5.2 Halbeisen's example

Here we construct an infinite-dimensional Alexandrov space $\check L$ with a point $p \in \check L$ such that the space of directions $\Sigma_p \check L$ and therefore the tangent space $\mathrm{T}_p \check L$ are not a length-spaces. The construction is just a little variation of one in [Halbeisen].

Let H be a Hilbert space and $\{\mathbf{e}_n\}$, $n = \mathbb{Z}_\geqslant$ be its orthonormal basis. Fix a small $\varepsilon > 0$ and consider two functions $f, \check f : H \to \mathbb{R}$

$$f(\mathbf{x}) = |\mathbf{x}|,$$

$$\check f(\mathbf{x}) = \max\left\{|\mathbf{x}|, \max_{n \in \mathbb{Z}_\geqslant}\{(1+\varepsilon)\langle \mathbf{x}, \mathbf{e}_n\rangle - \tfrac{1}{n}\}\right\}.$$

Both of these functions are convex and Lipschitz, therefore their graphs in $H \times \mathbb{R}$ equped with its intrinsic metric form infinte dimensional Alexandrov spaces, say L and $\check L$ (it is proved formally in 5.2.1).

Set p to be the origin of $H \times \mathbb{R}$. The space L is isometric to euclidean cone over $\Sigma_p L$ with vretex at p; $\Sigma_p L$ is isometric to a sphere in Hilbert space with radius $\frac{1}{\sqrt 2}$. Moreover, for any direction $\xi \in \Sigma_p L$ there is $\delta > 0$ such that the sector described by

$$\mathrm{Sect}_\delta(\xi) = \{x \in L_0 | \measuredangle_{L_0}(\uparrow_p^x, \xi) < \tfrac{\pi}{8}, |px| < \delta\}$$

also is a part of $\check L$. Note that for any two points q, r which lie in both L and $\check L$ we have

$$|qr|_{L_0} \geqslant |qr|_{\check L} > (1 - \tfrac{1}{100})|qr|_L,$$

the last inequality is valid sinse ε is small. These two facts imply that $\Sigma_p \check L$, as a set, coinsides with $\Sigma_p L$ and moreover the corresponding angle metrics $\measuredangle$ and $\check\measuredangle$ locally coniside on $\Sigma_p L$. Therefore the intrinsic metric induced by $\check\measuredangle$ on $\Sigma_p L$ coinsides with $\measuredangle$. Next we will show that for some pair of directrions, we have

$$\check\measuredangle(\xi_+, \xi_-) < \measuredangle(\xi_+, \xi_-).$$

and thus $\check\measuredangle$ is not intrinsic.

To see this, note that the following two rays in $H \times \mathbb{R}$

$$\gamma_+(t) = \tfrac{1}{\sqrt 2}(\mathbf{e}_0 t, t) \;\text{ and }\; \gamma_-(t) = \tfrac{1}{\sqrt 2}(-\mathbf{e}_0 t, t), \quad t \in [0, +\infty)$$

form unit-speed geodesics in both, L and $\check L$. Let $\xi_\pm$ be the directions of $\gamma_\pm$ at p. Denote by σ_n the half-planes in H which passes through $\mathbf{e}_n$ and has the boundary line spanned by $\mathbf{e}_0$. Consider sequence of 2-dimensional sectors $Q_n = \check L \cap (\sigma_n \times \mathbb{R})$. For each n, Q_n is bounded by two geodesic rays $\gamma_\pm$. Clearly $Q_n \xrightarrow[\mathrm{GH}]{} Q$, where Q is a euclidean anglein $\mathbb{R}^2$ with anglemeasure $\beta < \measuredangle(\xi_+, \xi_-) = \pi/\sqrt 2$. Indeed, Q_n is path-isometric to the subset of $\mathbb{R}^3$ described by

$$x \geqslant 0 \;\text{ and }\; z = \max\left\{\sqrt{x^2 + y^2}, (1+\varepsilon)y - \tfrac{1}{n}\right\}.$$

Thus, its limit Q is path-isometric to

$$x \geqslant 0 \;\text{ and }\; z = \max\left\{\sqrt{x^2 + y^2}, (1+\varepsilon)y\right\}.$$

In particular, for any $t, \tau \geqslant 0$,

$$|\gamma_+(t)\,\gamma_-(\tau)|^2_{\check{L}} \leqslant \lim_{n\to\infty} |\gamma_+(t)\,\gamma_-(\tau)|^2_{Q_n} = t^2 + \tau^2 - 2t\tau\cos\beta.$$

Therefore $\check{\measuredangle}(\xi_+, \xi_-) \leqslant \beta < \measuredangle(\xi_+, \xi_-)$.

5.2.1. Claim. *Let $S \subset H \times \mathbb{R}$ be a graph of convex Lipshitz function $f : H \to \mathbb{R}$. Then S, equiped with its intrinsic metric, forms an Alexandrov space with curvature $\geqslant 0$.*

Proof. For a subset $X \in H \times \mathbb{R}$, we will denote by $|**|_X$ the induced intrinsic distances in X.

Note that according to ??? any convex hypersurface in euclidean space equiped with intrinsic metric has curv $\geqslant 0$. Thus it is enough to show that for any 4-point set $\{x_0, x_1, x_2, x_3\} \subset S$, there is a finite-dimensional subspace $E \subset H \times \mathbb{R}$, such that $\{x_i\} \in E$ and $|x_i x_j|_{S\cap E}$ is arbitrary close to $|x_i x_j|_S$.

Clearly $|x_i x_j|_{S\cap E} \geqslant |x_i x_j|_S$, thus it is enough to show that for given $\varepsilon > 0$ one can choose E, so that

$$|x_i x_j|_{S\cap E} < |x_i x_j|_S + \varepsilon \qquad (*)$$

For each pair (x_i, x_j) choose a broken line β_{ij} connecting x_i, x_j, which lies under S (i.e. ouside of $\operatorname{Conv} S$) in $H \times \mathbb{R}$ and has length at most $|x_i x_j|_S + \varepsilon$. Take E to be the affine hull of all the vertxes in all β_{ij}. Then clearly $|x_i x_j|_{S\cap E} \leqslant$ lenght β_{ij} and thus we get $(*)$. ☐