Finiteness and duality on complex symplectic manifolds

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Abstract

For a complex compact manifold X, denote by $\mathcal{T}$ the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$.

- This category is a $\mathbb{C}$-triangulated category,
- this category is Ext-finite, that is,

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(F, G\,[n])$$

  is finite dimensional for any $F, G \in \mathcal{T}$,
- this category admits a Serre functor $S(\,\bullet\,)$ (see [1]), that is,

$$(\mathrm{Hom}_{\mathcal{T}}(F,G))^* \simeq \mathrm{Hom}_{\mathcal{T}}(G, S(F))$$

  where * is the duality functor for $\mathbb{C}$-vector spaces, and $S(F) = F \otimes \Omega_X[d_X]$.

By analogy with this situation, for a field $\mathbf{k}$, a $\mathbf{k}$-triangulated category $\mathcal{T}$ is said to be a Calabi-Yau category of dimension d if $\mathcal{T}$ is Ext-finite, admits a Serre functor and this Serre functor is a shift by d.

Here, we shall consider a complex symplectic manifold $\mathfrak{X}$. The natural base field is now $\widehat{\mathbf{k}} = \mathbb{C}[[\tau^{-1}, \tau]$ or a subfield $\mathbf{k}$ of $\widehat{\mathbf{k}}$. (Note that $\widehat{\mathbf{k}}$ and $\mathbf{k}$ may be considered as deformation-quantizations of $\mathbb{C}$.)

The manifold $\mathfrak{X}$ is endowed with the $\mathbf{k}$-algebroid stack $\mathcal{W}_{\mathfrak{X}}$ of deformation quantization, a variant of the sheaf of microdifferential operators on a cotangent bundle.

We shall consider the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects whit good cohomology (roughly speaking, coherent modules endowed with a good filtration on compact subsets) and its subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd,c}}(\mathcal{W}_{\mathfrak{X}})$ of objects with compact support.

We shall show that, under a natural properness condition, the composition $\mathcal{K}_2 \circ \mathcal{K}_1$ of two good kernels $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i^a})$ $(i = 1, 2)$ is a good kernel and that this composition commutes with duality.

As a particular case, we obtain that the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd,c}}(\mathcal{W}_{\mathfrak{X}})$ is Calabi-Yau of dimension $[d_{\mathfrak{X}}]$ over the field $\mathbf{k}$, where $d_{\mathfrak{X}}$ is the complex dimension of $\mathfrak{X}$.

Finally, we shall discuss a kind of Riemann-Roch theorem in this framework.

This paper summarizes various joint works with A. D'Agnolo [6], M. Kashiwara [13], P. Polesello [15] and J-P. Schneiders [19].

1 Review on deformation-quantization

The field k.

Let $\widehat{\mathbf{k}} := \mathbb{C}[[\tau^{-1}, \tau]$ be the field of formal Laurent series in τ^{-1}. We consider the filtered subfield $\mathbf{k}$ of $\widehat{\mathbf{k}}$ consisting of series $a = \sum_{-\infty<j\leq m} a_j\tau^j$ ($a_j \in \mathbb{C}$, $m \in \mathbb{Z}$) satisfying:

(1.1) there exist $C > 0$ with $|a_j| \leq C^{-j}(-j)!$ for all $j \leq 0$.

We denote by $\mathbf{k}_0$ the subring of $\mathbf{k}$ consisting of elements of order ≤ 0.

Affine case

When X is affine, one defines the filtered sheaf of $\mathbf{k}$-algebras $\mathcal{W}_{T^*X}$ as follows, a variant of the sheaf of microdifferential operators of Sato-Kashiwara-Kawai [16] (see also [10, 17] for an exposition).

- A section $P \in \mathcal{W}_{T^*X}$ of order $m \in \mathbb{Z}$ on $U \subset T^*X$ is given by its total symbol

  $$\sigma_{\mathrm{tot}}(P)(x; u, \tau) = \sum_{-\infty<j\leq m} p_j(x; u)\tau^j, \quad p_j \in \mathcal{O}_{T^*X}(U), \tag{1.2}$$

  whith the condition:

  $$\begin{cases}\text{for any compact subset } K \text{ of } U \text{ there exists a positive con-}\\ \text{stant } C_K \text{ such that } \sup_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j \leq 0.\end{cases} \tag{1.3}$$

- The total symbol of the product is given by the Leibniz rule:

  $$\sigma_{\mathrm{tot}}(P \circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!}\partial_u^\alpha(\sigma_{\mathrm{tot}}P)\partial_x^\alpha(\sigma_{\mathrm{tot}}Q).$$

Note that

- $\mathbf{k} = \mathcal{W}_{\mathrm{pt}}$,
- There is an embedding $\pi^{-1}\mathcal{D}_X \to \mathcal{W}_{T^*X}$ given by $x_i \mapsto x_i$ and $\partial_{x_i} \mapsto \tau u_i$.
- There is an $\mathbb{C}$-linear isomorphism of rings ${}^t\colon \mathcal{W}_{T^*X} \xrightarrow{\sim} (\mathcal{W}_{T^*X})^{\mathrm{op}}$ which satisfies: ${}^tx_i = x_i, {}^tu_i = u_i, {}^t\tau = -\tau$.

Note that many authors use the parameter $\hbar$ instead of τ^{-1}.

The ring $\mathcal{W}_{T^*X}$

Let $\mathfrak{X}$ be a complex symplectic manifold. We denote by $\mathfrak{X}^a$ the manifold $\mathfrak{X}$ endowed with the opposite symplectic form.

Definition 1.1. A W-ring on $\mathfrak{X}$ is a filtered sheaf of $\mathbf{k}$-algebras $\mathcal{W}_{\mathfrak{X}}$ such that for any $x \in \mathfrak{X}$ there exists an open neighborhood U of x and a symplectic isomorphism $\varphi\colon U \xrightarrow{\sim} V$ with V open in $T^*\mathbb{C}^n$ and an isomorphism of filtered sheaves of $\mathbf{k}$-algebras $\varphi_*\mathcal{W}_{\mathfrak{X}} \xrightarrow{\sim} \mathcal{W}_{T^*\mathbb{C}^n}$.

Note that for a W-ring $\mathcal{W}_{\mathfrak{X}}$:

- the sheaf of rings $\mathcal{W}_{\mathfrak{X}}$ is right and left coherent and Noetherian,
- $\operatorname{gr} \mathcal{W}_{\mathfrak{X}} \simeq \mathcal{O}_{\mathfrak{X}}[\tau^{-1}, \tau]$, in particular $\mathcal{W}_{\mathfrak{X}}(0)/\mathcal{W}_{\mathfrak{X}}(-1) \simeq \mathcal{O}_{\mathfrak{X}}$,
- denoting by $\sigma_0\colon \mathcal{W}_{\mathfrak{X}}(0) \to \mathcal{O}_{\mathfrak{X}}$ the natural map, we have for $P, Q \in \mathcal{W}_{\mathfrak{X}}(0)$, $\sigma_0(\tau[P,Q]) = \{\sigma_0(P), \sigma_0(Q)\}$,
- for any $\mathbf{k}$-algebra automorphism Φ of $\mathcal{W}_{\mathfrak{X}}$, there locally exists an invertible section P of $\mathcal{W}_{\mathfrak{X}}(0)$ such that $\Phi = \operatorname{Ad}(P)$. Moreover, P is unique up to a unique scalar multiple. Hence (denoting as usual by $A^\times$ the subgroup of invertible elements of a ring A):

$$\begin{array}{ccc} \mathcal{W}_{\mathfrak{X}}^\times(0)/\mathbf{k}_0^\times & \xrightarrow[\sim]{\operatorname{Ad}(\cdot)} & \operatorname{Aut}(\mathcal{W}_{\mathfrak{X}}(0)) \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{W}_{\mathfrak{X}}^\times/\mathbf{k}^\times & \xrightarrow[\sim]{\operatorname{Ad}(\cdot)} & \operatorname{Aut}(\mathcal{W}_{\mathfrak{X}}), \end{array}$$

- $(\mathcal{W}_{\mathfrak{X}})^{\mathrm{op}}$ is a $\mathcal{W}$-ring on $\mathfrak{X}^a$.

On a cotangent bundle T^*X one can construct a W-ring $\mathcal{W}_{T^*X}$ endowed with an anti-$\mathbf{k}$-linear anti-automorphism $P \mapsto {}^tP$. The section ${}^tP$ is called the adjoint of P.

However, on a complex symplectic manifold $\mathfrak{X}$ there do not exist W-rings in general.

2 The algebroid $\mathcal{W}_{\mathfrak{X}}$

Stacks and algebroids

See Giraud [7], Breen [2], Kashiwara [9], Kontsevitch [14], D'Agnolo-Polesello [5]. See also [12] for an exposition on stacks.

Consider

- a commutative unital ring $\mathbb{K}$,
- a topological space X,
- an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X,
- for $i \in I$, a sheaf of $\mathbb{K}$-algebras $\mathcal{A}_i$ on U_i,
- for $i, j \in I$, an isomorphism $f_{ij} \colon \mathcal{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{A}_i|_{U_{ij}}$.

The existence of a sheaf of $\mathbb{K}$-algebras $\mathcal{A}$ locally isomorphic to $\mathcal{A}_i$ requires the condition $f_{ij}f_{jk} = f_{ik}$ on triple intersections.
Let us weaken this last condition by assuming that there exist invertible sections $a_{ijk} \in \mathcal{A}_i^\times(U_{ijk})$ satisfying

$$\begin{cases} f_{ij}f_{jk} = \mathrm{Ad}(a_{ijk})f_{ik} \text{ on } U_{ijk}, \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl} \text{ on } U_{ijkl}. \end{cases}$$

We call

$$\{\{\mathcal{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I}\}$$

a $\mathbb{K}$-algebroid descent data on $\mathcal{U}$.

In this case, there exists a $\mathbb{K}$-algebroid stack $\mathcal{A}$ locally equivalent to the algebroids associated with the $\mathcal{A}_i$'s. More precisely, if A is an algebra, denote by A^+ the category with one object and having A as morphisms of this object. Consider the prestack on X given by $U_i \supset U \mapsto (\mathcal{A}_i(U))^+$. Then the the algebroid $\mathcal{A}$ is the stack associated with this prestack.

Although $\mathcal{A}$ is not a sheaf of algebras, modules over $\mathcal{A}$ are well-defined: they are described by pairs $\mathcal{M} = (\{\mathcal{M}_i\}_{i\in I}, \{\xi_{ij}\}_{i,j\in I})$, where $\mathcal{M}_i$ are $\mathcal{A}_i$-modules and $\xi_{ij} \colon {}_{f_{ji}}\mathcal{M}_j|_{U_{ij}} \to \mathcal{M}_i|_{U_{ij}}$ are isomorphisms of $\mathcal{A}_i$-modules satisfying

$$\xi_{ij} \circ \xi_{jk} = \xi_{ik} \circ a_{kji}^{-1}.$$

Here, ${}_{f_{ji}}\mathcal{M}_j$ is the $\mathcal{A}_i$-module deduced from the $\mathcal{A}_j$-module $\mathcal{M}_j|_{U_{ij}}$ by the isomorphism f_{ji}.

Twisted modules on complex manifolds

Let X be a topological space and let $\mathbf{c} \in H^2(X; \mathbb{C}_X^\times)$. By choosing an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X and a 2-cocycle $\{c_{ijk}\}_{i,j,k\in I}$ representing $\mathbf{c}$, one gets a descent data, hence an algebroid stack:

$$\mathbb{C}_{X,\mathbf{c}} := (\{\mathbb{C}_X|_{U_i}\}_i, \{f_{ij} = 1\}_{ij}, \{a_{ijk} = c_{ijk}\}_{ijk}).$$

A twisted sheaf with twist $\mathbf{c}$ is an object of $\mathrm{Mod}(\mathbb{C}_{X,\mathbf{c}})$.

Assume now that X is a complex manifold. Consider the short exact sequence

$$1 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \xrightarrow{d\log} d\mathcal{O}_X \to 0$$

which gives rise to the long exact sequence

$$H^1(X; \mathbb{C}_X^\times) \xrightarrow{\alpha} H^1(X; \mathcal{O}_X^\times) \xrightarrow{\beta} H^1(X; d\mathcal{O}_X) \xrightarrow{\gamma} H^2(X; \mathbb{C}_X^\times)$$

If $\mathcal{L}$ is a line bundle, it defines a class $[\mathcal{L}] \in H^1(X; \mathcal{O}_X^\times)$. For $\lambda \in \mathbb{C}$, one sets

$$\mathbf{c}_{\mathcal{L}}^\lambda = \gamma(\lambda \cdot \beta([\mathcal{L}])) \in H^2(X; \mathbb{C}_X^\times).$$

We shall apply this construction when $\mathcal{L} = \Omega_X$ and $\lambda = \frac{1}{2}$ and set for short:

$$\mathrm{Mod}(\mathbb{C}_{X,\frac{1}{2}}) = \mathrm{Mod}(\mathbb{C}_{X,\mathbf{c}_{\Omega_X}^{\frac{1}{2}}})$$

Quantization of symplectic manifolds

In 2000, M. Kontsevich [14] has proved the existence of a $\widehat{\mathbf{k}}$-algebroid stack $\mathcal{W}_{\mathfrak{X}}$ on any complex Poisson manifold $\mathfrak{X}$ in the formal case. The analytic case for symplectic manifolds has been treated with a different proof in [15]. Indeed, the story began in 1996 when M. Kashiwara [9] proved the existence of a canonical $\mathbb{C}$-algebroid stack $\mathcal{E}_{\mathfrak{Y}}$ over any complex contact manifold $\mathfrak{Y}$ locally equivalent to the stack associated with the sheaf of microdifferential operators of Sato-Kashiwara-Kawai [16] on the projective cotangent bundle P^*X to a complex manifold X.

In particular, for a complex symplectic manifold $\mathfrak{X}$, there is a canonical $\mathbf{k}$-algebroid stack $\mathcal{W}_{\mathfrak{X}}^{+}$ locally equivalent to the algebroid stack associated with the sheaf of algebras $\mathcal{W}_{T^*X}$. (The same result holds with $\mathcal{W}_{\mathfrak{X}}$ replaced with $\mathcal{W}_{\mathfrak{X}}(0)$ and $\mathbf{k}$ with $\mathbf{k}_0$.)

Notation 2.1. For short, as far as there is no risk of confusion, we shall write $\mathcal{W}_{\mathfrak{X}}$ instead of $\mathcal{W}_{\mathfrak{X}}^{+}$.

Let $\mathfrak{X}$ be a complex symplectic manifold. Then $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})$ is a well-defined Grothendieck abelian category and admits a bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$.

One proves as usual that the sheaf of algebras $\mathcal{W}_{T^*X}$ is coherent and the support of a coherent $\mathcal{W}_{T^*X}$-module is a closed complex analytic subvariety of T^*X. This support is involutive in view of Gabber's theorem. Hence, the (local) notions of a coherent or a holonomic $\mathcal{W}_{\mathfrak{X}}$-module make sense. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with coherent cohomologies, and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with Lagrangian supports in $\mathfrak{X}$,

Simple holonomic $\mathcal{W}_{\mathfrak{X}}$-modules

Definition 2.2. Let $\mathcal{L}$ be a coherent $\mathcal{W}_{\mathfrak{X}}$-module supported by a closed analytic Lagrangian variety Λ of $\mathfrak{X}$.

(a) Assume Λ is smooth. One says that $\mathcal{L}$ is simple along Λ if there exists locally a coherent $\mathcal{W}_{\mathfrak{X}}(0)$-submodule $\mathcal{L}(0)$ of $\mathcal{L}$ such that $\mathcal{L}(0)$ generates $\mathcal{L}$ over $\mathcal{W}_{\mathfrak{X}}$ and $\mathcal{L}(0)/\mathcal{L}(-1)$ is an invertible $\mathcal{O}_{\Lambda}$-module.

(b) A coherent $\mathcal{W}_{\mathfrak{X}}$-module supported by Λ is regular if it is locally a finite direct sum of simple modules at generic points of Λ.

It follows from Gabber's theorem (see [10, Th. 7.34]) that when Λ is smooth, a regular $\mathcal{W}_{\mathfrak{X}}$-module is locally a finite direct sum of simple modules.

Example 2.3. Let X be a complex manifold. We denote by $\mathcal{O}_X^{\tau}$ the $\mathcal{W}_{T^*X}$-module supported by the zero-section T_X^*X defined by $\mathcal{O}_X^{\tau} = \mathcal{W}_{T^*X}/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by the vector fields which annihilate the section $1 \in \mathcal{O}_X$. A section $f(x,\tau)$ of this module may be written as a series:

$$f(x,\tau) = \sum_{-\infty < j \le m} f_j(x)\tau^j, \quad m \in \mathbb{Z}, \tag{2.1}$$

the f_j's satisfying Condition (1.3). Then $\mathcal{O}_X^{\tau}$ is a simple $\mathcal{W}_{T^*X}$-module along T_X^*X.

One proves easily that any two $\mathcal{W}_{\mathfrak{X}}$-modules simple along Λ are locally isomorphic. Moreover, a smooth Lagrangian submanifold is locally isomorphic to the zero section X of T^*X. Hence, any simple $\mathcal{W}_{\mathfrak{X}}$-module is locally isomorphic to $\mathcal{O}_X^{\tau}$.

Theorem 2.4. [6] *Let* Λ *be a smooth Lagrangian submanifold of* $\mathfrak{X}$*. There is a* $\mathbf{k}$*-equivalence of stacks*

$$\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}}|_{\Lambda})_{\textit{reg-}\Lambda} \simeq \mathrm{Mod}(\mathbf{k}_{\Lambda} \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda,1/2})_{\textit{loc-sys}}. \tag{2.2}$$

Here, the left-hand side is the subcategory of $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}}|_{\Lambda})$ consisting of regular holonomic modules along Λ and the right-hand side is the subcategory of the category of twisted sheaves of $\mathbf{k}_{\Lambda}$-modules with twist $\mathbb{C}_{\Lambda,1/2}$ consisting of objects locally isomorphic to local systems over $\mathbf{k}$. The proof uses the corresponding theorem for contact manifolds due to Kashiwara.

Theorem 2.5. [13] *Let* Λ_i *(*$i = 0, 1$*) be two smooth Lagrangian submanifolds of* $\mathfrak{X}$ *and let* $\mathcal{L}_i$ *be a regular holonomic module along* Λ_i*. Then* $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$ *is* $\mathbf{k}$*-perverse.*

We conjecture that the hypothesis that the Λ_i are smooth can be deleted.

Operations on $\mathcal{W}_{\mathfrak{X}}$-modules

- There is a natural equivalence of algebroid stacks $\mathcal{W}_{\mathfrak{X}}^{\mathrm{op}} \simeq \mathcal{W}_{\mathfrak{X}^a}$.
- Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex symplectic manifolds and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{Y}})$. Their exterior product

$$\mathcal{M} \underline{\boxtimes} \mathcal{N} := \mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}} \boxtimes_{\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}}} (\mathcal{M} \boxtimes \mathcal{N})$$

  is well defined in $\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}})$.
- Denote by $\Delta_{\mathfrak{X}}$ the diagonal of $\mathfrak{X} \times \mathfrak{X}^a$. There is a canonical simple $\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}$-module $\mathcal{C}_{\Delta_{\mathfrak{X}}}$ along $\Delta_{\mathfrak{X}}$.
- Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$. One sets

$$\mathrm{D}'_{\mathrm{w}}\mathcal{M} := R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{C}_{\Delta_{\mathfrak{X}}}), \qquad \mathrm{D}_{\mathrm{w}}\mathcal{M} := \mathrm{D}'_{\mathrm{w}}\mathcal{M}\,[\frac{1}{2}d_{\mathfrak{X}}].$$

  ($d_{\mathfrak{X}}$ is the complex dimension of $\mathfrak{X}$.)
- Let $\mathcal{M}, \mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$. There is a natural isomorphism

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N}) \;\simeq\; R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}}(\mathcal{M} \underline{\boxtimes} \mathrm{D}'_{\mathrm{w}}\mathcal{N}, \mathcal{C}_{\Delta_{\mathfrak{X}}}).$$

3 Finiteness

Good $\mathcal{W}$-modules

The following definition adapt to $\mathcal{W}$-modules a definition of Kashiwara [10] for $\mathcal{D}$-modules.

Definition 3.1. (i) A coherent $\mathcal{W}_{\mathfrak{X}}$-module $\mathcal{M}$ is good if, for any open relatively compact subset U of $\mathfrak{X}$, there exists a coherent $\mathcal{W}_{\mathfrak{X}}(0)|_U$-module $\mathcal{M}_0$ contained in $\mathcal{M}|_U$ which generates $\mathcal{M}|_U$.

(ii) One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ consisting of good modules.

(iii) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects $\mathcal{M}$ such that $H^j(\mathcal{M})$ is good for all $j \in \mathbb{Z}$.

(iv) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd,c}}(\mathcal{W}_{\mathfrak{X}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with compact supports.

One proves that the category $\mathrm{Mod}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ is thick in $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ and one deduces that the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ is triangulated.

Main theorem

Consider three complex symplectic manifolds $\mathfrak{X}_i$ $(i = 1, 2, 3)$ and denote as usual by p_i and p_{ji} the projections defined on $\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1$.

For Λ_i a closed subset of $\mathfrak{X}_{i+1} \times \mathfrak{X}_i$ $(i = 1, 2)$, we set

$$\Lambda_2 \circ \Lambda_1 := p_{31}(p_{32}^{-1}\Lambda_2 \cap p_{21}^{-1}\Lambda_1).$$

For $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}_{i+1}\times\mathfrak{X}_i^a})$ $(1 \leq i \leq 2)$, we set

$$\mathcal{K}_2 \circ \mathcal{K}_1 := Rp_{31!}(p_{32}^{-1}\mathcal{K}_2 \overset{\mathrm{L}}{\otimes}_{p_2^{-1}\mathcal{W}_{\mathfrak{X}_2}} p_{21}^{-1}\mathcal{K}_1).$$

Theorem 3.2. [19] *Assume that p_{31} is proper on $p_{32}^{-1}\,\mathrm{supp}(\mathcal{K}_2)\cap p_{12}^{-1}\,\mathrm{supp}(\mathcal{K}_1)$. Then*

(i) *the object $\mathcal{K}_2 \circ \mathcal{K}_1$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}_3\times\mathfrak{X}_1^a})$,*

(ii) *there is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}_3^a\times\mathfrak{X}_1})$:*

$$\mathrm{D}_{\mathrm{w}}\mathcal{K}_2 \circ \mathrm{D}_{\mathrm{w}}\mathcal{K}_1 \xrightarrow{\sim} \mathrm{D}_{\mathrm{w}}(\mathcal{K}_2 \circ \mathcal{K}_1). \tag{3.1}$$

Application: Calabi-Yau category

Choosing $\mathfrak{X}_3 = \mathfrak{X}_1 = \{\mathrm{pt}\}$ in the main theorem, and using

$$\mathrm{RHom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N}) \simeq \mathrm{D}'_{\mathrm{w}}\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}}}\mathcal{N}$$

we get:

Theorem 3.3. *Let* $\mathfrak{X}$ *be a complex symplectic manifold and let* $\mathcal{M}$ *and* $\mathcal{N}$ *be two objects of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$. *Assume that* $\mathrm{supp}(\mathcal{M})\cap\mathrm{supp}(\mathcal{N})$ *is compact. Then*

(i) *the object* $\mathrm{RHom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{f}(\mathbf{k})$, (i.e., *has finite dimensional cohomology*),

(ii) *there is a natural isomorphism, functorial with respect to* $\mathcal{M}$ *and* $\mathcal{N}$*:*

$$\mathrm{RHom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N}) \simeq \big(\mathrm{RHom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{N},\mathcal{M}\,[d_{\mathfrak{X}}])\big)^{\star}.$$

Corollary 3.4. *Let* $\mathfrak{X}$ *be a complex symplectic manifold. Then* $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd,c}}(\mathcal{W}_{\mathfrak{X}})$ *is a Calabi-Yau category of dimension* $d_{\mathfrak{X}}$.

Sketch of proof

(i) FINITENESS:
For simplicity, we treat the absolute case and assume $\mathfrak{X}$ is compact. Using the hypothesis that $\mathcal{M}$ and $\mathcal{N}$ are good, we work with coherent $\mathcal{W}_{\mathfrak{X}}(0)$-modules $\mathcal{M}_0$ and $\mathcal{N}_0$. We first represent $\mathcal{M}_0$ and $\mathcal{N}_0$ by complexes whose components are finite direct sums of sheaves of the form $\mathcal{W}_{\mathfrak{X}}(0)_U$, ($U$ an open Stein subset of $\mathfrak{X}$) and morphisms are $\mathcal{W}_{\mathfrak{X}}(0)$-linear. Then $\mathrm{RHom}_{\mathcal{W}_{\mathfrak{X}}(0)}(\mathcal{M}_0, \mathcal{N}_0)$ is represented by a complex

$$L^\bullet(\mathcal{U}) := \cdots \to \bigoplus_{U_n \in \mathcal{U}_n} \mathcal{W}_{\mathfrak{X}}(0)_{U_n} \xrightarrow{d_n} \cdots \xrightarrow{d_1} \bigoplus_{U_0 \in \mathcal{V}_0} \mathcal{W}_{\mathfrak{X}}(0)_{U_0} \to 0$$

with $\mathbf{k}_0$-linear differential and the $\mathcal{U}_p$ are finite open coverings of $\mathfrak{X}$. Moreover, we may construct a similar complex $L^\bullet(\mathcal{V})$ such that, for each n, $\mathcal{V}_n$ is a refinement of $\mathcal{U}_n$, each $V \in \mathcal{V}_n$ is relatively compact in some $U \in \mathcal{U}_n$, and $L^\bullet(\mathcal{U}) \to L^\bullet(\mathcal{V})$ is a qis. Then the finiteness over $\mathbf{k}_0$ of the cohomology follows from the fact that the inclusion morphisms $\mathcal{W}_{\mathfrak{X}}(0)(U_n) \to \mathcal{W}_{\mathfrak{X}}(0)(V_n)$ are $\mathbf{k}_0$-nuclear, using a theorem of [8].
(ii) DUALITY:
To construct the duality morphism, one first constructs an isomorphism

$$\mathbf{k}_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq \mathcal{C}_{\Delta_{\mathfrak{X}^a}} \overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X} \times \mathfrak{X}^a}} \mathcal{C}_{\Delta_{\mathfrak{X}}}.$$

REMARK: one has to work with $\mathbf{k}_0$, not $\mathbf{k}$, because $\mathbf{k}_0$ is a "multiplicatively convex" algebra [8], that is, for each bounded set $B \subset \mathbf{k}_0$ there exists a constant $c > 0$ and a convex circled bounded set B' such that $B \subset c \cdot B'$ and $B' \cdot B' \subset B'$.

Application: tensor category of kernels

Let $\mathfrak{Y}$ be a complex symplectic manifold and let

$$\mathfrak{X} := \mathfrak{Y} \times \mathfrak{Y}^a. \tag{3.2}$$

Consider a family $\mathcal{S}$ of closed subsets of $\mathfrak{X} = \mathfrak{Y} \times \mathfrak{Y}$ with the following properties:

$$\begin{cases} \text{(i) if } A_1, A_2 \in \mathcal{S} \text{ then } A_1 \cup A_2 \in \mathcal{S}, \\ \text{(ii) the diagonal } \Delta_{\mathfrak{Y}} \text{ belongs to } \mathcal{S}, \\ \text{(iii) if } A_1, A_2 \in \mathcal{S} \text{ then } p_{13} \colon (p_{32}^{-1} A_2 \cap p_{21}^{-1} A_1) \to \mathfrak{X} \text{ is proper.} \end{cases}$$

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\mathcal{S}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with support in $\mathcal{S}$. By the main theorem, for $K_1, K_2 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\mathcal{S}}(\mathcal{W}_{\mathfrak{X}})$, $K_2 \circ K_1$ is well defined in $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\mathcal{S}}(\mathcal{W}_{\mathfrak{X}})$.

Theorem 3.5. *The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\mathcal{S}}(\mathcal{W}_{\mathfrak{X}})$ endowed with the product $\circ$ is a tensor category and the object $\mathcal{C}_{\Delta_{\mathfrak{Y}}}$ is a unit.*

The preceding construction may be generalized as follows. Consider a Lagrangian subvariety C-D-W

$$\Lambda \subset \mathfrak{X}^a \times \mathfrak{X}^a \times \mathfrak{X}.$$

satisfying

$$\text{The projection } p_3 \text{ is proper on } \Lambda. \tag{3.3}$$

Note that such Lagrangian varieties appear naturally in the study of symplectic groupoids (see for example [4]).

Definition 3.6. Let $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a\times\mathfrak{X}^a})$ supported by Λ. For $\mathcal{M}, \mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ we set:

$$\begin{aligned} \mathcal{M} \star \mathcal{N} &:= Rp_{3!}(\mathcal{K} \overset{\mathrm{L}}{\otimes}_{p_{12}^{-1}\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}}} p_{12}^{-1}(\mathcal{M} \boxtimes \mathcal{N})) \\ &= \mathcal{K} \circ (\mathcal{M} \underline{\boxtimes} \mathcal{N}). \end{aligned}$$

Applying the main theorem, we obtain:

Proposition 3.7. *The product $\star$ defines a functor*

$$\star \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}}) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}}).$$

4 Symplectic Riemann-Roch

Euler class of $\mathcal{W}$-modules

Let $\mathfrak{X}$ be complex symplectic manifold and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$. We have the chain of morphisms

$$\begin{aligned} R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{M}) &\xleftarrow{\sim} R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{C}_{\Delta_{\mathfrak{X}}})\overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}}}\mathcal{M} \\ &\simeq (R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{C}_{\Delta_{\mathfrak{X}}}) \otimes_{\mathbf{k}_{\mathfrak{X}}} \mathcal{M})\overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}}\mathcal{C}_{\Delta_{\mathfrak{X}}} \\ &\to \mathcal{C}_{\Delta_{\mathfrak{X}^a}}\overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}}\mathcal{C}_{\Delta_{\mathfrak{X}}} \simeq \mathbf{k}_{\mathfrak{X}}[d_{\mathfrak{X}}]. \end{aligned}$$

Let $V := \mathrm{supp}(\mathcal{M})$. We get a map

$$\mathrm{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{M}) \to H^{d_{\mathfrak{X}}}_V(\mathfrak{X};\mathbf{k}_{\mathfrak{X}}).$$

The image of $\mathrm{id}_{\mathcal{M}}$ gives an element

$$\mathrm{Eu}(\mathcal{M}) \in H^{d_{\mathfrak{X}}}_V(\mathfrak{X};\mathbf{k}_{\mathfrak{X}}). \tag{4.1}$$

We call the class $\mathrm{Eu}(\mathcal{M})$ in (4.1) the Euler class of $\mathcal{M}$.

Index theorem

We consider the situation of the main theorem. Hence, we have three complex symplectic manifolds $\mathfrak{X}_i$ ($i = 1, 2, 3$) and we have closed subsets Λ_i of $\mathfrak{X}_{i+1} \times \mathfrak{X}_i$ ($i = 1, 2$). We set for short $d_i := d_{\mathfrak{X}_i}$ ($i = 1, 2, 3$). Let $\lambda_i \in H^{d_{i+1}+d_i}_{\Lambda_i}(\mathfrak{X}_{i+1} \times \mathfrak{X}_i; \mathbf{k}_{\mathfrak{X}_{i+1}\times\mathfrak{X}_i})$. Assuming that p_{31} is proper on $p_{32}^{-1}(\Lambda_2) \cap p_{12}^{-1}(\Lambda_1)$, we set

$$\begin{aligned}
\Lambda_2 \circ \Lambda_1 &:= p_{31}(p_{32}^{-1}(\Lambda_2) \cap p_{12}^{-1}(\Lambda_1)),\\
\lambda_2 \circ \lambda_1 &:= \int_{\mathfrak{X}_2} (p_{32}^{-1}\lambda_2 \cup p_{21}^{-1}\lambda_1) \in H^{d_3+d_1}_{\Lambda_2\circ\Lambda_1}(\mathfrak{X}_3 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3\times\mathfrak{X}_1}).
\end{aligned}$$

Here, $\cup$ is the cup product and

$$\int_{\mathfrak{X}_2} : H^{d_3+2d_2+d_1}_{p_{32}^{-1}\Lambda_2\cap p_{21}^{-1}\Lambda_1}(\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3\times\mathfrak{X}_2\times\mathfrak{X}_1}) \to H^{d_3+d_1}_{\Lambda_2\circ\Lambda_1}(\mathfrak{X}_3 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3\times\mathfrak{X}_1})$$

is the Poincaré integration morphism.

Conjecture 4.1. [19] Let $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}_{i+1}\times\mathfrak{X}_i^a})$ ($1 \le i \le 2$) and assume that p_{31} is proper on $p_{32}^{-1}\operatorname{supp}(\mathcal{K}_2) \cap p_{12}^{-1}\operatorname{supp}(\mathcal{K}_1)$. Then

$$\mathrm{Eu}(\mathfrak{X}; \mathcal{K}_2 \circ \mathcal{K}_1) = \mathrm{Eu}(\mathcal{K}_2) \circ \mathrm{Eu}(\mathcal{K}_1).$$

As a particular case, one finds that for two objects $\mathcal{L}$ and $\mathcal{M}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{W}_{\mathfrak{X}})$ such that $\operatorname{supp}\mathcal{L} \cap \operatorname{supp}\mathcal{M}$ is compact, we have

$$\chi(\mathrm{R}\Gamma(\mathfrak{X}; \mathcal{L}\overset{\mathrm{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}}}\mathcal{M})) = \int_{\mathfrak{X}} \mathrm{Eu}(\mathcal{L}) \cup \mathrm{Eu}(\mathcal{M}).$$

A natural question would be to compute $\mathrm{Eu}(\mathcal{L})$ in terms of the Chern character of $\mathrm{gr}(\mathcal{L})$, a coherent module over $\mathcal{O}_{\mathfrak{X}}[\tau^{-1}, \tau]$.

In case of coherent $\mathcal{D}_X$-modules on a complex manifold X, the formula

$$\mathrm{Eu}(\mathcal{L}) = [\mathrm{Ch}(\mathrm{gr}\,\mathcal{L}) \cup \mathrm{Td}(TX)]^{d_{T^*X}}$$

had been conjectured in [18] and proved by [3]. A natural question would be to find a similar formula in the framework of symplectic manifolds.

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