Elliptic gamma functions, gerbes and triptic curves

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Paris, 18 January 2007

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based on joint work with Alexander Varchenko and
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Introduction

In conformal field theory based on quantum groups and statistical mechanics there appear linear difference equations with elliptic coefficients.

Idea: the step plays the role of a third period. Geometrically, one is lead to consider triptic curves $\mathbb{C}/\mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$.

Today we consider the simplest case of such a difference equation, the functional equation of the elliptic gamma function.

Jacobi's infinite product

In his *Fundamenta nova* Jacobi introduced the function

$$\Theta(t,q) = \prod_{n=0}^{\infty} (1 - q^{n+1}/t)(1 - q^n t), \qquad t \neq 0, |q| < 1.$$

The Jacobi product obeys the functional equation

$$\Theta(qt,q) = -t^{-1}\Theta(t,q).$$

This equation holds also for $|q| > 1$ if we set

$$\Theta(t,q) = \prod_{n=0}^{\infty} (1 - q^{-n}/t)^{-1}(1 - q^{-n-1} t)^{-1}, \qquad |q| > 1.$$

Jacobi and Hermite discovered transformation properties of Θ under $q \to q^{4\pi/\ln q}, t \to t^{-4\pi/\ln q}$ and more generally under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$

Geometric content: elliptic curves

Let $x_1, x_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. $E_{(x_1,x_2)} = \mathbb{C}/\mathbb{Z}\,x_1 + \mathbb{Z}\,x_2$ is an oriented elliptic curve.

$E_x \simeq E_{x'}$ iff $x' = \lambda A x$, $\lambda \in \mathbb{C}^\times$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Moduli space of oriented elliptic curves: $\mathcal{M} = Y/SL_2(\mathbb{Z})$

$Y = \{(x_1 : x_2) \in \mathbb{C}P^1 \,|\, x_1, x_2$ $\mathbb{R}$-linearly independent$\}$

$= \mathbb{C}P^1 - \mathbb{R}P^1 = H_+ \cup H_-$

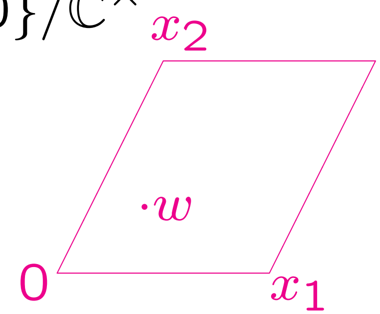
Universal oriented elliptic curve

The group $ISL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ acts on

$$X = \{(w, x_1, x_2) \,|\, \mathrm{Im}(x_1 \bar{x}_2) \neq 0\}/\mathbb{C}^\times$$

via $(A, n) \cdot (w, x) = (w + n_1 x_1 + n_2 x_2, Ax)$

$$\begin{array}{ll} \mathcal{E} = X/ISL_2(\mathbb{Z}) & \text{universal curve} \\ \downarrow & \\ \mathcal{M} = Y/SL_2(\mathbb{Z}) & \text{moduli space} \end{array}$$



Remarks: 1. X is the total space of the line bundle $O(1) \to \mathbb{C}P^1 - \mathbb{R}P^1$. (It is actually a trivial bundle over the union of contractible spaces $H_+ \cup H_-$)

2. These spaces are mildly singular. They should be treated as stacks.

The Jacobi product as a section of a line bundle over the universal elliptic curve

For $\operatorname{Im}\tau > 0$, let us write the theta product in additive coordinates:

$$\theta(z,\tau) = \prod_{n=0}^{\infty}(1-q^{n+1}/t)(1-q^n t), t = e^{2\pi i z}, q = e^{2\pi i\tau}$$

Extend to $\operatorname{Im}\tau \neq 0$ by $\theta(-z,-\tau) = \theta(z,\tau)^{-1}$.

Then $(w, x_1, x_2) \to \theta\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right)$ is a meromorphic function on X, a covering space of the universal elliptic curve $X/ISL_2(\mathbb{Z})$.

Transformation properties under $G = ISL_2(\mathbb{Z})$

$$\theta\left(\frac{w'}{x'_2}, \frac{x'_1}{x'_2}\right) = e^{2\pi i Q_g(w,x)}\theta\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right) \qquad (*)$$

$w' = w + n_1 x_1 + n_2 x_2$, $x' = Ax$, $g = (A, n) \in G = ISL_2(\mathbb{Z})$

$Q_g(w, x) \in \mathbb{Q}(x_1, x_2)[w]$ of degree 2 in w.

Meaning:

(a) $\phi = (e^{2\pi i Q_g(w,x)})_{g\in G}$ defines a G-equivariant line bundle L on X (a class in $H^1_G(X, \mathcal{O}^\times_X)$)

(b) θ is a G-equivariant meromorphic section of L. Namely if $\mathcal{M}$ denotes the sheaf of meromorphic functions, $\theta \in C^0_G(X, \mathcal{M}^\times)$ and (*) means $\delta\theta = \phi$. (In this case everything reduces to group cohomology)

Rational, trigonometric and elliptic gamma function

Euler 1729: $\Gamma(z+1) = z\,\Gamma(z)$

$$z! = \Gamma(z+1) = \prod_{j=1}^{\infty} \frac{j^{1-z}(j+1)^z}{j+z}$$

Jackson 1912: $\Gamma(z+\sigma,\sigma) = (1-e^{2\pi i z})\Gamma(z,\sigma)$

$$\Gamma(z,\sigma) = \prod_{j=0}^{\infty} \frac{1}{1-r^j t}, \qquad r = e^{2\pi i\sigma}, \quad t = e^{2\pi i z}$$

Ruijsenaars 1997: $\Gamma(z+\sigma,\tau,\sigma) = \theta(z,\tau)\Gamma(z,\tau,\sigma)$

$$\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \frac{1-q^{j+1}r^{k+1}t^{-1}}{1-q^j r^k t}, \quad q = e^{2\pi i\tau},\ r = e^{2\pi i\sigma},\ t = e^{2\pi i z}$$

“Modular” properties

Extend the definition of $\Gamma(z,\tau,\sigma)$ to a meromorphic function on $\mathbb{C}\times(\mathbb{C}-\mathbb{R})\times(\mathbb{C}-\mathbb{R})$:

$$\Gamma(z,-\tau,\sigma)=\Gamma(z+\tau,\tau,\sigma)^{-1},\qquad \Gamma(z,\tau,-\sigma)=\Gamma(z+\sigma,\tau,\sigma)^{-1}.$$

Then (G. F., A. Varchenko 2000)

$$\Gamma(z,\tau,\sigma)=\Gamma(z+\tau,\tau,\tau+\sigma)\Gamma(z,\tau+\sigma,\sigma).$$

$$\Gamma\left(\frac{w}{x_3},\frac{x_1}{x_3},\frac{x_2}{x_3}\right)\Gamma\left(\frac{w}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1}\right)\Gamma\left(\frac{w}{x_2},\frac{x_3}{x_2},\frac{x_1}{x_2}\right)=e^{-\pi i P_3(w,x)/3},$$

$$P_3(w,x)=\frac{w^3}{e_3}-\frac{3\,e_1}{2\,e_3}w^2+\frac{e_1^2+e_2}{2\,e_3}w-\frac{e_1\,e_2}{4\,e_3}.$$

$$e_1=x_1+x_2+x_3,\ e_2=x_1x_2+x_1x_3+x_2x_3,\ e_3=x_1x_2x_3.$$

Geometric content: triptic curves

A triptic curve is a stack of the form $E_x = \mathbb{C}/\mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$, where $x_1, x_2, x_3 \in \mathbb{C}$ span $\mathbb{C}$ over $\mathbb{R}$.

$E_x \simeq E_{x'}$ iff $x' = \lambda A x$ $\lambda \in \mathbb{C}^\times$, $A \in SL_3(\mathbb{Z})$. The moduli space of oriented triptic curves is $Y/SL_3(\mathbb{Z})$, $Y = \mathbb{C}P^2 - \mathbb{R}P^2$.

$ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ acts on $X = \{(w,x) \in \mathbb{C} \times \mathbb{C}^3 - \mathbb{C} \cdot \mathbb{R}^3\}/\mathbb{C}^\times =$ total space of $O(1) \to Y$.

$$\begin{array}{ll} \mathcal{E} = X/ISL_3(\mathbb{Z}) & \text{universal triptic curve} \\ \downarrow & \\ \mathcal{M} = Y/SL_3(\mathbb{Z}) & \text{moduli space} \end{array}$$

This time Y is topologically non-trivial: it retracts to the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 0$.

An $ISL_3(\mathbb{Z})$-equivariant cover of X

There is a good open cover of X labeled by Λ_{prim}, the set of primitive vectors in $\Lambda = \mathbb{Z}^3 \subset \mathbb{C}^3$. If $a \in \Lambda_{\text{prim}}$ let $H(a)$ be the oriented hyperplane in the dual lattice $\Lambda^\vee$ with equation $\langle \delta, a \rangle = 0$.

$$U_a = \{x \in Y = \mathbb{C}P^2 - \mathbb{R}P^2 \,|\, \text{Im}(\langle \alpha, x \rangle \overline{\langle \beta, x \rangle}) > 0\}$$

for any oriented basis α, β of $H(a)$. Let $V_a = p^{-1}(U_a) \subset X$.

Lemma $\mathcal{U} = (V_a)_{a \in \Lambda_{\text{prim}}}$ *is a good* $ISL_3(\mathbb{Z})$ *equivariant open cover of* X*.*

Let $\check{C}(\mathcal{U}, \mathcal{O}^\times)$, $\check{C}(\mathcal{U}, \mathcal{M}^\times)$ be the Čech complex of $\mathcal{U}$ with values in the sheaf of invertible holomorphic/meromorphic functions.

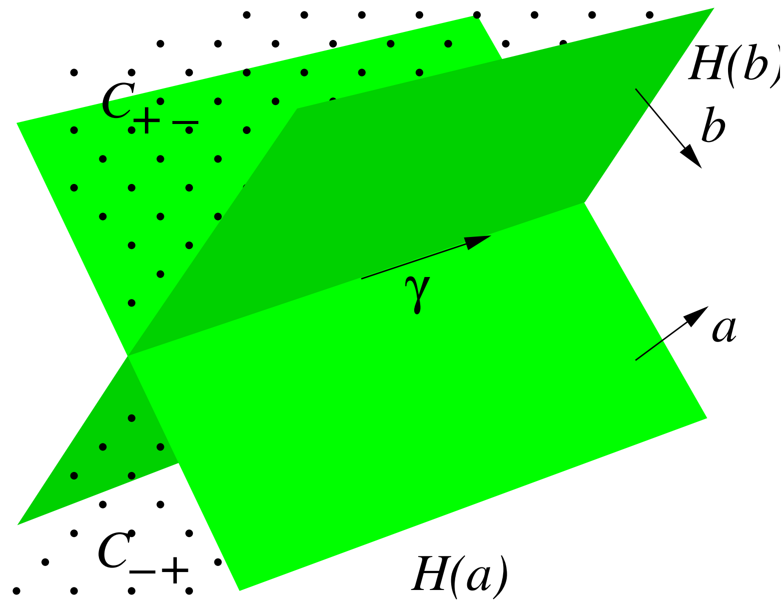
Gamma functions associated to pairs of primitive vectors

For $a, b \in \Lambda_{\mathrm{prim}}$ linearly independent set

$$\Gamma_{a,b}(w,x) = \frac{\prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma}(1 - e^{-2\pi i(\langle\delta,x\rangle - w)/\langle\gamma,x\rangle})}{\prod_{\delta \in C_{-+}(a,b)/\mathbb{Z}\gamma}(1 - e^{+2\pi i(\langle\delta,x\rangle - w)\,/\langle\gamma,x\rangle})}.$$

$H(a) \cap H(b) = \mathbb{Z}\gamma$. Set $\Gamma_{a,\pm a} = 1$.

$\Gamma_{a,b}$ is a meromorphic function on $V_a \cap V_b$. It reduces to $\Gamma\left(\frac{w}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$ if $(a,b) = (e_1, e_2)$.



Theorem $\Gamma_{a,b} = \Gamma_{b,a}^{-1}$ *and on* $V_a \cap V_b \cap V_c$*,*

$$\Gamma_{a,b}(w,x)\Gamma_{b,c}(w,x)\Gamma_{c,a}(w,x) = e^{-\pi i P_{a,b,c}(w,x)/3}$$

for some polynomial $P_{a,b,c}(w,x) \in \mathbb{Q}(x_1,x_2,x_3)[w]$ *of degree 3 in* w *with rational coefficients, holomorphic on* $V_a \cap V_b \cap V_c$*. Moreover* $\Gamma_{ga,gb}(w,gx) = \Gamma_{a,b}(w,x)$*,* $g \in SL_3(\mathbb{Z})$*.*

Consequences

(a) The invertible holomorphic functions $\phi_{a,b,c} = e^{-\pi i P_{a,b,c}/3}, a, b, c \in \Lambda_{\text{prim}}$ on $V_a \cap V_b \cap V_c$ form an $SL_3(\mathbb{Z})$-invariant Čech cocycle in $\check{C}^2(\mathcal{U}, \mathcal{O}^\times)$ on $X = O(1) \to \mathbb{C}P^2 - \mathbb{R}P^2$. It defines a holomorphic gerbe on the stack $X/SL_3(\mathbb{Z})$.

(b) $\Gamma = (\Gamma_{a,b})$ is a meromorphic section of this gerbe, namely an invariant cochain in $\check{C}^1(\mathcal{U}, \mathcal{M}^\times)$ such that $\delta\Gamma = \phi$

Including the translation subgroup

Let $\mu \in \Lambda^\vee = \mathbb{Z}^3$. Then

$$\frac{\Gamma_{a,b}(w,x)}{\Gamma_{a,b}(w+\langle \mu, x\rangle, x)} = \phi_{a,b}(\mu; w, x)\frac{\Delta_b(\mu; w, x)}{\Delta_a(\mu; w, x)}, \qquad (w,x) \in V_a \cap V_b,$$

for some meromorphic functions $\Delta_a(\mu;\ ) \in \mathcal{M}^\times(V_a)$ and holomorphic functions $\phi_{a,b}(\mu;\ ) \in \mathcal{O}^\times(V_a \cap V_b)$.

These identitities are part of a system of identities stating that (Γ, Δ) define a G-equivariant meromorphic section of the gamma gerbe $\mathcal{G}$ on the total space X of the line bundle $O(1) \to \mathbb{C}P^2 - \mathbb{R}P^2$. The gerbe is defined by an equivariant cocycle ϕ.

The gamma gerbe

Let $G = ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$. The complex

$$C^n_G(\mathcal{U}, \mathcal{F}) = \oplus_{p+q=n} C^p(G, \check{C}^q(\mathcal{U}, \mathcal{F})), \quad n = 0, 1, 2, \ldots$$

with total differential $D = \delta_G + (-1)^p \check{\delta}$ computes the equivariant cohomology of X with values in $\mathcal{F} = O^\times$ or $\mathcal{M}^\times$.

Theorem $\phi \in C^2_G(\mathcal{U}, \mathcal{O}^\times) = C^{0,2} \oplus C^{1,1} \oplus C^{2,0}$ is a 2-cocycle and thus defines a gerbe $\mathcal{G}$ on the stack X/G. The meromorphic cochain $(\Gamma, \Delta) \in C^1_G(\mathcal{U}, \mathcal{M}^\times) = C^{0,1} \oplus C^{1,0}$ obeys $D(\Gamma, \Delta) = \phi$ and thus defines a meromorphic section of $\mathcal{G}$.

Explicit formulae

In explicit terms, we have identities

$$\phi_{a,b,c}(y)\Gamma_{a,c}(y) = \Gamma_{a,b}(y)\Gamma_{b,c}(y), \qquad y \in V_a \cap V_b \cap V_c,$$

$$\phi_{a,b}(g;y)\Gamma_{g^{-1}a,g^{-1}b}(g^{-1}y)\Delta_b(g;y) = \Delta_a(g;y)\Gamma_{a,b}(y), \qquad y \in V_a \cap V_b,$$

$$\phi_a(g,h;y)\Delta_a(gh;y) = \Delta_a(g;y)\Delta_{g^{-1}a}(h;g^{-1}y), \qquad y \in V_a,$$

for all $a, b, c \in I$, $g, h \in G$.

	$\phi_{a,b,c}$		
$\uparrow \check{\delta}$	$\Gamma_{a,b}$	$\phi_{a,b}(g;\ )$	
		$\Delta_a(g;\ )$	$\phi_a(g,h;\ )$

$\xrightarrow{\delta_G}$

Characteristic class

Theorem *The Dixmier–Douady class* $[\phi] \in H^2_G(X, \mathcal{O}^\times)$ *of the gamma gerbe maps to a non-trivial class* $c \in H^3_G(X, \mathbb{Z})$*. There is an exact sequence*

$$0 \to \mathbb{Z} \to H^3_G(X, \mathbb{Z})/\mathit{torsion} \to H^3(\mathbb{Z}^3, \mathbb{Z}) \to 0,$$

and c *maps to a generator of* $H^3(\mathbb{Z}^3, \mathbb{Z}) \simeq \mathbb{Z}$*.*

It is well-known that the theta function bundle is hermitian. The same holds for the gamma gerbe:

Theorem *The gamma gerbe* $\mathcal{G}$ *has a hermitian structure compatible with the complex structure and thus admits a connective structure.*