcal{M}, \mathcal{N})$ belongs to $D_{f}^b(k)$, (i.e., has finite dimensional cohomology),

(ii) there is a natural isomorphism, functorial with respect to $\mathcal{M}$ and $\mathcal{N}$:

$$\text{RHom}_{W_{x}}(\mathcal{M}, \mathcal{N}) \simeq (\text{RHom}_{W_{x}}(\mathcal{N}, \mathcal{M}[d_{x}]))^*.$$  

**Corollary 3.4.** Let $\mathfrak{X}$ be a complex symplectic manifold. Then $D_{\text{gd,c}}^b(W_{x})$ is a Calabi-Yau category of dimension $d_{x}$. 
Sketch of proof

(i) FINITENESS:
For simplicity, we treat the absolute case and assume $X$ is compact. Using the hypothesis that $\mathcal{M}$ and $\mathcal{N}$ are good, we work with coherent $\mathcal{W}_X(0)$-modules $\mathcal{M}_0$ and $\mathcal{N}_0$. We first represent $\mathcal{M}_0$ and $\mathcal{N}_0$ by complexes whose components are finite direct sums of sheaves of the form $\mathcal{W}_X(0)_U$, ($U$ an open Stein subset of $X$) and morphisms are $\mathcal{W}_X(0)$-linear. Then $\text{RHom}_{\mathcal{W}_X(0)}(\mathcal{M}_0, \mathcal{N}_0)$ is represented by a complex

$$L^\bullet(\mathcal{U}) := \cdots \to \bigoplus_{U_n \in \mathcal{U}_n} \mathcal{W}_X(0)_{U_n} \xrightarrow{d_n} \cdots \xrightarrow{d_1} \bigoplus_{U_0 \in \mathcal{V}_0} \mathcal{W}_X(0)_{U_0} \to 0$$

with $k_0$-linear differential and the $\mathcal{U}_p$ are finite open coverings of $X$. Moreover, we may construct a similar complex $L^\bullet(\mathcal{V})$ such that, for each $n$, $\mathcal{V}_n$ is a refinement of $\mathcal{U}_n$, each $V \in \mathcal{V}_n$ is relatively compact in some $U \in \mathcal{U}_n$, and $L^\bullet(\mathcal{U}) \to L^\bullet(\mathcal{V})$ is a qis. Then the finiteness over $k_0$ of the cohomology follows from the fact that the inclusion morphisms $\mathcal{W}_X(0)(U_n) \to \mathcal{W}_X(0)(V_n)$ are $k_0$-nuclear, using a theorem of [8].

(ii) DUALITY:
To construct the duality morphism, one first constructs an isomorphism

$$k_X[d_X] \simeq C_{\Delta_X} \otimes_{\mathcal{W}_X \times \mathcal{X}} C_{\Delta_X}.$$  

REMARK: one has to work with $k_0$, not $k$, because $k_0$ is a “multiplicatively convex” algebra [8], that is, for each bounded set $B \subset k_0$ there exists a constant $c > 0$ and a convex circled bounded set $B'$ such that $B \subset c \cdot B'$ and $B' \cdot B' \subset B'$. 

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Application: tensor category of kernels

Let $\mathcal{Y}$ be a complex symplectic manifold and let

$$\mathfrak{X} := \mathcal{Y} \times \mathcal{Y}^a.$$  

(3.2)

Consider a family $\mathcal{S}$ of closed subsets of $\mathfrak{X} = \mathcal{Y} \times \mathcal{Y}$ with the following properties:

- (i) if $A_1, A_2 \in \mathcal{S}$ then $A_1 \cup A_2 \in \mathcal{S}$,
- (ii) the diagonal $\Delta_{\mathcal{Y}}$ belongs to $\mathcal{S}$,
- (iii) if $A_1, A_2 \in \mathcal{S}$ then $p_{13} : (p_{32}^{-1}A_2 \cap p_{21}^{-1}A_1) \to \mathfrak{X}$ is proper.

Denote by $D_{gd,S}(W_{\mathfrak{X}})$ the full triangulated subcategory of $D_{gd}(W_{\mathfrak{X}})$ consisting of objects with support in $\mathcal{S}$. By the main theorem, for $K_1, K_2 \in D_{gd,S}(W_{\mathfrak{X}})$, $K_2 \circ K_1$ is well defined in $D_{gd,S}(W_{\mathfrak{X}})$.

**Theorem 3.5.** The category $D_{gd,S}(W_{\mathfrak{X}})$ endowed with the product $\circ$ is a tensor category and the object $\mathcal{C}_{\Delta_{\mathcal{Y}}}$ is a unit.

The preceding construction may be generalized as follows. Consider a Lagrangian subvariety $C-D-W$ satisfying

$$\Lambda \subset \mathfrak{X}^a \times \mathfrak{X}^a \times \mathfrak{X}.$$  

(3.3)

The projection $p_3$ is proper on $\Lambda$.

Note that such Lagrangian varieties appear naturally in the study of symplectic groupoids (see for example [4]).

**Definition 3.6.** Let $\mathcal{K} \in D_{gd}(W_{\mathfrak{X} \times \mathfrak{X}^a \times \mathfrak{X}^a})$ supported by $\Lambda$. For $\mathcal{M}, \mathcal{N} \in D_{gd}(W_{\mathfrak{X}})$ we set:

$$\mathcal{M} * \mathcal{N} := Rp_3!(\mathcal{K} \otimes_{p_{12}} p_{31}^{-1}(\mathcal{M} \boxtimes \mathcal{N})).$$

Applying the main theorem, we obtain:

**Proposition 3.7.** The product $*$ defines a functor

$$\star : D_{gd}(W_{\mathfrak{X}}) \times D_{gd}(W_{\mathfrak{X}}) \to D_{gd}(W_{\mathfrak{X}}).$$
4 Symplectic Riemann-Roch

Euler class of $\mathcal{W}$-modules

Let $\mathfrak{X}$ be complex symplectic manifold and let $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{W}_\mathfrak{X})$. We have the chain of morphisms

$$R\text{Hom}_{\mathcal{W}_\mathfrak{X}}(\mathcal{M}, \mathcal{M}) \xrightarrow{\sim} R\text{Hom}_{\mathcal{W}_\mathfrak{X}}(\mathcal{M}, \mathcal{C}_{\Delta_x}) \otimes_{\mathcal{W}_\mathfrak{X}} \mathcal{M}$$

$$\simeq (R\text{Hom}_{\mathcal{W}_\mathfrak{X}}(\mathcal{M}, \mathcal{C}_{\Delta_x}) \otimes_{k_x} \mathcal{M}) \otimes_{\mathcal{W}_{\mathfrak{X} \times x}} \mathcal{C}_{\Delta_x}$$

$$\rightarrow \mathcal{C}_{\Delta_x} \otimes_{\mathcal{W}_{\mathfrak{X} \times x}} \mathcal{C}_{\Delta_x} \simeq k_x [d_x].$$

Let $V := \text{supp}(\mathcal{M})$. We get a map

$$\text{Hom}_{\mathcal{W}_\mathfrak{X}}(\mathcal{M}, \mathcal{M}) \rightarrow H^{dx}_V(\mathfrak{X}; k_x).$$

The image of $\text{id}_\mathcal{M}$ gives an element

$$\text{Eu}(\mathcal{M}) \in H^{dx}_V(\mathfrak{X}; k_x).$$

We call the class $\text{Eu}(\mathcal{M})$ in (4.1) the Euler class of $\mathcal{M}$. 
Index theorem

We consider the situation of the main theorem. Hence, we have three complex symplectic manifolds \( \mathfrak{X}_i \) \((i = 1, 2, 3)\) and we have closed subsets \( \Lambda_i \) of \( \mathfrak{X}_{i+1} \times \mathfrak{X}_i \) \((i = 1, 2)\). We set for short \( d_i := d_{\mathfrak{X}_i} \) \((i = 1, 2, 3)\). Let \( \lambda_i \in H^{d_i}_{\Lambda_i} (\mathfrak{X}_{i+1} \times \mathfrak{X}_i; k_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i}) \). Assuming that \( p_{31} \) is proper on \( p_{32}^{-1} (\Lambda_2) \cap p_{12}^{-1} (\Lambda_1) \), we set

\[
\Lambda_2 \circ \Lambda_1 := p_{31} (p_{32}^{-1} (\Lambda_2) \cap p_{12}^{-1} (\Lambda_1)),
\]

\[
\lambda_2 \circ \lambda_1 := \int_{\mathfrak{X}_2} (p_{32}^{-1} \lambda_2 \cup p_{21}^{-1} \lambda_1) \in H^{d_3 + d_1}_{\Lambda_2 \circ \Lambda_1} (\mathfrak{X}_3 \times \mathfrak{X}_1; k_{\mathfrak{X}_3 \times \mathfrak{X}_1}).
\]

Here, \( \cup \) is the cup product and

\[
\int_{\mathfrak{X}_2} : H^{d_3 + 2d_2 + d_1}_{p_{32}^{-1} \Lambda_2 \cap p_{21}^{-1} \Lambda_1} (\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1; k_{\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1}) \to H^{d_3 + d_1}_{\Lambda_2 \circ \Lambda_1} (\mathfrak{X}_3 \times \mathfrak{X}_1; k_{\mathfrak{X}_3 \times \mathfrak{X}_1})
\]

is the Poincaré integration morphism.

**Conjecture 4.1.** [19] Let \( \mathcal{K}_i \in D_{\text{gd}}^b (\mathcal{W}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i}) \) \((1 \leq i \leq 2)\) and assume that \( p_{31} \) is proper on \( p_{32}^{-1} \text{supp}(\mathcal{K}_2) \cap p_{12}^{-1} \text{supp}(\mathcal{K}_1) \). Then

\[
\text{Eu}(\mathfrak{X}; \mathcal{K}_2 \circ \mathcal{K}_1) = \text{Eu}(\mathcal{K}_2) \circ \text{Eu}(\mathcal{K}_1).
\]

As a particular case, one finds that for two objects \( \mathcal{L} \) and \( \mathcal{M} \) in \( D_{\text{gd}}^b (\mathcal{W}_{\mathfrak{X}}) \) such that \( \text{supp} \mathcal{L} \cap \text{supp} \mathcal{M} \) is compact, we have

\[
\chi (R \Gamma (\mathfrak{X}; \mathcal{L} \otimes_{\mathcal{W}_{\mathfrak{X}}} \mathcal{M})) = \int_{\mathfrak{X}} \text{Eu} (\mathcal{L}) \cup \text{Eu} (\mathcal{M}).
\]

A natural question would be to compute \( \text{Eu}(\mathcal{L}) \) in terms of the Chern character of \( \text{gr}(\mathcal{L}) \), a coherent module over \( \mathcal{O}_{\mathfrak{X}}[\tau^{-1}, \tau] \).

In case of coherent \( \mathcal{D}_{\mathfrak{X}} \)-modules on a complex manifold \( \mathfrak{X} \), the formula

\[
\text{Eu}(\mathcal{L}) = \left[ \text{Ch}(\text{gr} \mathcal{L}) \cup \text{Td}(TX) \right]^{d_{\mathfrak{X}}}_{d_{\mathfrak{X}}}
\]

had been conjectured in [18] and proved by [3]. A natural question would be to find a similar formula in the framework of symplectic manifolds.
References


