

On stability of minimal spheres and a two-dimensional version of Syngé's theorem

By

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1. Introduction. In this note, we will refine an elegant argument of Aminov [1] on the stability of minimal 2-spheres in a Riemannian manifold, which together with recent results of Sacks, Uhlenbeck, Meeks, and Yau [7] on the existence of stable minimal 2-spheres, yields the following theorem:

Theorem. *Let M be an n -dimensional compact Riemannian manifold with positive curvature operators, where $n \geq 3$. Then $\pi_2(M) = 0$.*

Recall that a Riemannian manifold M is said to have positive curvature operators if the linear transformation $\hat{R}: A^2 T_p M \rightarrow A^2 T_p M$ defined by

$$\langle \hat{R}(x \wedge y), z \wedge w \rangle = \langle R(x, y)w, z \rangle,$$

where R is the Riemann-Christoffel curvature tensor of M , has positive eigenvalues. Using linear Hodge theory, it can be shown [5] that if a compact connected oriented Riemannian manifold has positive curvature operators, then its real cohomology groups are isomorphic to those of a sphere. It might be conjectured that a compact simply connected Riemannian manifold with positive curvature operators is, in fact, homeomorphic to a sphere. To prove this conjecture when $n = \dim M \geq 4$ it would suffice to show that M has the integer homology of a sphere by a short argument based upon the positive resolution of the Poincaré conjecture. The above theorem, based upon the nonlinear theory of minimal surfaces (or from a different perspective, upon the nonlinear theory of harmonic maps), gives a little additional evidence for the validity of the conjecture by showing that $H_2(M; \mathbb{Z}) = 0$.

We can regard the theorem as a version of Syngé's theorem with π_1 replaced by π_2 . (See [3, p. 243] for the classical version of Syngé's theorem relying on the existence of smooth closed geodesics, and [9, § 7] for a version of Syngé's theorem (due to Simons) utilizing minimal hypersurfaces.) The proof is based upon the theory of holomorphic vector bundles over the Riemann sphere [6], a theory which was used effectively by Siu and Yau [10] in their resolution of the Frankel conjecture. In a sense, our theorem is optimal, in that the complex projective space has nonnegative curvature operators, but nonzero π_2 [5].

2. Proof of the theorem. Assume that $\pi_2(M) \neq 0$. According to Meeks and Yau [7, Theorem 5] there is a smooth harmonic map $f: S^2 \rightarrow M$ which minimizes energy among all noncontractible smooth maps from S^2 to M . This map f must be a stable branched minimal immersion.

Let f^*TM denote the pullback of the tangent bundle of M to S^2 . f^*TM inherits a fiber metric \langle, \rangle from the Riemannian metric of M and a connection ∇ from the Levi-Civita connection of M (see [8, p. 292, Lemma 3]). It will be convenient to give the complexified bundle $f^*TM \otimes \mathbb{C}$ a holomorphic structure. Note that the fiber metric \langle, \rangle can be extended in two useful ways to the complexification,

- as a complex bilinear form $(,)$, or
- as a Hermitian inner product $\langle\langle, \rangle\rangle$ which is conjugate linear in the second variable,

the relationship between these two being

$$\langle\langle Y_1, Y_2 \rangle\rangle = (Y_1, \bar{Y}_2), \text{ for } Y_1, Y_2 \in A^\circ(f^*TM) = \{\text{smooth sections of } f^*TM\}.$$

The connection on $f^*(TM)$ extends to a connection (also denoted by ∇) on $f^*TM \otimes \mathbb{C}$ which is unitary with respect to $\langle\langle, \rangle\rangle$. Regarding S^2 as a complex manifold, we let $A^{p,q}(f^*TM \otimes \mathbb{C})$ denote the space of (p, q) - forms on S^2 with coefficients in $f^*TM \otimes \mathbb{C}$. The connection ∇ decomposes into

$$\begin{aligned} \nabla': A^\circ(f^*TM \otimes \mathbb{C}) &\rightarrow A^{1,0}(f^*TM \otimes \mathbb{C}), \\ \nabla'': A^\circ(f^*TM \otimes \mathbb{C}) &\rightarrow A^{0,1}(f^*TM \otimes \mathbb{C}). \end{aligned}$$

It now follows from e.g. Atiyah [2, p. 46] that there is a unique holomorphic structure on $f^*TM \otimes \mathbb{C}$ such that $\nabla'' = \bar{\partial}$. We can regard $(,)$ as a section of the holomorphic bundle

$$(f^*TM \otimes \mathbb{C})^* \otimes (f^*TM \otimes \mathbb{C})^*$$

and since $(,)$ is parallel with respect to the connection ∇ , it is annihilated by $\nabla'' = \bar{\partial}$. In other words, $(,)$ defines a holomorphic orthogonal complex structure on $f^*TM \otimes \mathbb{C}$.

According to Grothendieck [6], $f^*TM \otimes \mathbb{C}$ decomposes into a Whitney sum of holomorphic line bundles

$$f^*TM \otimes \mathbb{C} = L_1 \oplus L_2 \oplus \dots \oplus L_n.$$

After a possible reordering, we can arrange that

$$c_1(L_1) \cong c_1(L_2) \cong \dots \cong c_1(L_n),$$

where $c_1 =$ Chern class. The ordered sequence of integers $(c_1(L_1), \dots, c_1(L_n))$ is then an invariant of the holomorphic vector bundle $f^*TM \otimes \mathbb{C}$ [6, p. 126], and since $(,)$ defines a holomorphic vector bundle isomorphism from $f^*TM \otimes \mathbb{C}$ to its dual,

$$c_1(L_i) = -c_1(L_{n-i}).$$

Note that each line bundle summand L_i with $c_1(L_i) \cong 0$ possesses a nonzero holomorphic section.

Let $z = x + iy$ be a holomorphic coordinate on S^2 -{point}. Since $f: S^2 \rightarrow M$ minimizes energy, if Y is a smooth section of $f^* TM$, $I(Y, Y) \cong 0$, where I is the index from given by the explicit formula ([4, p. 11]),

$$I(Y, Y) = \int_{S^2} \{ \langle \nabla_{\partial/\partial x} Y, \nabla_{\partial/\partial x} Y \rangle + \langle \nabla_{\partial/\partial y} Y, \nabla_{\partial/\partial y} Y \rangle - \langle R(Y, \partial/\partial x) \partial/\partial x, Y \rangle - \langle R(Y, \partial/\partial y) \partial/\partial y, Y \rangle \} dx \wedge dy.$$

(For simplicity of notation, we write $\partial/\partial x$ instead of $f_*(\partial/\partial x)$ for the section of $f^* TM$ determined by the vector field $\partial/\partial x$ on S^2 .) The index form can be extended to a Hermitian symmetric bilinear form on sections of $f^* TM \otimes \mathbb{C}$ by extending the curvature tensor R to be complex linear and replacing $\langle \cdot, \cdot \rangle$ by $\langle\langle \cdot, \cdot \rangle\rangle$ in the above formula.

Lemma. *If Z is a smooth section of $f^* TM \otimes \mathbb{C}$,*

$$\int_{S^2} \{ \langle\langle \nabla_{\partial/\partial x} Z, \nabla_{\partial/\partial x} Z \rangle\rangle + \langle\langle \nabla_{\partial/\partial y} Z, \nabla_{\partial/\partial y} Z \rangle\rangle \} dx \wedge dy = 4 \int_{S^2} \langle\langle \nabla_{\partial/\partial \bar{z}} Z, \nabla_{\partial/\partial \bar{z}} Z \rangle\rangle dx \wedge dy + i \int_{S^2} \langle\langle R(\partial/\partial x, \partial/\partial y) Z, Z \rangle\rangle dx \wedge dy.$$

Proof. Integrate by parts as follows:

$$\begin{aligned} \int_{S^2} \|\nabla_{\partial/\partial \bar{z}} Z\|^2 dx \wedge dy &= - \int_{S^2} \langle\langle \nabla_{\partial/\partial \bar{z}} \nabla_{\partial/\partial \bar{z}} Z, Z \rangle\rangle dx \wedge dy \\ &= -(1/4) \int_{S^2} \langle\langle \nabla_{\partial/\partial x - i \partial/\partial y} \nabla_{\partial/\partial x + i \partial/\partial y} Z, Z \rangle\rangle dx \wedge dy \\ &= -(1/4) \int_{S^2} \langle\langle (\nabla_{\partial/\partial x} \nabla_{\partial/\partial x} + \nabla_{\partial/\partial y} \nabla_{\partial/\partial y}) Z, Z \rangle\rangle dx \wedge dy \\ &\quad - (i/4) \int_{S^2} \langle\langle (\nabla_{\partial/\partial x} \nabla_{\partial/\partial y} - \nabla_{\partial/\partial y} \nabla_{\partial/\partial x}) Z, Z \rangle\rangle dx \wedge dy \\ &= (1/4) \int_{S^2} \{ \|\nabla_{\partial/\partial x} Z\|^2 + \|\nabla_{\partial/\partial y} Z\|^2 \} dx \wedge dy \\ &\quad - (i/4) \int_{S^2} \langle\langle R(\partial/\partial x, \partial/\partial y) Z, Z \rangle\rangle dx \wedge dy. \quad \text{q.e.d.} \end{aligned}$$

In view of the lemma, the index form on $f^* TM \otimes \mathbb{C}$ can be written

$$I(Z, Z) = 4 \int_{S^2} \|\nabla_{\partial/\partial \bar{z}} Z\|^2 dx \wedge dy + i \int_{S^2} \langle\langle R(\partial/\partial x, \partial/\partial y) Z, Z \rangle\rangle dx \wedge dy - \int_{S^2} \{ \langle\langle R(Z, \partial/\partial x) \partial/\partial x, Z \rangle\rangle + \langle\langle R(Z, \partial/\partial y) \partial/\partial y, Z \rangle\rangle \} dx \wedge dy.$$

If $Z = X + iY$, where X and Y are sections of the real vector bundle $f^* TM$,

$$\begin{aligned} i \langle\langle R(\partial/\partial x, \partial/\partial y) Z, Z \rangle\rangle &= 2 \langle\langle R(\partial/\partial x, \partial/\partial y) X, Y \rangle\rangle \\ &= -2 \langle\langle R(X, \partial/\partial x) \partial/\partial y, Y \rangle\rangle + 2 \langle\langle R(X, \partial/\partial y) \partial/\partial x, Y \rangle\rangle, \end{aligned}$$

the last equality using the Bianchi symmetry of the curvature tensor. Hence, if Z is a holomorphic section of $f^* TM \otimes \mathbb{C}$,

$$\begin{aligned} I(Z, Z) &= - \int_{S^2} \{ \langle\langle \hat{R}(X \wedge \partial/\partial x + Y \wedge \partial/\partial y), X \wedge \partial/\partial x + Y \wedge \partial/\partial y \rangle\rangle \\ &\quad + \langle\langle \hat{R}(X \wedge \partial/\partial y - Y \wedge \partial/\partial x), X \wedge \partial/\partial y - Y \wedge \partial/\partial x \rangle\rangle \} dx \wedge dy. \end{aligned}$$

Under the assumption that \hat{R} is positive-definite,

$$I(Z, Z) \geq 0 \Rightarrow X \wedge \partial/\partial x + Y \wedge \partial/\partial y = 0 = X \wedge \partial/\partial y - Y \wedge \partial/\partial x.$$

To derive a contradiction and thus finish the proof of the theorem, it therefore suffices to show that there exists a holomorphic section $Z = X + iY$ of $f^*TM \otimes \mathbb{C}$ such that

$$Z \wedge (\partial/\partial z) = (1/2)(X + iY) \wedge (\partial/\partial x - i\partial/\partial y) \neq 0.$$

(Here $\partial/\partial z$ stands for the f_* image of the vector field $\partial/\partial z$ on S^2 .) But this is clear, since the direct sum decomposition of $f^*TM \otimes \mathbb{C}$ possesses at least two holomorphic line bundles with nonnegative Chern classes, and hence $f^*TM \otimes \mathbb{C}$ must possess at least two linearly independent holomorphic sections.

References

- [1] JU. A. AMINOV, On the instability of a minimal surface in an n -dimensional Riemannian space of positive curvature. *Math. USSR-Sb.* **29** (3), 359–375 (1976).
- [2] M. F. ATIYAH, *Geometry of Yang-Mills fields*. Lezioni Fermiane, Scuola Normale Superiore, Pisa 1979.
- [3] R. L. BISHOP and R. J. CRITTENDEN, *Geometry of manifolds*. New York 1964.
- [4] J. EELLS and L. LEMAIRE, A report on harmonic maps. *Bull. London Math. Soc.* **10**, 1–68 (1978).
- [5] S. GALLOT and D. MEYER, Opérateur de courbure et Laplacien des formes différentielles d'une variété Riemannienne. *J. Math. Pures Appl.* **54**, 259–284 (1975).
- [6] A. GROTHENDIECK, Sur la classification des fibres holomorphes sur la sphère de Riemann. *Amer. J. Math.* **79**, 121–138 (1957).
- [7] W. H. MEEKS III and S. T. YAU, Topology of three dimensional manifolds and the embedding problems in minimal surface theory. *Ann. of Math.* **112**, 441–484 (1980).
- [8] J. W. MILNOR and J. D. STASHEFF, *Characteristic classes*. Princeton, New Jersey 1974.
- [9] L. SIMON, Survey lectures on minimal submanifolds, pp. 3–52 in E. Bombieri (ed.) *Seminar on minimal submanifolds*. *Ann. of Math. Studies* **103**, Princeton, New Jersey 1983.
- [10] Y. T. SIU and S. T. YAU, Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.* **59**, 189–204 (1980).

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