Homological perturbation theory and Homological mirror symmetry: the $\mathbb{R}^2$ case

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Higher Structures in Geometry and Physics

~ In honor of Murray Gerstenhaber’s 80th and Jim Stasheff’s 70th birthday! ~

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**Motivation:** To construct an explicit example of $A_\infty$-structures associated to geometry (Fukaya category of Lagrangian submanifolds) (resolving the transversality problem)

Another motivation: Homological mirror symmetry (Kontsevich’94) of tori (mention at the end of this talk).

**Plan of talk:**

- Def. of $A_\infty$-algebras, $A_\infty$-categories, etc.
- A Fukaya category $Fuk(\mathbb{R}^2)$ on $\mathbb{R}^2$ and the deRham model $C_{DR}(\mathbb{R})$
- Main theorem on the homotopy equivalence between $Fuk(\mathbb{R}^2)$ and $C_{DR}(\mathbb{R})$
- Idea of the proof.
Def. \([A_\infty\text{-algebra (Stasheff’63)}]\)

\((V, m := \{m_n\}_{n \geq 1})\) is an \(A_\infty\)-algebra \(\iff\)

\[V = \bigoplus_{r \in \mathbb{Z}} V^r : \mathbb{Z}\text{-graded vector space,}\]

\[m := \{m_n : V^\otimes n \rightarrow V\}_{n \geq 1} : a\ collection\ of\ degree\ (2 - n)\ multilinear\ maps\ s.t.\]

\[0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm m_k(v_1, \cdots, v_j, \]

\[m_l(v_{j+1}, \cdots, v_{j+l}), v_{j+l+1}, \cdots, v_n),\]

for \(n = 1, 2, \ldots,\) where \(v_i \in V^{\deg v_i}, i = 1, \ldots, n,\) and \(|m_n| = (2 - n)\) implies

\[|m_n(v_1, \ldots, v_n)| = (2 - n) + |v_1| + \cdots + |v_n|.\]
The $A_\infty$-relations for $n = 1, 2, 3$:

for $m_1 = d$, $m_2 = \cdot$, $x, y, z \in V$:

i) \[ d^2 = 0 , \]

ii) \[ d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) , \]

iii) \[ (x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z). \]

i) $\Leftrightarrow$ $(V, d)$ forms a complex.

ii) $\Leftrightarrow$ Leibniz rule of $d$ w.r.t. to product $\cdot$.

iii) $\cdot$ is associative up to homotopy.

In particular, if $m_3 = 0$, the product $\cdot$ is strictly associative. An $A_\infty$-algebra $(V, m)$ with $m_3 = m_4 = \cdots = 0$ is called a differential graded (DG) algebra.
Def. \( [\mathcal{A}_\infty\text{-morphism}] \)

Given two \( \mathcal{A}_\infty\text{-algebras} \) \((V, m)\) and \((V', m')\), an \( \mathcal{A}_\infty\text{-morphism} \) \( f : (V, m) \to (V', m') \) is a collection of degree \((1 - k)\) multilinear maps

\[ f := \{ f_k : V^\otimes k \to V' \}_{k \geq 1} \text{ s.t.} \]

\[
\sum_{i \geq 1} \sum_{k_1 + \cdots + k_n = n} \pm m'_i (f_{k_1} \otimes \cdots \otimes f_{k_i})(v_1, \ldots, v_n)
\]

\[
= \sum_{i+1+j=k} \sum_{i+l+j=n} \pm f_k (1^\otimes i \otimes m_l \otimes 1^\otimes j)(v_1, \ldots, v_n)
\]

for \( n = 1, 2, \ldots \).

**Note:** the above relation for \( n = 1 \) implies 
\( f_1 : V \to V' \) forms a chain map

\[ f_1 : (V, m) \to (V', m'). \]
Def. An $A_\infty$-morphism $f : (V, m) \to (V', m')$ is called an $A_\infty$-quasi-isomorphism iff $f_1 : (V, m_1) \to (V', m'_1)$ induces an isom. on the cohomologies of the two complexes.

Rem. For a given $A_\infty$-quasi-isomorphism $f : (V, m) \to (V', m')$, there always exists an inverse $A_\infty$-quasi-isomorphism

$$f' : (V', m') \to (V, m).$$

Thus, $A_\infty$-quasi-isomorphisms define (homotopy) equivalence between $A_\infty$-algebras.
We need a categorical version of these terminologies.

**Def. \([A_\infty\text{-category (Fukaya’93)}]\)**

An \(A_\infty\text{-category} \mathcal{C} \iff \)

\[ \text{Ob}(\mathcal{C}) = \{a, b, \cdots\} : \text{a set of objects} \]

\[ V_{ab} := \text{Hom}_\mathcal{C}(a, b) : \mathbb{Z}\text{-graded vector space} \]

for \(\forall a, b \in \text{Ob}(\mathcal{C})\)

a collection of multilinear maps

\[ m := \{m_n : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \to V_{a_1a_{n+1}}\}_{n \geq 1} \]

degree \((2 - n)\) defining an \(A_\infty\)-structure.

In particular, \(\mathcal{C}\) with \(m_3 = m_4 = \cdots = 0\) is called a **DG-category**.
Def. Given two $A_\infty$-categories $\mathcal{C}$ and $\mathcal{C}'$, $f := \{f, f_1, f_2, \ldots \} : \mathcal{C} \to \mathcal{C}'$ is an $A_\infty$-functor \iff

\[ f : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}') \text{ a map of objects;} \]

a collection of multilinear maps

\[ f_k : \text{Hom}_\mathcal{C}(a_1, a_2) \otimes \cdots \otimes \text{Hom}_\mathcal{C}(a_k, a_{k+1}) \]

\[ \to \text{Hom}_{\mathcal{C}'}(f(a_1), f(a_{k+1})), \quad k = 1, 2, \ldots \]

degree $(1 - k)$ satisfying the defining equation of an $A_\infty$-morphism.

We call $f$ homotopy equivalence iff $f : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}')$ is bijection and $f_1 : \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_{\mathcal{C}'}(f(a), f(b))$ induces an isom. on the cohomologies for $\forall a, b \in \text{Ob}(\mathcal{C})$. 
Fukaya category and its deRham model

Fukaya category $F_{uk}(\mathbb{R}^2, \mathcal{F}_N)$ for $\mathbb{R}^2$

Let $F_{uk}(\mathbb{R}^2, \mathcal{F}_N)$ be an $A_\infty$-category satisfying the following two conditions:

For a fixed integer $N \geq 2$, let $\{f_1, ..., f_N\}$ be a collection of functions on $\mathbb{R}$ s.t.

$$L_a : y = \frac{df_a}{dx} = t_ax + s_a , \quad t_a, s_a \in \mathbb{R}$$

is a line in $\mathbb{R}^2$ with coordinates $(x, y)$ ($a = 1, ..., N$).

Denote by $\mathcal{F}_N := \{f_1, ..., f_N\} = \{1, ..., N\}$ such a collection satisfying:

- $t_a \neq t_b$ for $\forall a, b \in \mathcal{F}_N$.

- Not more than three lines intersect at the same point in $\mathbb{R}^2$. 

**Condition 1** \( \forall a \neq b \in S_N, \)

- \( V^0_{ab} = \mathbb{R} \cdot [v_{ab}], \quad V^1_{ab} = 0, \quad (t_a < t_b), \)
- \( V^0_{ab} = 0, \quad V^1_{ab} = \mathbb{R} \cdot [v_{ab}], \quad (t_a > t_b). \)

Here, \([v_{ab}]\) is the base of \( V_{ab} \) labeled by the intersection point \( v_{ab}(= v_{ba}) \) of \( L_a \) and \( L_b \).

![Diagram of intersecting lines](image)

**Condition 2 (Transversal \( A_\infty \)-products)**

For a fixed \( n \geq 2 \) and \( a_1, \ldots, a_{n+1} \in S_N \) s.t.

\[
a_j \neq a_k \text{ for } j \neq k = 1, \ldots, n + 1,
\]

\( m_n : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \to V_{a_1a_{n+1}} \) is

\[
m_n([v_{a_1a_2}], \ldots, [v_{a_na_{n+1}}]) = c_{a_1 \cdots a_{n+1}}[v_{a_1a_{n+1}}]
\]
where, if \( \vec{v} := (v_{a_1a_2}, \ldots, v_{a_n,a_{n+1}}, v_{a_{n+1}a_1}) \) forms a clockwise convex \((n + 1)\)-gon,

\[
c_{a_1 \ldots a_k} = \pm e^{-\text{Area}(v)}
\]

for \( \text{Area}(\vec{v}) \) the area of the \((n + 1)\)-gon

and \( c_{a_1 \ldots a_{n+1}} = 0 \) otherwise.

\( m_1 : V_{ab} \to V_{ab} \) is set to be \( m_1 = 0 \ \forall a \neq b. \)
For transversal $A_\infty$-products, the $A_\infty$-relation follows from a polygon which has one nonconvex vertex.

There exist two ways to divide such a polygon into two convex polygons.

In this figure, the ways of dividing the area $X + Y + Z$ into two are

(i) $X + (Y + Z)$ or (ii) $(X + Y) + Z$. 
Corresponding to (i) and (ii) one has

\[(i) : \quad + m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi})
\]
\[= e^{-X-(Y+Z)}v_{ai}, \]

\[(ii) : \quad - m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi})
\]
\[= -e^{-(X+Y)-Z}v_{ai}. \]

Thus, we obtain

\[0 = + m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi})
\]
\[\quad - m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}), \]

which is just one of the \(A_\infty\)-relations.
On the other hand, we define a DG-category $\mathcal{C}_{DR}(\mathbb{R}, \mathcal{F}_N)$ as follows:

**Def. $[\mathcal{C}_{DR}(\mathbb{R}, \mathcal{F}_N)]$**

- $\text{Ob}(\mathcal{C}_{DR}(\mathbb{R}, \mathcal{F}_N)) = \mathcal{F}_N$;

- $\forall a, b \in \mathcal{F}_N$, $\text{Hom}(a, b) = \bigoplus_{r=0,1} \Omega^r_{ab}(\mathbb{R})$, $\Omega^0_{ab} := \mathcal{S}(\mathbb{R})$, $\Omega^1_{ab} := \mathcal{S}(\mathbb{R}) \cdot dx$,

where, $\mathcal{S}(\mathbb{R})$ is the Schwartz space,

and $dx$ is the base of one-form on $\mathbb{R}$;

- a differential $d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}$ by

$$d_{ab} := d - df_{ab} \wedge,$$

where $f_{ab} := f_a - f_b$;

- a product $\Omega^r_{ab} \otimes \Omega^r_{bc} \to \Omega^r_{ac} + r_{bc}$ by the usual wedge product $\wedge$. 

Thm. \exists an \( A_\infty \)-category \( \mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N) \) s.t.

- \( \text{Ob}(\mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N)) = \mathcal{F}_N \);

- \( \text{Hom}_{\mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N)}(a, b) \) satisfies the condition 1 \( \forall a \neq b \in \mathcal{F}_N \);

- the \( A_\infty \)-structure \( \{m_k\}_{k \geq 1} \) of \( \mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N) \) satisfies the condition 2;

- \( \mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N) \) is homotopic to \( \mathcal{C}_{\text{DR}}(\mathbb{R}, \mathcal{F}_N) \) as \( A_\infty \)-categories.

We can prove this by constructing such an \( A_\infty \)-category \( \mathrm{Fuk}(\mathbb{R}^2, \mathcal{F}_N) =: \mathcal{C}(\mathcal{F}_N) \) explicitly based on Kontsevich-Soibelman’00’s proposal combining homological perturbation theory (HPT) and Harvey-Lawson’01’s argument on Morse theory.
Idea of the proof

The construction of the $A_\infty$-category $\mathcal{C}(\mathcal{F}_N)$ is divided into 2 steps.

I. Apply HPT to $\mathcal{C}_{DR}(\mathbb{R}, \mathcal{F}_N) = \mathcal{C}_{DR}(\mathfrak{F}_N)$ and construct a one parameter family of $A_\infty$-categories $\tilde{\mathcal{C}}_\epsilon(\mathcal{F}_N)$ which are homotopic to $\mathcal{C}_{DR}(\mathfrak{F}_N)$.

II. Consider the limit $\tilde{\mathcal{C}}(\mathcal{F}_N) := \lim_{\epsilon \to 0} \tilde{\mathcal{C}}_\epsilon(\mathcal{F}_N)$ and find the minimum subcategory $\mathcal{C}(\mathcal{F}_N) \subset \tilde{\mathcal{C}}(\mathcal{F}_N)$ with the same objects $\mathcal{F}_N$. 


I. HPT and the $A_\infty$-category $\tilde{C}_e(\mathcal{F}_N)$

A version of homological perturbation theory (developed by Gugenheim, Lambe, Stasheff, Huebschmann, Kadeishvili, ... ) we shall employ is as follows.

**Thm.** Given an $A_\infty$-algebra $(V, m)$, suppose we have linear maps $h : V^r \to V^{r-1}$ and $P : V^r \to V^r$ satisfying

$$dh + hd = Id_V - P, \quad P^2 = P, \quad (d := m_1).$$

Then, there is a canonical way to construct an $A_\infty$-structure $m'$ on $P(V)$ s.t. $(P(V), m')$ is homotopy equivalent to $(V, m)$.

Note that $h$ gives a Hodge decomposition of $(V, d)$ if $dP = 0$, where $P(V) = H(V)$. 
Apply this HPT to $\mathcal{C}_{DR}(\mathcal{F}_N)$.

Construct $h_{ab}$ on $\text{Hom}_{\mathcal{C}_{DR}(\mathcal{F}_N)}(a, b) = \Omega_{ab}$.

- For any $a \in \mathcal{F}_N$, we set $h_{aa} = 0$.
- For $a \neq b \in \mathcal{F}_N$, fix $\epsilon \in (0, 1]$ and define $d_{\epsilon;ab}^\dagger : \Omega_{ab}^r \rightarrow \Omega_{ab}^{r-1}$ by
  $$d_{\epsilon;ab}^\dagger = \epsilon d^\dagger - \iota_{\text{grad}(f_{ab})}.$$

Can show that $H_{\epsilon} := d_{ab}d_{\epsilon;ab}^\dagger + d_{\epsilon;ab}^\dagger d_{ab}$ has only non-negative real eigenvalues.

In particular,

[ for $\epsilon = 1$ ], $H_1$ is the Hamiltonian of a harmonic oscillator,

[ for $\epsilon = \text{`0'}$ ], $H_0 = e^{f_{ab}} \mathcal{L}_{\text{grad}(f_{ab})} e^{-f_{ab}}$.

(cf. $d_{ab} := d - df_{ab} \wedge = e^{f_{ab}} \cdot d \cdot e^{-f_{ab}}$. )
Let $\psi_t : \Omega^r_{ab} \to \Omega^r_{ab}, \ t \in [0, \infty)$, be a linear map satisfying $\psi_0 = Id$ and
\[
\frac{d\psi_t}{dt} = -H_\varepsilon\psi_t.
\]

Then, we obtain
\[
d_{ab}h_{\varepsilon;ab} + h_{\varepsilon;ab}d_{ab} = \text{Id}_{\Omega_{ab}} - P_{\varepsilon;ab},
\]
\[
h_{\varepsilon;ab} := \int_0^\infty dt \ d_{\varepsilon;ab}^\dagger \psi_t, \quad P_{\varepsilon;ab} := \lim_{t \to \infty} \psi_t.
\]
Here $P_{\epsilon;ab}$ defines a projection:

$$P_{\epsilon;ab} \Omega_{ab}^0 = \text{Ker}(d_{ab} : \Omega_{ab}^0 \to \Omega_{ab}^1),$$

$$P_{\epsilon;ab} \Omega_{ab}^1 = \text{Ker}(d_{\epsilon;ab}^\dagger : \Omega_{ab}^1 \to \Omega_{ab}^0).$$

Choose bases $e_{\epsilon;ab}$ of $P_{\epsilon;ab} \Omega_{ab}^r$, $r = 0, 1$, by

$$e_{\epsilon;ab} = \text{const} \cdot e^{f_{ab}}, \quad t_a < t_b$$

(Gaussian normalize so that $e_{\epsilon;ab}(x_{ab}) = 1$)

$$e_{\epsilon;ab} = \text{const} \cdot e^{-\frac{1}{\epsilon}(f_{ab})} dx, \quad t_a > t_b.$$  

(Gaussian normalize so that $\int_{-\infty}^{\infty} e_{\epsilon;ab} = 1$)

In the limit $\epsilon \to 0$, the degree one base $e_{\epsilon;ab}$ ($t_a > t_b$) becomes the delta function localized at the point $x_{ab} (= x(v_{ab}))$. 
In the limit $\epsilon \to 0$, $h_{ab} := \lim_{\epsilon \to 0} h_{\epsilon;ab}$ and $P_{ab} := \lim_{\epsilon \to 0} P_{\epsilon;ab}$ turn out to be

$$h_{ab} = \int_0^\infty dte^{f_{ab}}\varphi_t^*(e^{-f_{ab}} \iota_{\text{grad}(f_{ab})}),$$

$$P_{ab} = \lim_{t \to \infty} e^{f_{ab}}\varphi_t^* e^{-f_{ab}},$$

where $\varphi_t : \mathbb{R} \to \mathbb{R}$ is the flow defined by

$$\frac{d\varphi_t}{dt} = \text{grad}(f_{ab}), \quad \varphi_0 = \text{Id}.$$

Let us consider the following case:

$$h_{ab}(\delta(x-p)dx)$$

$$= \int_0^\infty dte^{f_{ab}}\varphi_t^* e^{-f_{ab}}\delta(x-p)\frac{df_{ab}}{dx}(x)$$

$$= e^{f_{ab}}(\varphi_t^* e^{-f_{ab}})|_{\varphi_t(x)=p(x)}.$$
\[ h_{ab}(\delta(x - p)dx) \text{ for } t_a < t_b \text{ and } x_{ab} < p \]

Flow of \( \text{grad}(f_{ab}) \)

\( \text{step function} \) twisted by \( e^{f_{ab}} \)

* Now, let us derive the \( A_\infty \)-products \( \{m'_n\} \) of \( \tilde{\mathcal{C}}(\mathcal{F}_N) \) with the identifications

\[
\lim_{\epsilon \to 0} P_{\epsilon;ab} \Omega_{ab} =: \text{Hom}_{\tilde{\mathcal{C}}(\mathcal{F}_N)}(a, b) \simeq V_{ab},
\]

\[
\lim_{\epsilon \to 0} e_{\epsilon;ab} = e_{ab} \iff [v_{ab}]
\]

for \( a \neq b \).
• Example for $m_3'(e_{ab}, e_{bc}, e_{cd})$

HPT implies $m_3'(e_{ab}, e_{bc}, e_{cd}) =$

$$= -e^{-(X+Y+Z)} \cdot e_{ad}.$$
• An example of non-transversal product:

\[ m'_3(e_{ab}, e_{bc}, e_{cd}, e_{da}) \]

By observations as above, we will define

\[ V_{aa} = \text{Hom}_{C(\mathscr{F}_N)}(a, a) \supset \text{Hom}_{\tilde{C}(\mathscr{F}_N)}(a, a) \]

by introducing \( \vartheta_{ab} = \vartheta_{vab} \) (step function),

etc., as its generators.
II. Subcategory of $\tilde{C}(\mathfrak{F}_N) := \lim_{\epsilon \to 0} \tilde{C}_\epsilon(\mathfrak{F}_N)$

Consider the minimum subcategory $\mathcal{C}(\mathfrak{F}_N) \subset \tilde{C}(\mathfrak{F}_N)$ with the same set of objects $\mathfrak{F}_N$ and

$$\text{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, b) = \text{Hom}_{\tilde{C}(\mathfrak{F}_N)}(a, b) = V_{ab}$$

for $a \neq b$.

Then, for any $a \in \mathfrak{F}_N$, $V_{aa}$ (with comm. DGA structure) is defined purely algebraically by the following idea.
For any $v \in \mathfrak{F}_N - \{a\}$,

- introduce degree zero generator $\vartheta_v$ and degree one generator $\delta_v$ which are supposed to be

$$\delta_{vab} = \lim_{\epsilon \to 0}(\epsilon e_{\epsilon;ab} \wedge \epsilon e_{\epsilon;ba}),$$

$$\vartheta_{vab}(x) = \int_{-\infty}^{x} dx' \delta_{vab}(x').$$

- appropriate relations

$$\vartheta_v \cdot \vartheta_{v'} = \vartheta_{v'} \text{ for } x(v) < x(v'), \text{ etc.},$$

- $V_{aa}^0$ and $V_{aa}^1$ are the degree zero and one vector space of elements generated by $\vartheta_v$, $\delta_v$ s.t. they are zero at $x = \pm \infty$.

- differential $d : V_{aa}^0 \to V_{aa}^1$ by extending $d(\vartheta_v) = \delta_v$ by the Leibniz rule.

**Note.** $(\vartheta_v)^2 \neq \vartheta_v$, etc.,
• More examples of non-transversal $A_\infty$-products of $C(\mathcal{F}_N)$

For $t_a < t_b$,

$$m'_2((\vartheta_{v_{ab}})^n, [v_{ab}]) = m'_2([v_{ab}], (\vartheta_{v_{ab}})^n) = \frac{1}{2^n}[v_{ab}],$$

$$m'_3([v_{ba}], (\vartheta_{v_{ab}})^n, [v_{ab}]) = \frac{1}{n + 1}\vartheta_{v_{ab}}(1 - (\vartheta_{v_{ab}})^n),$$

for $n \geq 1$, 

...
$k$ elements at $v_{bc}$

\[ m'_{2+k}(\left[v_{ab}, \delta_{v_{bc}}, \ldots, \delta_{v_{bc}}, \left[v_{bc}\right], \delta_{v_{bc}}, \ldots, \delta_{v_{bc}}, \left[v_{cd}\right]\right) ] = \frac{(-1)^k}{k!} e^{-(X+Y+Z)} \cdot \left[v_{ad}\right]. \]
The precise proof of the main theorem is given by defining $\mathcal{C'}_{DR}(\mathcal{F}_N)$ and $\tilde{\mathcal{C}}_{DR}(\mathcal{F}_N)$ s.t.

\[ \mathcal{C}(\mathcal{F}_N) \xrightarrow{\text{HPT}} \mathcal{C'}_{DR}(\mathcal{F}_N) \xrightarrow{\text{HPT}} \tilde{\mathcal{C}}_{DR}(\mathcal{F}_N) \xrightarrow{\text{HPT}} \mathcal{C}_{DR}(\mathcal{F}_N) \]

Applying HPT for $\mathcal{C'}_{DR}(\mathcal{F}_N)$ gives $\mathcal{C}(\mathcal{F}_N)$. 
Future directions

- Generalization to higher dimensional case
  (though not so straightforward)

- The $\mathbb{R}^{2n}$ case can be applied to the $T^{2n}$ case.

(Note. In this case, each object has identity morphism.)

$\Rightarrow$ $\circ$ application to homological mirror symmetry for tori;

$\Rightarrow$ $\circ$ can produce geometric examples of finite dim. $A_{\infty}$-algebra

$\circ$ (Noncommutative, etc.,) deformation of these $A_{\infty}$-categories ??

$\circ$ building block to more general mfds ?