Chapter 6

Tangent space

6.1 Tangent and ultratangent space

In this section we compare to notions of tangent space for spaces with curvature bounded below

(i) the tangent space, $\mathrm{T}_p = \operatorname{Cone}\Sigma_p$, defined in section 2.5

(ii) maybe add $\mathrm{T}_p L^{\omega}$???

(iii) ultratangent space for an ultrafilter ω, T_p^{ω} which is defined in section 2.8.

Both of these notions are analogous to tangent space of Riemannian manifold. If the dimension is finite, then these two notions coinside, see 6.3.1. In infinite-dimensional case, they are different and both useful; often lack of a property in one is compensated by the other.

It is clear from the definition that tangent space has cone structure. On the other hand, in general, ultratangent space does not have a cone structure; a Hilbert's cube $Q = \prod_{n=1}^{\infty}[0, 2^{-n}] \in \mathrm{Alex}[0,\infty]$ gives an example.

In general, the metric on T_p might be not intrinsic (see Halbeisen's example, section 5.2). In this case, by definition $\mathrm{T}_p \notin \mathrm{Alex}[0,\infty]$; for T_p^{ω} the situation is better:

6.1.1. Theorem. *If $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$ then $\mathrm{T}_p^{\omega} \in \mathrm{Alex}[0,\infty]$.*

Proof. Since $L \in \mathrm{Alex}[\kappa,\infty]$, then for its blowup nL, we have $nL \in \mathrm{Alex}[\kappa/n^2,\infty]$. Thus, from 3.1.2, $(nL, p) \xrightarrow{\omega} (\mathrm{T}_p^{\omega}, o_p)$ implies $\mathrm{T}_p^{\omega} \in \mathrm{Alex}[0,\infty]$. ☐

The next theorem shows that $\mathrm{T}_p =$ can be (and often will be) considered as a subset of T_p^{ω}. That also implies that 3-angles comparison (3.1.1) as well as n-point comparison (18.1.1) holds in T_p.

6.1.2. Theorem. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$, then there is an isometric embedding $\imath : \mathrm{T}_p \hookrightarrow \mathrm{T}_p^{\omega}$ such that $(nL, p; q_n) \xrightarrow{\omega} (\mathrm{T}_p^{\omega}, o_p; \imath(v))$ iff $\mathrm{T}_p(L^{\omega}) \ni n\log_p q_n \xrightarrow{\omega} v \in \mathrm{T}_p$.*

Proof. Let us first define $\imath : \mathrm{T}'_p \to \mathrm{T}_p^{\omega}$, where $\mathrm{T}'_p = \operatorname{Cone}\Sigma'_p \subset \mathrm{T}_p$ the subcone of geodesic vectors (see section 2.5). Given $v \in \mathrm{T}'_p$ the exponent $v_n = \exp_p(\tfrac{1}{n} \cdot v)$

is defined for all large n. Pass to the ω-limits $(nL, p; v_n) \xrightarrow{\omega} (\mathrm{T}_p^\omega, o_p; v_\omega)$. Set $\imath(v) = v_\omega$.

Since angles between geodesics in L are well defined (see ???), for any $v, w \in \mathrm{T}'_p$ we have $n|v_n w_n| \to |vw|$. Thus $|v_\omega w_\omega| = |vw|$, i.e. $\imath$ is a global isometry of T'_p. Since T'_p is dense in T_p, we can extend $\imath$ to a global isometry $\mathrm{T}_p \to \mathrm{T}_p^\omega$. $\square$

6.1.3. Theorem. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$. Given two tangent vectors $u, v \in \mathrm{T}_p$, consider model triangle $[\tilde{o}_p\tilde{u}\tilde{v}] = \tilde{\triangle}_0(o_p uv)$ and denote by* Sect(uv) *the region in $\mathbb{E}^2$ bounded by rays $\tilde{o}_p\tilde{u}$ and $\tilde{o}_p\tilde{v}$.*

Then for any geodesic $[uv]$ in T_p^ω there is a global isometric embedding $\imath :$ Sect$(uv) \to \mathrm{T}_p^\omega$ *such that $\imath(\tilde{o}_p) = o_p$, $\imath([\tilde{u}\tilde{v}]) = [uv]$.*

Proof. Follows directly from the lemma on flat triangle (18.4.3). $\square$

6.2 Linear subspace of tangent space

Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$. A vector $v \in \mathrm{T}_p$ is called *invertable* if there is $w \in \mathrm{T}_p$ such that $\measuredangle(v, w) = \pi$.

Given a point p, the subcone of all invertable vectors of T_p will be called *linear subspace* of T_p and denoted by Lin_p. The reason for such a name is coming from the following theorem.

For tangent vectors $v, v' \in \mathrm{T}_p$, we will write $v' = -v$ if $\measuredangle(v, v') = \pi$ and $|v| = |v'|$. Clearly $-v$ is well defined if and only if $v \in \mathrm{Lin}_p$.

6.2.1. Theorem. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$ then* Lin_p *forms a subcone of T_p isometric to a Hilbert space.*

6.2.2. Corollary. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L^\bullet(x_1, x_2, \dots, x_n)$ then there is a subcone $E \subset \mathrm{T}_p$ which isometric to a Euclidean space such that* $\log_p x_i \in E$ *for every $i \in [1..n]$.*

Proof. By the definition of $L^\bullet$ (3.1.5), $\log_p x_i \in \mathrm{Lin}_p$ for each $i \in [1..n]$. It remains to apply theorem 6.2.1. $\square$

The main difficulty in the proof of theorem 6.2.1comes from the fact that in general $\mathrm{T}_p \notin \mathrm{Alex}[0,\infty]$ (see Habeisen's example, section 5.2). Otherwise the statement would follow directly from the Splitting theorem (16.1.1). Infact the proof of this theorem is a far walk around, in the proof we use construction of gradient, as well as Splitting theorem, proof of which use gradient flow. Thus in order to understand our proof completely one needs to read most of chapter **??**.

First we give a construction of a tangent vector w for given two vectors u, v. If the tangent space is Euclidean, then $w = -u - v$.

6.2.3. Anti-sum lemma. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$.*

Assume two vectors $u, v \in \mathrm{T}_p$. Then there is unique element $w \in \mathrm{T}_p$ such that

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle \geqslant 0 \quad \textit{and} \quad \langle u, w\rangle + \langle v, w\rangle + \langle w, w\rangle = 0$$

for any $x \in \mathrm{T}_p$.

Proof. Choose two sequences of points $a_n, b_n \in L^\bullet(p)$ such that $\uparrow_p^{a_n} \to u/|u|$ and $\uparrow_p^{b_n} \to v/|v|$. Consider sequence of functions $f_n = |u| \operatorname{dist}_{a_n} + |v| \operatorname{dist}_{b_n}$. Note that $d_p f_n(x) = -|u|\langle \uparrow_p^{a_n}, x\rangle - |v|\langle \uparrow_p^{b_n}, x\rangle$, thus $d_p f_n(x)$ converges to $-\langle u, x\rangle - \langle v, x\rangle$ uniformly for $x \in \Sigma_p$. Therefore, according to ???, sequence $\nabla_p f_n$ converges; set $w = \lim_n \nabla_p f_n$.

From definition of gradient

$$\langle w, w\rangle = \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle = \lim_{n\to\infty} d_p f_n(\nabla_p f_n) = -\langle u, w\rangle - \langle v, w\rangle,$$

$$\langle w, x\rangle = \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle \geqslant \lim_{n\to\infty} d_p f_n(x) = -\langle u, x\rangle - \langle v, x\rangle.$$

□

Proof of 6.2.1. Let us first show that $\mathrm{Lin}_p \in \mathrm{Alex}[0,\infty]$.

Note that $\mathrm{T}_p^\omega \in \mathrm{Alex}[0,\infty]$ (see 6.1.1) and Lin_p is a closed subset of T_p^ω. Thus. to show that $\mathrm{Lin}_p \in \mathrm{Alex}[0,\infty]$ it is sufficient to show that the metric on Lin_p induced from T_p^ω is intrinsic.

Indeed, given two vectors $x, y \in \mathrm{Lin}_p$, let us apply lemma 6.2.3 for two vectors $u = \frac{1}{2}\cdot(-x)$ and $v = \frac{1}{2}\cdot(-y)$ and show that $w \in \mathrm{T}_p$ is a midpoint of $[xy]$. First note that

$$|w|^2 = -\langle w, u\rangle - \langle w, v\rangle = \tfrac{1}{2}\langle w, x\rangle + \tfrac{1}{2}\langle w, y\rangle.$$

Therefore,

$$\begin{aligned}
|xw|^2 + |wy|^2 &= 2|w|^2 + |x|^2 + |y|^2 - 2\langle w, x\rangle - 2\langle w, y\rangle = \\
&= |x|^2 + |y|^2 - \langle w, x\rangle - \langle w, y\rangle \leqslant \\
&\leqslant |x|^2 + |y|^2 + \langle u, x\rangle + \langle v, x\rangle + \langle u, y\rangle + \langle v, y\rangle = \\
&= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2 - \langle x, y\rangle = \\
&= \tfrac{1}{2}|xy|^2.
\end{aligned}$$

Thus $|xw| = |wy| = \frac{1}{2}|xy|$, i.e. w is a midpoint for x and y.

Note that for any $v \in \mathrm{Lin}_p$ there is a line ℓ passing trough v and o_p, thus applying 16.1.2, we get that Lin_p is isometric to a Hilbert space. □

6.2.4. Open question. *Let* $L \in \mathrm{Alex}[\kappa,\infty]$ *and* $p \in L$.

Is it true that T_p *spits isometrically*

$$\mathrm{T}_p = \mathrm{Lin}_p^\perp \oplus \mathrm{Lin}_p$$

where $\mathrm{Lin}_p^\perp = \{\, v \in \mathrm{T}_p \mid \langle v, x\rangle = 0$ *for any* $x \in \mathrm{Lin}_p \,\}$*?*

Compare to theorem 6.3.6

6.3 Finite dimensional case.

For finite dimensional case, the above constructions can be essentially simplified by using the following theorem. One should note however, our way to define dimension of Alexandrov space depends on the theorems on tangent cone of general case. Thus, even if you are interested only in finite dimensional case the previous sections of this chapter can not be avoided.

6.3.1. Theorem. *Let $L \in \mathrm{Alex}^m[\kappa,\infty]$. Then for any point $p \in L$, $\mathrm{T}_p \in \mathrm{Alex}^m[0,\infty]$ and moreover $\mathrm{T}_p = \mathrm{T}_p^\varpi$.*

According to ???, this theorem impyes that if $m \geqslant 2$ then for any $p \in L$, $\Sigma_p \in \mathrm{Alex}^{m-1}[1,\infty]$. That gives a possibility of making definitions and proving theorems by induction on the dimension of the space.

6.3.2. Lemma. *Let $L \in \mathrm{Alex}^m[\kappa,\infty]$, $m < \infty$ then for any $p \in L$, Σ_p is compact and T_p is proper.*

The prof is very similar to the proof of 8.2.9.

Proof. Assume Σ_p is not compact. Then for some fixed $\varepsilon > 0$ and any $N \in \mathbb{Z}_>$ one could find N directions $\xi_i \in \Sigma_p$, $i \in [1..N]$ such that for any $i \neq j$ we have $\measuredangle(\xi_i, \xi_j) > \varepsilon$. For each ξ_i, choose $x_i \in L$ so that $\uparrow_p^{x_i} \approx \xi_i$, one can do this so that in addition $\tilde\measuredangle_\kappa(p\,^{x_i}_{x_j}) > \varepsilon$ for all i, j.

For any point q sufficiently close to p we also have $\tilde\measuredangle_\kappa(q\,^{x_i}_{x_j}) > \varepsilon$ for all $i, j \in [1..N]$. According to 8.2.2(b′), one can choose $q \approx p$ so that Σ_q is isometric to $\mathbb{S}^{m-1}$.

Thus, one arrives to a contradiction for $N > \operatorname{pack}_\varepsilon \mathbb{S}^{m-1}$.

Finally, since $\mathrm{T}_p = \operatorname{Cone}\Sigma_p$, we get that T_p is a proper space. □

Proof of 6.3.1. According to theorem 6.1.1, there is a natural isometric embedding $\imath : \mathrm{T}_p \hookrightarrow \mathrm{T}_p^\varpi$. Thus it is enough to show that $\imath(\mathrm{T}_p) = \mathrm{T}_p^\varpi$. I.e., given $v_\varpi \in \mathrm{T}_p^\varpi$ we have to construct $v \in \mathrm{T}_p$ such that $\imath(v) = v_\varpi$.

Fix $v_\varpi \in \mathrm{T}_p^\varpi$, and choose a sequence $v_n \in L^\bullet(p)$, such that $n|pv_n|$ is bounded and $(nL, p; v_n) \xrightarrow{\varpi} (\mathrm{T}^\varpi, o_p; v_\varpi)$. Set

$$v = \varpi\text{-}\lim_{n\to\infty} n \cdot \log_p v_n.$$

Since T_p is proper (6.3.2) and $n|pv_n|$ is bounded, this limit is well defined.

From the construction of $\imath$, we get??? that $\imath(v) = v_\varpi$ □

6.3.3. Theorem. *Let $L_n \in \mathrm{Alex}^m[\kappa,\infty]$ and $(L_n, p_n) \xrightarrow[\mathrm{GH}]{} (L, p)$.*

Assume $\Sigma_{p_n} L_n \xrightarrow[\mathrm{GH}]{} \Sigma$ then $\Sigma \geqslant \Sigma_p$.

6.3.4. Corollary. *Let $L \in \mathrm{Alex}^m[\kappa,\infty]$, then for any $p \in L$,*

$$\mathbb{S}^{m-1} \geqslant \Sigma_p.$$

6.3.5. Corollary. *Let $L \in \mathrm{Alex}^m[\kappa,\infty]$, then for any converging sequence of points $p_n \to p$ in L, if $\Sigma_{p_n} \xrightarrow[\mathrm{GH}]{} \Sigma$ then*

$$\Sigma \geqslant \Sigma_p.$$

Moreover, if $\uparrow_p^{p_n} \to \xi \in \Sigma_p$ then $\Sigma \geqslant \Sigma_\xi \mathrm{T}_p$.

Proof of 6.3.5.??? □

Proof of 6.3.3. Given a finite ε-net $\{\xi_i\}$ in Σ_p, we will show that there is a collection of points $\{\zeta_i\}$ in Σ such that $|\zeta_i\zeta_j| > \measuredangle(\xi_i,\xi_j)-\varepsilon$ for any pair (i,j). Once it is done, one can construct a noncontracting map $\Sigma_p\to\Sigma$ by passing to a partial limit of maps $\xi_i\mapsto\zeta_i$.

For each ξ_i, choose $x_i\in L$ so that $\uparrow_p^{x_i}\approx\xi_i$, one can do this so that $\tilde\measuredangle_\kappa(p\,{}^{x_i}_{x_j}) > \measuredangle(\xi_i,\xi_j)-\varepsilon$ for all pairs (i,j). For each x_i, choose a sequence $x_{i,n}\in L_n$ so that $x_{i,n}\to x_i$.

Passing to a subsequence of Σ_{p_n}, we can assume that $\uparrow_{p_n}^{x_{i,n}}$ converges for any i. Set $\zeta_i=\lim_n\uparrow_{p_n}^{x_{i,n}}\in\Sigma$. Then

$$|\zeta_i\zeta_j|_\Sigma \geqslant \varliminf_{n\to\infty}\measuredangle(\uparrow_{p_n}^{x_{i,n}},\uparrow_{p_n}^{x_{j,n}}) \geqslant \varliminf_{n\to\infty}\tilde\measuredangle_\kappa(p_n\,{}^{x_{i,n}}_{x_{j,n}}) = \tilde\measuredangle_\kappa(p\,{}^{x_i}_{x_j}) \geqslant \measuredangle(\xi_i,\xi_j)-\varepsilon$$

□

6.3.6. Theorem. *Let $L\in\mathrm{Alex}^m[\kappa,\infty]$, $m<\infty$ $p\in L$. Then*

$$\mathrm{T}_p \overset{\text{iso}}{=\!=} \mathrm{Lin}_p^\perp\oplus\mathrm{Lin}_p,$$

where Lin_p is the linear subspace of T_p (defined in section) and $\mathrm{Lin}_p^\perp$ are subcone of T_x perpendicular to Lin_p.

Proof. Follows from splitting theorem 16.1.1 and 6.3.1. □