17.3 On area of boundary

The following probelem was suggested by A. Lytchak. It looks simple, but we could not make a proof without use of gradient exponent.

This problem would have followed from conjecture **??** (that boundary of an Alexandrov's space is an Alexandrov's space), but before this conjecture has been proved, any partial result is of some interest, see also remark on page 143.

17.3.1. Claim. *Let* $L \in \mathrm{Alex}^m[1,\infty]$. *Then*

$$\mathrm{vol}_{m-1}\,\partial L \leqslant \mathrm{vol}_{m-1}\,\mathbb{S}^{m-1}.$$

Let us first prepare a proposition:

17.3.2. Proposition. *The inverse of the gradient exponential map* $\mathrm{gexp}_p^{-1}(\kappa;*)$ *is uniquely defined inside any minimizing geodesic starting at* p.

Proof. Let $\gamma : [0,t_0] \to L$ be a unit-speed minimizing geodesic, $\gamma(0) = p$, $\gamma(t_0) = q$. From the angle comparison we get that $|\nabla_x \mathrm{dist}_p| \geqslant -\cos\tilde\measuredangle_\kappa(x\,{}^p_q)$. Therefore, for any ζ we have

$$|p\alpha_\zeta(t)|^+_t \geqslant -|\alpha^+_\zeta(t)|\cos\tilde\measuredangle_\kappa(\alpha_\zeta(t)\,{}^p_q) \ \text{ and } \ |\alpha_\zeta(t)q|^+_t \geqslant -|\alpha^+_\zeta(t)|.$$

Therefore, $\tilde\measuredangle_\kappa(q\,{}^{\alpha_\zeta(t)}_p)$ is nondecreasing in t, hence the result. □

Proof of 17.3.1. Let $z \in L$ be the point at maximal distance from ∂L, in particular it realizes maximum of $f = \sin\circ\,\mathrm{dist}_{\partial L}$. From theorem 9.3.2, $f(z) \leqslant 1$ and $f'' + f \leqslant 0$.

Note that $L \subset \overline{\mathrm{Ball}}(\frac{\pi}{2}, z)$, otherwise if $y \in L$ with $|yz| > \pi/2$, then appling inequality $f'' + f \leqslant 0$ to the geodesic $[zy]$, we get $d_z f(\uparrow^y_z) > 0$, i.e. z is not a maximum of f.

From this it follows that gradient exponent

$$\mathrm{gexp}_z(1;*) : (\overline{\mathrm{Ball}}(\tfrac{\pi}{2}, o_z), \mathfrak{s}) \to L$$

is a short onto map.

Moreover,

$$\partial L \subset \mathrm{gexp}_z(\partial\overline{\mathrm{Ball}}(\tfrac{\pi}{2}, o_z)).$$

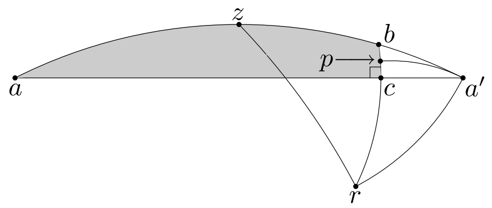
Indeed, gexp gives a homotopy equivalence $\partial\overline{\mathrm{Ball}}(\pi/2, o_z)) \to L\backslash z$. Clearly, $\Sigma_z = \partial(\overline{\mathrm{Ball}}(\pi/2, o_z), \mathfrak{s})$ has no boundary, therefore $H_{m-1}(\partial L, \mathbb{Z}_2) \neq 0$, see [Grove–Petersen 93, lemma 1]. Hence for any point $x \in \partial L$, any minimizing geodesic zx must have a point of the image $\mathrm{gexp}_z(\partial\overline{\mathrm{Ball}}(\frac{\pi}{2}, o_z))$ but, as it is shown in proposition 17.3.2, it can only be its end x.

Now since

$$\mathrm{gexp}_z(1;*) : (\overline{\mathrm{Ball}}(\tfrac{\pi}{2}, o_z), \mathfrak{s}) \to L$$

is short and $(\partial\overline{\mathrm{Ball}}(\frac{\pi}{2}, o_z), \mathfrak{s})$ is isometric to $\Sigma_z L$ we get $\mathrm{vol}\,\partial L \leqslant \mathrm{vol}\,\Sigma_z L$ and clearly, $\mathrm{vol}\,\Sigma_z L \leqslant \mathrm{vol}\,\mathbb{S}^{m-1}$. □

Remark. Among other corollaries of conjecture **??**, it is expected that if $L \in \mathrm{Alex}^m[1,\infty]$ then ∂L, equipped with induced intrinsic metric, admits a non-contracting map to $\mathbb{S}^{m-1}$. In particular, its intrinsic diameter is at most π, and perimeter of any triangle in ∂L is at most 2π. This does not follow from the proof above, since in general $\mathrm{gexp}_z(1;\partial\overline{\mathrm{Ball}}(\pi/2,o_z)\not\subset\partial L$, i.e. $\mathrm{gexp}_z(1;\partial\overline{\mathrm{Ball}}(\pi/2,o_z)$ might have some creases left inside of L, which might be used as a shortcut for curves with ends in ∂L.



The existence of such creases one can see already in dimension 2: Cut a spherical triangle $[abc]$, with right angle at c and $|ab|>\frac{\pi}{2}$. Let us glue space L from two copies of $[abc]$ along sides $[ab]$ and $[bc]$.

According to doubling theorem ???, $L\in\mathrm{Alex}^2[1,\infty]$ and its boundary consists of copies of side $[ac]$. The point z, the farest point from the boundary, lies on side $[ab]$ and $|az|=\frac{\pi}{2}$. Let us show that

$$p=\mathrm{gexp}_z(1;\tfrac{\pi}{2}\cdot\uparrow_z^b)\notin\partial L.$$

Clearly $p\in[bc]$, thus we need to show only that $p\neq c$. One can check it by calculations, but also one can see it using the following additional construction in $\mathbb{S}^2$: Let a' be the opposite point to a in $\mathbb{S}^2$ and r be a point on the extension of $[bc]$ behind c such that $|zr|_{\mathbb{S}^2}=\frac{\pi}{2}$. Then $p\in[bc]$ and from the fact that $\mathrm{gexp}_z(1;*)$ is short, one can see that $|pr|=|a'r|$. Clearly $|a'r|>|cr|$; therefore $p\neq c$; i.e. $p\notin\partial L$.