Chapter 11

Gradient flow

The most of technique developed in this section works in more general setting. Roughly, if your space has well defined angles then you have big chances for to apply gradient flow in there. However we were not trying hard to make these statements most general.

Starting with [Sharafutdinov], the technique of gradient flow was used in comparison geometry, it was used further in ???. In [Perelman–Petrunin QG], it was adapted to Alexandrov spaces with curvature bounded below. Bit later, independently in [Jost] and [Mayer], it had been used in spaces with curvature bounded above. These two approaches were unified and generalized to a wide class of metric spaces in [Lytchak] and yet developed further in [Ohta]. The part of this work related to upper curvature bound is summarized and generalized in [AGS] (but this book it is not at all reader-friendly).

Gradient flow provide a usefull tool in Alexandrov's geometry, which we will use everywhere in the book; we even use it to prove very basic things which are usually proved by other means.

One of the technical difficulties in Alexandrov's geometry comes from nonextendability of geodesics. In particular, for $L \in \mathrm{Alex}[\kappa,\infty]$, the exponential map, $\exp_p : \mathrm{T}_p \to L$, if defined the usual way, can be undefined in an arbitrary small neighborhood of origin o_p. Here we construct its analog, the *gradient exponential map* (gexp_p), which practically solves this problem. It has many important properties of the ordinary exponential map and in certain respects it is "better", even in Riemannian case.

This chapter can be divided in three main parts: (1) gradient part (sections 11.1–11.3) which describes "gradient vector field" and its properties; (2) gradient curves ans gradient flow (sections 11.4–11.6) Radial curves and gradient exponent (sections 11.7–11.8).

11.1 Gradient: Definition and existence

11.1.1. Definition of gradient. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \looparrowright \mathbb{R}$ and for a point $p \in \mathrm{Dom}\, f$ the differential $d_p f : \mathrm{T}_p \to \mathbb{R}$ is well defined.*

A tangent vector $g \in \mathrm{T}_p$ is called a gradient of f at p *(in short: $g = \nabla_p f$) if*

(i) $d_p f(w) \leqslant \langle g, w\rangle$ for any $w \in \mathrm{T}_p$, and

(ii) $d_p f(g) = \langle g, g\rangle$.

11.1.2. Existence and uniqueness. *Assume* $L \in \mathrm{Alex}[\kappa,\infty]$ *and* $f : L \circ\!\!\!\to \mathbb{R}$ *be locally Lipschitz and semiconcave. Then for any point* $p \in \operatorname{Dom} f$*, there is unique gradient* $\nabla_p f \in \mathrm{T}_p$.

Proof. Let us first prove uniqueness. If $g, g' \in \mathrm{T}_p$ both can serve as gradients then

$$\langle g, g\rangle = d_p f(g) \leqslant \langle g, g'\rangle \ \text{ and } \ \langle g', g'\rangle = d_p f(g') \leqslant \langle g, g'\rangle,$$

i.e.

$$|gg'|^2 = \langle g, g\rangle - 2\langle g, g'\rangle + \langle g', g'\rangle \leqslant 0.$$

To prove existence, note first that if $d_p f \leqslant 0$ then one can take $\nabla_p f = o_p$.

Otherwise, if $s = \sup_{\xi\in\Sigma_p} d_p(\xi) > 0$, it is sufficient to show that there is $\overline{\xi} \in \Sigma_p$ such that $d_p(\overline{\xi}) = s$. Indeed, if $\overline{\xi}$ does exist, then applying lemma 7.8.3 for $u = \overline{\xi}$, $v = \varepsilon \cdot w$ with $\varepsilon \to 0+$, we get

$$d_p f(w) \leqslant \langle w, s{\cdot}\overline{\xi}\rangle$$

for any $w \in \mathrm{T}_p$. I.e., one can take $\nabla_p f = s \cdot \overline{\xi}$.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $d_p f(\xi_n) \to s$. Applying lemma 7.8.3 again for $u = \xi_n$, $v = \xi_m$, we get

$$s \geqslant \frac{d_p(\xi_n) + d_p(\xi_m)}{\sqrt{2 + 2\cos\measuredangle(\xi_n, \xi_m)}}.$$

Therefore, $\measuredangle(\xi_n, \xi_m) \to 0$ as $m, n \to \infty$. Thus (ξ_n) is a Cauchy sequence and one can take $\overline{\xi} = \lim_n \xi_n$. □

11.2 Gradient: Basic calculus

The next lemma roughly states that gradient of function points in the direction of its maximal slope and if slope is almost maximal then it is almost direction of gradient.

11.2.1. Lemma. *Let* $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \circ\!\!\!\to \mathbb{R}$ *be locally Lipschitz and semiconcave and* $p \in \operatorname{Dom} f$.

Assume $|\nabla_p f| > 0$*, set* $\overline{\xi} = \frac{1}{|\nabla_p f|}{\cdot}\nabla_p f$ *then*

(i) If for $v \in \mathrm{T}_p$*, we have* $|v| \leqslant 1 + \varepsilon$ *and* $d_p(v) > |\nabla_p f|(1 - \varepsilon)$*, then*

$$\left|\overline{\xi}\, v\right| < 100\sqrt{\varepsilon}.$$

(ii) If $v_n \in \mathrm{T}_p$ *be a sequence of vectors such that* $\overline{\lim}_n |v_n| \leqslant 1$ *and* $\underline{\lim}_n d_p f(v_n) \geqslant |\nabla_p f|$ *then*

$$\lim_{n\to\infty} v_n = \overline{\xi}.$$

(iii) $\overline{\xi}$ *is the unique maximum direction for the restriction* $d_p f|_{\Sigma_p}$*. In particular,*

$$|\nabla_p f| = \sup_{\xi\in\Sigma_p f} d_p f.$$

Proof. According to definition of gradient,

$$|\nabla_p f|(1-\varepsilon) < d_p f(v) \leqslant \langle v, \nabla_p f\rangle = |v|\cdot|\nabla_p f|\cos\measuredangle(\nabla_p f, v)$$

thus

$$\big||v|-1\big| < 2\varepsilon \quad\text{and}\quad \cos\measuredangle(\nabla_p f, v) > (1-2\varepsilon),$$

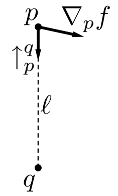
hence we get (i). Statements (ii) and (iii) follow directly from (i). ☐

Remark. Note that according to ???, if $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \circ\!\!\to \mathbb{R}$ is semiconcave and $\varphi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and semiconcave then $\varphi\circ f$ is also semiconcave. As a corollary of the above theorem we also get that $\nabla_x(\varphi\circ f) = \varphi^+(f(x))\cdot\nabla_x f$ for any $x \in \operatorname{Dom} f$.

The following inequalities describe an important property of the "gradient vector field" which will be used throughout this paper.

11.2.2. Lemma. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \circ\!\!\to \mathbb{R}$ satisfies $f'' + \varkappa f \leqslant \lambda$ and $[pq] \subset \operatorname{Dom} f$. Then*

$$\langle \uparrow_p^q, \nabla_p f\rangle \geqslant \frac{f(q) - f(p)\operatorname{cs}_\varkappa \ell - \lambda \operatorname{md}_\varkappa \ell}{\operatorname{sn}_\varkappa \ell},$$



where $\ell = |pq|$.

In particular,

- *if $\varkappa = 0$,*
$$\langle \uparrow_p^q, \nabla_p f\rangle \geqslant \Big(f(q) - f(p) - \lambda\tfrac{\ell^2}{2}\Big)/\ell;$$
- *if $\varkappa = 1$, $\lambda = 0$ we have*
$$\langle \uparrow_p^q, \nabla_p f\rangle \geqslant (f(q) - f(p)\cos\ell)/\sin\ell;$$
- *if $\varkappa = -1$, $\lambda = 0$ we have*
$$\langle \uparrow_p^q, \nabla_p f\rangle \geqslant (f(q) - f(p)\operatorname{ch}\ell)/\operatorname{sh}\ell;$$

Proof of 11.2.2. Since $f'' + \varkappa f \leqslant \lambda$, we have

$$\begin{aligned} f(q) &= \gamma_{[pq]}(\ell) \\ &\leqslant (f\circ\gamma_{[pq]})^+(0)\operatorname{sn}_\varkappa \ell + f(p)\operatorname{cs}_\varkappa\ell + \lambda\operatorname{md}_\varkappa\ell \end{aligned}$$

Note that $\gamma_{[pq]}(0) = p$, $\gamma_{[pq]}(\ell) = q$, $\gamma^+_{[pq]}(0) = \uparrow_p^q$. Thus,

$$\begin{aligned} \langle \uparrow_p^q, \nabla_p f\rangle &\geqslant d_p f(\uparrow_p^q) = \\ &= (f\circ\gamma_{[pq]})^+(0) \geqslant \\ &\geqslant \frac{f(q) - f(p)\operatorname{cs}_\varkappa \ell - \lambda\operatorname{md}_\varkappa \ell}{\operatorname{sn}_\varkappa \ell}. \end{aligned}$$ ☐

The following corollary states that gradient vector field is monotonic in the sense similar to definition of *monotone operators* (see [Phelps]).

11.2.3. Corollary (monotonicity of gradient). *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \circ\!\!\to \mathbb{R}$ be locally Lipschitz and λ-concave and $[pq] \subset \operatorname{Dom} f$. Then*

$$\langle \uparrow_p^q, \nabla_p f\rangle + \langle \uparrow_q^p, \nabla_q f\rangle \geqslant -\lambda\ell.$$

where $\ell = |pq|$.

Proof. This inequality is a sum of two inequalities from 11.2.2 □

11.2.4. Lemma. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f, g : L \circ\!\!\to \mathbb{R}$ and $p \in \operatorname{Dom} f \cap \operatorname{Dom} g$. Then*

$$|\nabla_p f\, \nabla_p g|^2 \leqslant (|\nabla_p f| + |\nabla_p g|) \sup_{\xi\in\Sigma_p} \{\, |d_p f(\xi) - d_p f(\xi)|\,\}.$$

In particular, if $f_n : L \circ\!\!\to \mathbb{R}$ is a sequence of locally Lipschitz and semiconcave subfunctions such that $p \in \operatorname{Dom} f_n$ for each n and $d_p f_n$ converges uniformly on Σ_p then sequence $\nabla_p f_n \in T_p$ converges.

Proof. Set $s = \sup_{\xi\in\Sigma_p}\{\, |d_p f(\xi) - d_p f(\xi)|\,\}$. Clearly for any $v \in T_p$, we have

$$|d_p f(v) - d_p g(v)| \leqslant s|v|.$$

From the definition of gradient (11.1.1) we have:

$$\begin{aligned} d_p f(\nabla_p g) &\leqslant \langle \nabla_p f, \nabla_p g\rangle, & d_p g(\nabla_p f) &\leqslant \langle \nabla_p f, \nabla_p g\rangle,\\ d_p f(\nabla_p f) &= \langle \nabla_p f, \nabla_p f\rangle, & d_p g(\nabla_p g) &= \langle \nabla_p g, \nabla_p g\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} |\nabla_p f\, \nabla_p g|^2 &= \langle \nabla_p f, \nabla_p f\rangle + \langle \nabla_p g, \nabla_p g\rangle - 2\langle \nabla_p f, \nabla_p g\rangle \leqslant\\ &\leqslant d_p f(\nabla_p f) + d_p g(\nabla_p g) - d_p f(\nabla_p g) - d_p g(\nabla_p f) \leqslant\\ &\leqslant s(|\nabla_p f| + |\nabla_p g|). \end{aligned}$$

□

11.3 Gradient: Semicontinuity

In this section we collect number of corollaries of the following simple lemma.

11.3.1. Ultra limit of |gradient|. *Assume*

(i) $L_n \in \mathrm{Alex}[\kappa,\infty]$ *and* $(L_n, p_n) \xrightarrow{\varpi} (L_\varpi, p_\varpi)$*;*

(ii) $f_n : L_n \circ\!\!\to \mathbb{R}$ *and* $f_\varpi : L_\varpi \circ\!\!\to \mathbb{R}$ *are locally Lipschitz and λ-concave and* $f_n \xrightarrow{\varpi} f_\varpi$*;*

(iii) $x_n \in \operatorname{Dom} f_n$ *and* $x_n \xrightarrow{\varpi} x_\varpi \in \operatorname{Dom} f_\varpi$.

Then

$$|\nabla_{x_\varpi} f_\varpi| \leqslant \varpi\text{-}\lim_{n\to\infty} |\nabla_{x_n} f_n|.$$

Proof. Fix an $\varepsilon > 0$ and choose $y_\omega \in \operatorname{Dom} f_\omega$ near x_ω such that

$$|\nabla_{x_\omega} f_\omega| - \varepsilon < \frac{f_\omega(y_\omega) - f_\omega(x_\omega)}{|x_\omega y_\omega|}.$$

Choose $y_n \in L_n$ such that $y_n \xrightarrow{\omega} y_\omega$. If $|x_\omega y_\omega|$ is sufficiently small, the λ-concavity of f_n implies that

$$|\nabla_{x_\omega} f_\omega| - 2\varepsilon < d_{x_n} f_n(\uparrow_{x_n}^{y_n}),$$

for ω-almost all n. Hence, $|\nabla_p f| - 2\varepsilon \leqslant \omega\text{-}\lim_{n\to\infty} |\nabla_{p_n} f_n|$ for any $\varepsilon > 0$; i.e.

$$|\nabla_p f| \leqslant \omega\text{-}\lim_{n} |\nabla_{p_n} f_n|. \qquad \square$$

Note that isometric embedding $\imath : L \hookrightarrow L^\omega$ induces an embedding $\imath_p : \mathrm{T}_p L \hookrightarrow \mathrm{T}_p L^\omega$. Thus, we can (and will) consider $\mathrm{T}_p L$ as a subcone of $\mathrm{T}_p L^\omega$.

11.3.2. Corollary. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \multimap \mathbb{R}$ be locally Lipschitz and semiconcave. Then for any point $p \in \operatorname{Dom} f$ we have*

$$\nabla_p f = \nabla_p f^\omega.$$

Proof. Note that $\operatorname{Dom} f \subset \operatorname{Dom} f^\omega \subset L^\omega$, applying 11.3.1 for $L_n = L$ and $x_n = x$, we get $|\nabla_x f| \geqslant |\nabla_x f^\omega|$. On the other hand, $f = f^\omega|_L$, hence $d_p f = d_p f^\omega|_{\mathrm{T}_p L}$ and thus, from 11.2.1(iii) $|\nabla_x f| \leqslant |\nabla_x f^\omega|$. Therefore $|\nabla_x f| = |\nabla_x f^\omega|$.

Further,

$$\begin{aligned}|\nabla_x f|^2 &= d_p f(\nabla_x f)\\ &= d_p f^\omega(\nabla_x f) \leqslant\\ &\leqslant \langle \nabla_x f^\omega, \nabla_x f\rangle =\\ &= |\nabla_x f^\omega| \cdot |\nabla_x f| \cos\measuredangle(\nabla_x f^\omega, \nabla_x f).\end{aligned}$$

Together with $|\nabla_x f^\omega| = |\nabla_x f|$ it implies that $\measuredangle(\nabla_x f^\omega, \nabla_x f) = 0$, and the statement follows. $\square$

In particular, we have lower-semicontinuity of the function $x \mapsto |\nabla_x f|$:

11.3.3. Semicontinuity of |gradient|. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \multimap \mathbb{R}$ be locally Lipschitz and semiconcave. Then the function $x \mapsto |\nabla_x f|$ is lower-semicontinuos; i.e. for any sequence $x_n \to x \in \operatorname{Dom} f$, we have*

$$|\nabla_x f| \leqslant \varliminf_{n\to\infty} |\nabla_{x_n} f|.$$

Proof. According to 11.3.2, $|\nabla_x f| = |\nabla_x f^\omega|$. Applying 11.3.1 for $x_n \to x$, we get that $\omega\text{-}\lim_n |\nabla_{x_n} f| \geqslant |\nabla_x f^\omega| = |\nabla_x f|$. Passing to arbitrary subsequence of (x_n) we obtain the result. $\square$

11.4 Gradient-like curves

Gradient-like curves provide a technical tool which will be used later in the construction of gradient curves, which will appear to be special reparametrization of gradient-like curves.

11.4.1. Definition. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be locally Lipschitz and semiconcave.*

A Lipschitz curve $\hat\alpha : [s_{\min}, s_{\max}) \to \operatorname{Dom} f$ will be called f-gradient-like curve *if*

$$\hat\alpha^{+} = \tfrac{1}{|\nabla f|} \cdot \nabla f;$$

i.e., for any $s \in [s_{\min}, s_{\max})$, $\hat\alpha^{+}(s)$ is well defined and $\hat\alpha^{+}(s) = \frac{1}{|\nabla_{\hat\alpha(s)} f|} \cdot \nabla_{\hat\alpha(s)} f$.

Note that in particular this definition implies that $|\nabla_p f| > 0$ for any point p on f-gradient-like curve. The next theorem gives a weaker equivalent definition.

11.4.2. Theorem. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be locally Lipschitz and semiconcave and $|\nabla_p f| > 0$ for any $p \in \operatorname{Dom} f$.*

A curve $\hat\alpha : [s_{\min}, s_{\max}) \to \operatorname{Dom} f$ is an f-gradient-like curve if and only if it is 1-Lipschitz and

$$\varliminf_{s \to s_0+} \frac{f \circ \hat\alpha(s) - f \circ \hat\alpha(s_0)}{s - s_0} \geqslant |\nabla_{\hat\alpha(s_0)} f| \qquad (*)$$

for almost all $s_0 \in [s_{\min}, s_{\max})$.

Proof. The "only if" part follows directly from definition. To prove the "if" part, note that for any $s_0 \in [s_{\min}, s_{\max})$ we have

$$\varliminf_{s \to s_0+} \frac{f \circ \hat\alpha(s) - f \circ \hat\alpha(s_0)}{s - s_0} \geqslant \varliminf_{s \to s_0+} \fint_{s_0}^{s} |\nabla_{\hat\alpha(\text{ş})} f| \, d\,\text{ş} \geqslant |\nabla_{\hat\alpha(s_0)} f|;$$

the first inequality follows from $(*)$ and the second from lower-semicontinuity of function $x \mapsto |\nabla_x f|$, see 11.3.3. Thus, the result follows form 11.2.1. □

The following theorem is similar to [Mayer, ???] and [Ohta, 5.7].

11.4.3. Theorem. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be locally Lipschitz and λ-concave. Assume $\hat\alpha : [0, s_{\max}) \to \operatorname{Dom} f$ is an f-gradient-like curve then $f \circ \hat\alpha : [0, s_{\max}) \to \mathbb{R}$ is λ-concave.*

11.4.4. Corollary. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be a locally Lipschitz and semiconcave function and $\hat\alpha : [0, s_{\max}) \to \operatorname{Dom} f$ be an f-gradient-like curve. Then function $s \mapsto |\nabla_{\hat\alpha(s)} f|$ is right-continuous, i.e. for any $s_0 \in [0, s_{\max})$ we have*

$$|\nabla_{\hat\alpha(s_0)} f| = \lim_{s \to s_0+} |\nabla_{\hat\alpha(s)} f|.$$

Proof. Applying 11.4.3 locally, we have that $f \circ \hat\alpha(s)$ is semiconcave. The statement follows since $(f \circ \hat\alpha)^{+}(s) = d_p f\Big(\frac{1}{|\nabla_{\hat\alpha(s)} f|} \cdot \nabla_{\hat\alpha(s)} f\Big) = |\nabla_{\hat\alpha(s)} f|$ □

Proof of 11.4.3. For any $s > s_0$,

$$\begin{aligned}(f\circ\hat\alpha)^+(s_0) &= |\nabla_{\hat\alpha(s_0)}f| \geqslant \\ &\geqslant d_{\hat\alpha(s_0)}f\left(\uparrow_{\hat\alpha(s_0)}^{\hat\alpha(s)}\right) \geqslant \\ &\geqslant \frac{f(\hat\alpha(s)) - f(\hat\alpha(s_0))}{|\hat\alpha(s)\,\hat\alpha(s_0)|} - \frac{\lambda}{2}|\hat\alpha(s)\,\hat\alpha(s_0)|.\end{aligned}$$

Set $\lambda_+ = \max\{0,\lambda\}$. Since $s - s_0 \geqslant |\hat\alpha(s)\,\hat\alpha(s_0)|$, for any $s > s_0$ we have

$$(f\circ\hat\alpha)^+(s_0) \geqslant \frac{f(\hat\alpha(s)) - f(\hat\alpha(s_0))}{s - s_0} - \frac{\lambda_+}{2}(s - s_0) \qquad (*)$$

Thus $f\circ\hat\alpha$ is λ_+-concave. That finishes the proof for $\lambda \geqslant 0$; in case $\lambda < 0$ we get only that $f\circ\hat\alpha$ is 0-concave.

Note that $|\hat\alpha(s)\,\hat\alpha(s_0)| = s - s_0 - o(s - s_0)$, thus

$$(f\circ\hat\alpha)^+(s_0) \geqslant \frac{f(\hat\alpha(s)) - f(\hat\alpha(s_0))}{s - s_0} - \frac{\lambda}{2}(s - s_0) + o(s - s_0). \qquad (**)$$

Together, $(*)$ and $(**)$ imply that $f\circ\hat\alpha$ is λ-concave. □

11.5 Gradient curves: Definition

In this section we define gradient curves and tie them tightly to gradient-like curves which were introduced in section 11.4.

11.5.1. Definition. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, and $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be a locally Lipschitz and semiconcave function.*

A locally Lipschitz curve $\alpha : [t_{\min}, t_{\max}) \to \operatorname{Dom} f$ will be called f-gradient curve *if*

$$\alpha^+ = \nabla f;$$

i.e. for any $t \in [t_{\min}, t_{\max})$, $\alpha^+(t)$ is well defined and $\alpha^+(t) = \nabla_{\alpha(t)}f$.

The next lemma states that gradient and gradient-like curves are special reparametrizations of each-other.

11.5.2. Lemma. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \mathrel{\circ\!\!\to} \mathbb{R}$ be a locally Lipschitz and semiconcave subfunction such that $|\nabla_p f| > 0$ for any $p \in \operatorname{Dom} f$.*

Assume $\alpha : [0, t_{\max}) \to \operatorname{Dom} f$ be a locally Lipschitz curve and $\hat\alpha : [0, s_{\max}) \to \operatorname{Dom} f$, be its reparametrization by arc-length, so $\alpha = \hat\alpha\circ\sigma$ for some homeomorphism $\sigma : [0, t_{\max}) \to [0, s_{\max})$, then

$$\alpha^+ = \nabla f \qquad \Leftrightarrow \qquad \hat\alpha^+ = \frac{1}{|\nabla f|}\cdot\nabla f \quad \textit{and} \quad \sigma^{-1}(s) = \int_0^s \frac{1}{(f\circ\hat\alpha)'(\text{ş})}d\text{ş}.$$

Proof $(\Rightarrow)$. According to 15.1.1,

$$\sigma'(t) \overset{\text{a.e.}}{=\!=} |\alpha^+(t)| = |\nabla_{\alpha(t)}f|. \qquad (*)$$

Note that

$$(f\circ\alpha)'(t)\overset{\text{a.e.}}{=\!=}(f\circ\alpha)^+(t)=|\nabla_{\alpha(t)}f|^2.$$

Thus, setting $s=\sigma(t)$,

$$(f\circ\hat\alpha)'(s)\overset{\text{a.e.}}{=\!=}\frac{(f\circ\alpha)'(t)}{\sigma'(t)}\overset{\text{a.e.}}{=\!=}|\nabla_{\alpha(t)}f|=|\nabla_{\hat\alpha(s)}f|.$$

Thus, from 11.4.2, it follows that $\hat\alpha(t)$ is an f-gradient-like curve, i.e.

$$\hat\alpha^+=\frac{1}{|\nabla f|}\cdot\nabla f.$$

In particular, $(f\circ\hat\alpha)^+(s)=|\nabla_{\hat\alpha^+(s)}f|$ and from $(*)$

$$\sigma^{-1}(s)=\int\limits_0^s\frac{1}{|\nabla_{\hat\alpha(\c{s})}f|}d\c{s}=\int\limits_0^s\frac{1}{(f\circ\hat\alpha)'(\c{s})}d\c{s}.$$

$(\Leftarrow)$. Clearly,

$$\sigma(t)=\int\limits_0^t(f\circ\hat\alpha)^+(\sigma(\c{t}))\ d\,\c{t}=\int\limits_0^t|\nabla_{\alpha(\c{t})}f|\ d\,\c{t}.$$

According to 11.4.4, the function $s\mapsto|\nabla_{\hat\alpha(s)}f|$ is right-continuous. Therefore the same is true for function $t\mapsto|\nabla_{\hat\alpha\circ\sigma(t)}f|=|\nabla_{\alpha(t)}f|$. Hence, for any $t_0\in[0,t_{\max})$ we have

$$\sigma^+(t_0)=\lim_{t\to t_0+}\fint\limits_{t_0}^{t}|\nabla_{\alpha(\c{t})}f|\,d\,\c{t}=|\nabla_{\alpha(t_0)}f|.$$

Thus, we have

$$\alpha^+(t_0)=\sigma^+(t_0)\cdot\hat\alpha^+(\sigma(t_0))=\nabla_{\alpha(t_0)}f.\qquad\square$$

11.6 Gradient curves: Existence and uniqueness

In general, “past” of gradient curves can not be determined by present. For example, consider concave function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=-|x|$; two curves $\alpha(t)=\min\{0,t\}$ and $\beta(t)=0$ are f-gradient and $\alpha(t)=\beta(t)=0$ for all $t\geqslant0$, however $\alpha(t)\ne\beta(t)$ for all $t<0$.

The next theorem shows that “future” gradient curve is unique.

11.6.1. Picard’s theorem. *Let* $L\in\mathrm{Alex}[\kappa,\infty]$, $f:L\,\circ\!\!\to\mathbb{R}$ *be semiconcave. Assume* $\alpha,\beta:[0,t_{\max})\to\operatorname{Dom}f$ *be two* f*-gradient curves such that* $\alpha(0)=\beta(0)$ *then* $\alpha(t)=\beta(t)$ *for any* $t\in[0,t_{\max})$.

As well as the standard Picard’s theorem, the proof follows directly from the following estimate:

11.6.2. Distance estimate. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $f : L \to \mathbb{R}$ be a locally Lipschitz and λ-concave function. Assume $\alpha, \beta : [0, t_{\max}) \to L$ be two f-gradient then for any $t \in [0, t_{\max})$*

$$|\alpha(t)\ \beta(t)| \leqslant e^{\lambda t}|\alpha(0)\ \beta(0)|.$$

Moreover, the same conclusion holds for a for locally Lipschitz and λ-concave subfunction $f : L \hookrightarrow \mathbb{R}$ if for any $t \in [0, t_{\max})$ there is a geodesic $[\alpha(t)\,\beta(t)]$ in $\operatorname{Dom} f$.

Proof. Let us fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t. In case L is not geodesic space, pass to its ω-product L^ω.

Set $\ell(t) = |\alpha(t)\ \beta(t)|$, from the first variation formula (2.4.3) and the estimate in 11.2.3 we get

$$\ell^+(t) \leqslant -\langle \uparrow_{\alpha(t)}^{\beta(t)}, \nabla_{\alpha(t)} f\rangle - \langle \uparrow_{\beta(t)}^{\alpha(t)}, \nabla_{\beta(t)} f\rangle \leqslant \lambda \ell(t).$$

Hence the result. ☐

The proof of the following theorem was given in [Perelman–Petrunin QG], then it was essentially simplified in [Lytchak] by using ultralimits.

11.6.3. Existence. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $f : L \hookrightarrow \mathbb{R}$ be locally Lipschitz and λ-concave . Then for any $p \in \operatorname{Dom} f$ there is an f-gradient curve $\alpha : [0, \varepsilon) \to L$ with $\alpha(0) = p$ and $\varepsilon = \varepsilon(p) > 0$.*

Proof. Note that if $|\nabla_p f| = 0$ then one can take constant curve $\alpha(t) = p$. Otherwise, take $\varepsilon > 0$, such that $\operatorname{Ball}(\varepsilon, p) \subset \operatorname{Dom} f$ and $|\nabla_x f| > \varepsilon$ for all $x \in \operatorname{Ball}(\varepsilon, p)$ (that is possible due to 11.3.3).

The curve $\alpha : [0, \varepsilon) \to L$ will be constructed in three steps: first we construct an f^ω-gradient-like curve $\hat\alpha_\omega$ in L^ω as an ω-limit of certain sequence of broken geodesics in L; second, we parametrize $\hat\alpha$ as in 11.5.2, to obtain an f^ω-gradient curve α_ω in L^ω; third, applying uniqueness (11.6.1) together with 2.8.4 we obtain that α_ω lies in $L \subset L^\omega$ and therefore one can take $\alpha = \alpha_\omega$.

Note that if L is proper then $L = L^\omega$ and $f^\omega = f$; thus, in this case, the third step is not necessary.

Step 1. Applying open-close argument, for any $n \in \mathbb{Z}_>$ we can construct a unit-speed curve $\hat\alpha_n : [0, \varepsilon] \to L$ starting at p, with a partition of $[0, \varepsilon)$ into countable number of half-open intervals $[\sigma_i, \bar\sigma_i)$ so that for each i we have

(i) $\hat\alpha_n([\sigma_i, \bar\sigma_i])$ is a minimizing geodesic and $\bar\sigma_i - \sigma_i < \frac{1}{n}$.

(ii) $f \circ \hat\alpha_n(\bar\sigma_i) - f \circ \hat\alpha_n(\sigma_i) > (\bar\sigma_i - \sigma_i)|\nabla_{\hat\alpha_n(\sigma_i)} f| - \frac{1}{n}$.

Pass to a subsequence of $(\hat\alpha_n)$ such that $f \circ \hat\alpha_n$ uniformly converges; set

$$h(s) = \lim_{n\to\infty} f \circ \hat\alpha_n(s).$$

Set $\hat\alpha_\omega = \omega\text{-}\lim_n \hat\alpha_n$, it is a curve in L^ω starting at $p \in L \subset L^\omega$.

Clearly $\hat\alpha_\omega$ is 1-Lipschitz. From (ii) and 11.3.1, we get $(f^\omega \circ \hat\alpha_\omega)^+(\sigma) \geqslant |\nabla_{\hat\alpha_\omega(\sigma)} f^\omega|$. Therefore, according to 11.4.2, $\hat\alpha_\omega : [0, \varepsilon) \to L^\omega$ is an f^ω-gradient-like curve.

Step 2. Clearly $h(s) = f^\omega \circ \alpha_\omega$. Therefore, according to 11.4.3, h is λ-concave. Thus, we can define a homeomorphism $\sigma : [0,\delta] \to [0,\varepsilon]$ with the following Lebesgue integral:

$$\sigma^{-1}(s) = \int\limits_0^s \frac{1}{h'(\text{ş})} d\text{ş} \qquad (*)$$

According to 11.5.2, $\alpha(t) = \hat\alpha \circ \sigma(t)$ is an f^ω-gradient curve in L^ω.

Step 3. Since $\nabla_p f = \nabla_p f^\omega$ for any $p \in L \subset L^\omega$, it is sufficient to show that $\alpha_\omega \subset L$. Assume contrary, then according to 2.8.4 there is a subsequence $\hat\alpha_{n_k}$ such that

$$\hat\alpha_\omega \ne \hat\alpha'_\omega \overset{\text{def}}{=\!=} \omega\text{-}\lim_{k\to\infty} \hat\alpha_{n_k}.$$

Clearly $h(s) = f^\omega \circ \hat\alpha_\omega = f^\omega \circ \hat\alpha'_\omega$. Thus, for $\sigma : [0,\delta] \to [0,\varepsilon]$ defined by $(*)$, we have that both curves $\hat\alpha_\omega \circ \sigma$ and $\hat\alpha'_\omega \circ \sigma$ are f^ω-gradient. Thus from uniqueness of future (11.6.1), we get $\hat\alpha_\omega \circ \sigma = \hat\alpha'_\omega \circ \sigma$ and therefore $\hat\alpha_\omega = \hat\alpha'_\omega$, a contradiction. ☐

11.7 Radial curves

If one ignore parametrization, the radial curves trying to escape from given point p using greedy algorithm; they are always go in the direction which increase the distance to p with the maximal rate. But, what is more important, is the special parametrization of these curves, which make them to behave as geodesic in a natural comparison sense.

First let us define $\varkappa$-radial curves; these curves work best in spaces with curvature $\geqslant \varkappa$, but we define them for arbitrary lower curvature bound.

11.7.1. Definition. *Assume $L \in \mathrm{Alex}[\kappa,\infty]$, $\varkappa \in \mathbb{R}$, $p \in L$. A curve $\sigma : [s_{\min}, s_{\max}) \to L$ is called* $\varkappa$-radial curve with respect to p *if $s_{\min} = |p\sigma(s_{\min})| \in (0, \frac{\varpi_\varkappa}{2})$, and it satisfies the following differential equation*

$$\sigma^+(s) = \frac{\operatorname{tg}_\varkappa |p\sigma(s)|}{\operatorname{tg}_\varkappa s} \cdot \nabla_{\sigma(s)} \operatorname{dist}_p . \qquad (\mathrm{rad})_\varkappa$$

for any $s \in [s_{\min}, s_{\max})$, here $\operatorname{tg}_\varkappa x = \frac{\operatorname{sn}_\varkappa x}{\operatorname{cs}_\varkappa x}$.

If $x = \sigma(s_{\min})$, we say that σ is initiated *in x.*

Note that according to definition $s_{\max} \leqslant \frac{\varpi_\varkappa}{2}$. Note also that any geodesic $\gamma_{[px]}$ satisfies equation $(\mathrm{rad})_\varkappa$ for any $\varkappa \in \mathbb{R}$ (if $|px| < \frac{\varpi_\varkappa}{2}$). Here is a more precise statement:

11.7.2. Proposition. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $\sigma : [s_{\min}, s_{\max}) \to L$ be a $\varkappa$-radial curve with respect to p then for any $s \in [s_{\min}, s_{\max})$, we have $|p\sigma(s)| \leqslant s$.*

Moreover, if for some s_0 we have $|p\sigma(s_0)| = s_0$ then the restriction $\sigma|_{[s_{\min}, s_0]}$ is a minimizing geodesic which contained in a minimizing geodesic $[p\,\sigma(s_0)]$.

Proof. Follows directly from the definition. ☐

11.7.3. Existence and uniqueness. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $\varkappa \in \mathbb{R}$, $p, x \in L$ and $s_{\min} = |px| < \frac{\varpi_\varkappa}{2}$. Then there is unique $\varkappa$-radial curve $\sigma : [s_{\min}, \frac{\varpi_\kappa}{2}) \to L$ with respect to p which starts at x.*

Proof. Let us define one parameter family of smooth real functions $\chi_\kappa : [0, \frac{\varpi_\kappa}{2}) \to \mathbb{R}$

$$\chi_\kappa(t) = \int\limits_0^t \mathrm{tg}_\kappa \text{\c{t}}\ \ d\,\text{\c{t}}.$$

Clearly χ_κ is increasing. Thus???, the composition $f = \chi_\kappa \circ \mathrm{dist}_p : \mathrm{Ball}(\frac{\varpi_\kappa}{2}, p) \to \mathbb{R}$ is semiconcave.

Therefore, according to 11.6.3, there is an f gradient curve $\alpha : [0, t_{\max}) \to L$ which starts at x; i.e. $\alpha(0) = x$.

Now consider solution of differential equation $\tau(t)$, $\tau' = \mathrm{tg}^2_\kappa(\tau)$ and $\tau(0) = r$ (that is also a gradient curve in $\mathbb{R}$ for function $\chi_\kappa : [0, \varpi) \to \mathbb{R}$). Direct calculations show that composition $\alpha \circ \tau^{-1} : [r, \infty) \to L$ forms a κ-radial curve w.r.t. p.

Now assume σ_1, σ_2 be two $\varkappa$-radial curves with respect to p which start at x. Then compositions $\sigma_i \circ \tau$ both give f-gradient curves, thus $\sigma \circ \tau \equiv \sigma \circ \tau$. Therefore $\sigma_1(s) = \sigma_2(s)$ once both sides defined. ☐

The next corollary shows that $[px]$ can be extended behind x as a minimizing geodesic then it coinsides with radial curve starting at x w.r.t. p.

11.7.4. Corollary. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $\varkappa \in \mathbb{R}$, $p, x \in L$, $|px| < \frac{\varpi_\varkappa}{2}$ and $x' \in [px]$. Let σ be a radial curve w.r.t. p which is initiated at x' then*

$$\gamma_{[px]}(s) = \sigma(s)$$

at each $s \in \mathrm{Dom}(\gamma_{[px]}) \cap \mathrm{Dom}(\sigma)$.

Proof. Follows directly from 11.7.2 and 11.7.3. ☐

Here is the main theorem of this section

11.7.5. Radial comparison. *Let $L \in \mathrm{Alex}[\kappa,\infty]$ and $p \in L$. Assume $\rho : [r_{\min}, \frac{\varpi_\kappa}{2}) \to L$ and $\sigma : [s_{\min}, \frac{\varpi_\kappa}{2}) \to L$ be two radial curves w.r.t. p. Then for any $r \in [r_{\min}, \frac{\varpi_\kappa}{2})$ and $s \in [s_{\min}, \frac{\varpi_\kappa}{2})$, we have*[1]

$$\tilde\measuredangle_\kappa\{|\rho(r)\sigma(s)|; r, s\} \leqslant \tilde\measuredangle_\kappa\left(p\,{}^{\rho(r_{\min})}_{\sigma(s_{\min})}\right).$$

In particular, for any point $q \neq p$ and $s \in [s_{\min}, \frac{\varpi_\kappa}{2})$ we have

$$\tilde\measuredangle_\kappa\{|q\sigma(s)|; |pq|, \} \leqslant \tilde\measuredangle_\kappa\left(p\,{}^{q}_{\sigma(s_{\min})}\right).$$

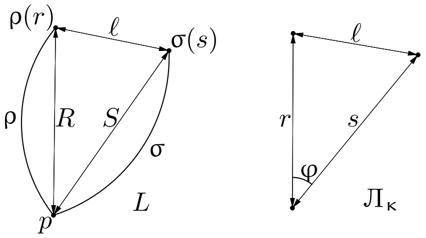
The proof of radial comparison (11.7.5) is an application of 11.2.2 plus trigonometric manipulations. We prove it first the simplest case $\kappa = 0$ and then harder case $\kappa \neq 0$. The arguments for case $\kappa \neq 0$ are nearly the same, but formulas are different and.

---

[1] Note that

$$\tilde\measuredangle_\kappa\left(p\,{}^{\rho(r_{\min})}_{\sigma(s_{\min})}\right) = \tilde\measuredangle_\kappa\{|\rho(r_{\min})\sigma(s_{\min})|; r_{\min}, s_{\min}\},$$

see section 2.4 and definition of radial curve (11.7.1).



We proof case $\kappa = 0$ separately since it is easier to follow. In fact we do not really need it, once the case $\kappa \neq 0$ is proved, the case $\kappa = 0$ can be obtained by a limit procedure.

Proof of case $\kappa = 0$. Set

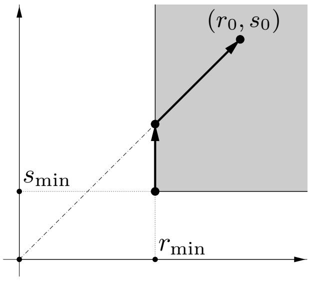
$$R = R(r) = |p\,\rho(r)|,$$
$$S = S(s) = |p\,\sigma(s)|,$$

$$\ell = \ell(r,s) = |\rho(r)\,\sigma(s)|, \qquad \varphi = \varphi(r,s) = \tilde\measuredangle_0\{\ell(r,s);r,s\}.$$

The statement follows from the following three inequalities:

$$\frac{\partial^+\varphi}{\partial r}(s_{\min},r) \leqslant 0, \qquad \frac{\partial^+\varphi}{\partial s}(s,r_{\min}) \leqslant 0 \qquad (*)^\varphi_0$$

$$s\cdot\frac{\partial^+\varphi}{\partial s} + r\cdot\frac{\partial^+\varphi}{\partial r} \leqslant 0. \qquad (**)^\varphi_0$$



Indeed, one can connect $(s_{\min}, r_{\min})$ and (s_0, r_0) in $[s_{\min},\infty)\times[r_{\min},\infty)$ by a join of coordinate line and a segment defined by $r/s = r_0/s_0$. According to $(*)^\varphi_0$ and $(**)^\varphi_0$, the value of φ does not increase while pair (r,s) moving along this join. Thus $\varphi(r_0,s_0) \leqslant \varphi(r_{\min},s_{\min})$.

It remains to show $(*)^\varphi_0$ and $(**)^\varphi_0$. First let us rewrite these inequalities in terms of ℓ, that will be the inequalities which we will prove further.

$$\frac{\partial^+\ell}{\partial s}(s,r_{\min}) \leqslant \cos\tilde\measuredangle_0\{r_{\min};s,\ell\}, \quad \frac{\partial^+\ell}{\partial r}(s_{\min},r) \leqslant \cos\tilde\measuredangle_0\{s_{\min};r,\ell\}, \qquad (*)^\ell_0$$

$$s\cdot\frac{\partial^+\ell}{\partial s} + r\cdot\frac{\partial^+\ell}{\partial r} \leqslant s\cdot\cos\tilde\measuredangle_0\{r;s,\ell\} + r\cdot\cos\tilde\measuredangle_0\{s;r,\ell\} = \ell. \qquad (**)^\ell_0$$

Set

$$f = \tfrac12\mathrm{dist}_p^2. \qquad (A)_0$$

Clearly f is 1-concave and

$$\rho^+(r) = \tfrac{1}{r}\cdot\nabla_{\rho(r)}f \ \text{ and } \ \sigma^+(s) = \tfrac{1}{s}\cdot\nabla_{\sigma(s)}f. \qquad (B)_0$$

Thus from 11.2.2, we have

$$\frac{\partial^+\ell}{\partial r} = -\tfrac{1}{r}\langle\nabla_{\rho(r)}f, \uparrow_{\rho(r)}^{\sigma(s)}\rangle \leqslant \frac{\ell^2+R^2-S^2}{2\ell r}. \qquad (C)_0$$

Since $R(r) \leqslant r$ and $S(s_{\min}) = s_{\min}$, we get

$$\begin{aligned}\frac{\partial^+\ell}{\partial r}(r,s_{\min}) &\leqslant \frac{\ell^2+r^2-s_{\min}^2}{2\ell r} = \\ &= \cos\tilde\measuredangle_0\{s_{\min};r,\ell\},\end{aligned} \qquad (D)_0$$

which is the first inequality in $(*)^{\ell}_{0}$. By switching places of ρ and σ we obtain the second inequality in $(*)^{\ell}_{0}$. Further, summing together $(C)_0$ with its mirror-inequality for $\frac{\partial^+\ell}{\partial s}$, we get

$$r\frac{\partial^+\ell}{\partial r}+s\frac{\partial^+\ell}{\partial s}\leqslant\frac{\ell^2+R^2-S^2}{2\ell}+\frac{\ell^2+S^2-R^2}{2\ell}=\ell \qquad (E)_0$$

which is $(**)^{\ell}_{0}$. ☐

Proof of case $\kappa\neq 0$. Set as before

$$\begin{aligned}
R&=R(r)=|p\,\rho(r)|, & \ell&=\ell(r,s)=|\rho(r)\,\sigma(s)|\\
S&=S(s)=|p\,\sigma(s)|, & \varphi&=\varphi(r,s)=\tilde{\measuredangle}_\kappa\{\ell(r,s);r,s\}.
\end{aligned}$$

The statement follows from the following three inequalities:

$$\frac{\partial^+\varphi}{\partial r}(s_{\min},r)\leqslant 0,\qquad \frac{\partial^+\varphi}{\partial s}(s,r_{\min})\leqslant 0 \qquad (*)^{\varphi}_{\pm}$$

$$\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\frac{\partial^+\varphi}{\partial s}+\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\frac{\partial^+\varphi}{\partial r}\leqslant 0 \qquad (**)^{\varphi}_{\pm}$$

Indeed, functions $s\mapsto\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S$ and $r\mapsto\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R$ are Lipschitz. Thus there is a solution for differential equation

$$(r',s')=(\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S,\ \operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R)$$

with any initial data. $(r_0,s_0)\in[r_{\min},\frac{\varpi_\kappa}{2})\times[s_{\min},\frac{\varpi_\kappa}{2})$. (Unlike case $\kappa=0$ the solution can not be written explicitly.)

Since $\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S,\ \operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R>0$, this solution $(r(t),s(t))$ must meet one of coordinate rays $\{r_{\min}\}\times[s_{\min},\frac{\varpi_\kappa}{2})$ or $[r_{\min},\frac{\varpi_\kappa}{2})\times\{s_{\min}\}$. I.e., one can connect pair $(s_{\min},r_{\min})$ to (s_0,r_0) by a join of coordinate line and the solution $(r(t),s(t))$. According to $(*)^{\varphi}_{\pm}$ and $(**)^{\varphi}_{\pm}$, the value of φ does not increase while pair (r,s) moving along this join. Thus $\varphi(r_0,s_0)\leqslant\varphi(r_{\min},s_{\min})$.

As before we rewrite the inequalities $(*)^{\varphi}_{\pm}$ and $(**)^{\varphi}_{\pm}$ in terms of ℓ:

$$\frac{\partial^+\ell}{\partial s}(s,r_{\min})\leqslant\cos\tilde{\measuredangle}_\kappa\{r_{\min};s,\ell\},\quad \frac{\partial^+\ell}{\partial r}(s_{\min},r)\leqslant\cos\tilde{\measuredangle}_\kappa\{s_{\min};r,\ell\}, \qquad (*)^{\ell}_{\pm}$$

$$\begin{aligned}
&\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\frac{\partial^+\ell}{\partial s}+\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\frac{\partial^+\ell}{\partial r}\leqslant\\
&\qquad\leqslant\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\cos\tilde{\measuredangle}_\kappa\{r;s,\ell\}+\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\cos\tilde{\measuredangle}_\kappa\{s;r,\ell\}
\end{aligned} \qquad (**)^{\ell}_{\pm}$$

Further, set

$$f=-\tfrac{1}{\kappa}\operatorname{cs}_\kappa(\operatorname{dist}_p)=\operatorname{md}_\kappa(\operatorname{dist}_p)-\tfrac{1}{\kappa}. \qquad (A)_{\pm}$$

Clearly $f''+\kappa f\leqslant 0$ and

$$\rho^+(r)=\frac{1}{\operatorname{tg}_\kappa r\cdot\operatorname{cs}_\kappa R}\cdot\nabla_{\rho(r)}f\ \text{ and }\ \sigma^+(s)=\frac{1}{\operatorname{tg}_\kappa s\cdot\operatorname{cs}_\kappa S}\cdot\nabla_{\sigma(s)}f. \qquad (B)_{\pm}$$

Thus from 11.2.2, we have

$$\begin{aligned}\frac{\partial^+\ell}{\partial r} &= -\frac{1}{\operatorname{tg}_\kappa r\cdot\operatorname{cs}_\kappa R}\langle\nabla_{\rho(r)}f,\uparrow_{\rho(r)}^{\sigma(s)}\rangle\leqslant\\ &\leqslant\frac{1}{\operatorname{tg}_\kappa r\cdot\operatorname{cs}_\kappa R}\cdot\frac{\operatorname{cs}_\kappa S-\operatorname{cs}_\kappa R\operatorname{cs}_\kappa\ell}{\kappa\cdot\operatorname{sn}_\kappa\ell}=\\ &\leqslant\frac{\frac{\operatorname{cs}_\kappa S}{\operatorname{cs}_\kappa R}-\operatorname{cs}_\kappa\ell}{\kappa\cdot\operatorname{tg}_\kappa r\cdot\operatorname{sn}_\kappa\ell}.\end{aligned}\qquad (C)_\pm$$

Note that for all $\kappa\neq0$, the function $x\mapsto\frac{1}{\kappa\cdot\operatorname{cs}_\kappa x}$ is increasing. Thus, since $R(r)\leqslant r$ and $S(s_{\min})=s_{\min}$, we get

$$\begin{aligned}\frac{\partial^+\ell}{\partial r}(r,s_{\min}) &\leqslant\frac{\frac{\operatorname{cs}_\kappa s_{\min}}{\operatorname{cs}_\kappa r}-\operatorname{cs}_\kappa\ell}{\kappa\cdot\operatorname{tg}_\kappa r\cdot\operatorname{sn}_\kappa\ell}=\\ &=\frac{\operatorname{cs}_\kappa s_{\min}-\operatorname{cs}_\kappa\ell\operatorname{cs}_\kappa r}{\kappa\cdot\operatorname{sn}_\kappa r\cdot\operatorname{sn}_\kappa\ell}=\\ &=\cos\tilde\measuredangle_\kappa\{s_{\min};r,\ell\},\end{aligned}\qquad (D)_\pm$$

which is the first inequality in $(*)^\ell_\pm$ for $\kappa\neq0$. By switching places of ρ and σ we obtain the second inequality in $(*)^\ell_\pm$. Further, summing together $(C)_\pm$ with its mirror-inequality for $\frac{\partial^+\ell}{\partial s}$, we get

$$\begin{aligned}&\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\frac{\partial^+\ell}{\partial r}+\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\frac{\partial^+\ell}{\partial s}\leqslant\\ &\leqslant\frac{\operatorname{cs}_\kappa S\cdot\operatorname{cs}_\kappa r-\operatorname{cs}_\kappa\ell\cdot\operatorname{cs}_\kappa R\cdot\operatorname{cs}_\kappa r}{\kappa\cdot\operatorname{sn}_\kappa\ell}+\frac{\operatorname{cs}_\kappa R\cdot\operatorname{cs}_\kappa s-\operatorname{cs}_\kappa\ell\cdot\operatorname{cs}_\kappa S\cdot\operatorname{cs}_\kappa s}{\kappa\cdot\operatorname{sn}_\kappa\ell}=\\ &=\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\frac{\operatorname{cs}_\kappa s-\operatorname{cs}_\kappa\ell\cdot\operatorname{cs}_\kappa r}{\kappa\cdot\operatorname{sn}_\kappa r\cdot\operatorname{sn}_\kappa\ell}+\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\frac{\operatorname{cs}_\kappa r-\operatorname{cs}_\kappa\ell\cdot\operatorname{cs}_\kappa s}{\kappa\cdot\operatorname{sn}_\kappa s\cdot\operatorname{sn}_\kappa\ell}=\\ &=\operatorname{sn}_\kappa r\cdot\operatorname{cs}_\kappa R\cdot\cos\tilde\measuredangle_\kappa\{r;s,\ell\}+\operatorname{sn}_\kappa s\cdot\operatorname{cs}_\kappa S\cdot\cos\tilde\measuredangle_\kappa\{s;r,\ell\}\end{aligned}\qquad (E)_\pm$$

which is $(**)^\ell_\pm$. □

Remark. If $\kappa\geqslant0$ then κ-radial curves are the only curves which can be defined using equation

$$\sigma^+(s)=\chi(s,|p\,\sigma(s)|)\cdot\nabla_{\sigma(s)}\operatorname{dist}_p$$

and satisfy radial comparison 11.7.5 (see exercise 7). In case $\kappa<0$, there are different ways to define such curves. In particular one can take curves defined by simpler equation

$$\sigma^+(s)=\frac{1}{\operatorname{sn}_\kappa s}\cdot\nabla_{\sigma(s)}(\operatorname{md}_\kappa\circ\operatorname{dist}_p).$$

Among all curves of that type, the κ-radial curves defined in 11.7.1 maximize the value $|p\,\sigma(s)|$.

11.8 Gradient exponent

The following theorem follows easily from radial comparison (11.7.5).

11.8.1. Gradient exponent. *Let $L \in \mathrm{Alex}[\kappa,\infty]$, $p \in L$ and*

$$I_\kappa = \left[\begin{array}{ll} [0,\infty) & \textit{if } \ \kappa \leqslant 0 \\ [0,\tfrac{\varpi_\kappa}{2}] & \textit{if } \ \kappa > 0 \end{array}\right.$$

Then there is a short map[2]

$$\mathrm{gexp}_p = \mathrm{gexp}_{p;\kappa} : \Sigma_p \times_{\mathrm{sn}_\kappa} I_\kappa \to L$$

such that for any $q \in \overline{\mathrm{Ball}}(\tfrac{\varpi_\kappa}{2}, p)$ then $\mathrm{gexp}_p((\uparrow_p^q, |pq|)) = q$.

Proof. Given $\xi \in \Sigma_p$, concider a sequence of points $x_n \in L$ so that $\uparrow_p^{x_n} \to \xi \in \Sigma_p$. Let $r_n = |px_n|$ and $\sigma_n : [r_n, \tfrac{\varpi_\kappa}{2}) \to L$ be the radial curve w.r.t. p initiated at x_n. Directly from radial comparison (11.7.5), we have that $\sigma_n : [r_n, \tfrac{\varpi_\kappa}{2}) \to L$ converge to a curve $\sigma_\xi : (0, \tfrac{\varpi_\kappa}{2}) \to L$ and this limit is independent from the choice of the sequence x_n. The domain of σ_ξ can be extended continuously by $\sigma_\xi(0) = p$ and if $\kappa > 0$ to $\sigma_\xi(\tfrac{\varpi_\kappa}{2})$. Thus we can think that σ_ξ is defined on I_κ.

If $q \in \overline{\mathrm{Ball}}(\tfrac{\varpi_\kappa}{2}, p)$ and $\xi = \uparrow_p^q$ then taking $x_n \in [pq]$ it follows that $\sigma_\xi(|pq|) = q$.

Thus we can define a map $\mathrm{gexp}_p((\xi, t)) = \sigma_\xi(t)$. According to 11.7.5 the map $\mathrm{gexp}_p : \Sigma_p \times_{\mathrm{sn}_\kappa} I_\kappa \to L$ is short map and from above for any $q \in \overline{\mathrm{Ball}}(\tfrac{\varpi_\kappa}{2}, p)$ then $\mathrm{gexp}_p((\uparrow_p^q, |pq|)) = q$. ☐

---

[2] In most of the cases κ will be lower bound for curvature of L, thus we can omit this in the notation