

Chapter 3

Games with definition

We will start with global definition through 3-angle comparison; in section 3.2, we give number of equivalent angle comparison definitions. These definitions give the easiest way to adapt your Euclidean intuition to Alexandrov's world. Later we will give few equivalent global definitions through concavity of distance function (section 3.3), through development (section ??) and through extension of short map on a 4-point subsets (section 3.4). Finally we give local versions of each definition in section 3.5 and prove the globalization theorem.

3.1 The definition.

3.1.1. Definition of Alexandrov space. *A length-space L is called Alexandrov space with curvature $\geq \kappa$ (for shortness $L \in \text{Alex}[\kappa, \infty]$) if for any quadruple $p, x_1, x_2, x_3 \in L$ we have*

$$\tilde{\Delta}_\kappa(p_{x_1}) + \tilde{\Delta}_\kappa(p_{x_2}) + \tilde{\Delta}_\kappa(p_{x_3}) \leq 2\pi. \quad (\sphericalangle_\kappa)$$

if all model angles $\tilde{\Delta}_\kappa(p_{x_j})$ are well defined.

???We say that L is Alexandrov space with curvature bounded below (for shortness CBB-space) if there is $\kappa \in \mathbb{R}$, such that $L \in \text{Alex}[\kappa, \infty]$.

Remarks

- Alexandrov spaces with curvature bounded below will mostly denoted by L , it is for Lower curvature bound.
- If $\kappa > 0$, in the definition of spaces with curvature $\geq \kappa$, most authors assume in addition that $\text{diam } L \leq \varpi_\kappa$, or equivalently, that L is not isometric to one of exceptional spaces, see 3.7.1.
- If $\kappa_2 > 0$, the inclusion

$$\text{Alex}[\kappa_1, \infty] \subset \text{Alex}[\kappa_2, \infty] \quad \text{if } \kappa_1 < \kappa_2.$$

is not at all trivial; it will be shown only in 3.5.5.

- The inequality (\sphericalangle_κ) will be called *3-angles comparison*.

It is straightforward to check the following claim, for definitions of ω -limit and ω -product, see section 2.8.

3.1.2. Claim. *Let $(L_n, \star_n) \xrightarrow{\omega} (L_\omega, \star_\omega)$ and $\kappa_n \xrightarrow{\omega} \kappa_\omega$ then*

$$L_n \in \text{Alex}[\kappa_n, \infty] \implies L_\omega \in \text{Alex}[\kappa_\omega, \infty].$$

Moreover, for any complete metric space L ,

$$L \in \text{Alex}[\kappa, \infty] \iff L^\omega \in \text{Alex}[\kappa, \infty].$$

Note that a general complete length space might have no geodesics???. For Alexandrov spaces the situation is different, all Alexandrov's spaces are almost geodesic in the sence of the following definition:

3.1.3. Definition. *A complete length space X is called almost geodesic if for any point $p \in X$ there is a dence G_δ -set $E \subset X$ such that for any $q \in E$ there is unique geodesic $[pq]$.*

Remark for AKP??? It is not clear showld we include "unique geodesic" in this definition. If so, we probably should change the name to something like almost-star-shaped or something like that.

3.1.4. Theorem [Plaut 02, Th. 27]. *Any Alexandrov space with curvature bounded below is almost geodesic.*

3.1.5. Definition. *Let X be a metric space and $p \in X$, let us call $q \in X$ p -straight (or $q \in X^\bullet(p)$) if*

$$\overline{\lim}_{r \rightarrow q} \frac{|pr| - |pq|}{|qr|} = 1.$$

For a finite set of points $\{x_1, x_2, \dots, x_n\}$, we will use short notation

$$X^\bullet(x_1, x_2, \dots, x_n) = \bigcap_{i=1}^n X^\bullet(x_i).$$

3.1.6. Lemma. *Let X be complete length space, then for any $p \in X$ the set of $X^\bullet(p) \subset X$ is a dence G_δ -set.*

Proof. Set

$$\Omega_n = \{q \in X \mid \exists r \in X : (1 - \frac{1}{n})|qr| < |pr| - |pq| < \frac{1}{n}\}.$$

Clearly Ω_n is open; let us show that it is dense in X . Assume contrary, then there is a point $x \in X$ such that $\text{Ball}(\varepsilon, x) \cap \Omega_n = \emptyset$ for some $\varepsilon > 0$. Since metric on L is intrinsic, for any $\delta > 0$, there exist a point $y \in L$ such that $|xy| < \varepsilon/2 + \delta$ and $|py| < |px| - \varepsilon/2 + \delta$. For small enough ε and δ , that implies $(1 - \frac{1}{n})|yx| < |px| - |py| < \frac{1}{n}$, i.e. $y \in \Omega_n$, a contradiction.

Since $X^\bullet(p) = \bigcap_{n \in \mathbb{N}} \Omega_n$, the result follows. \square

Proof of theorem 3.1.4. Let L be an Alexandrov space with curvature $\geq \kappa$.

We will show that if $q \in L^\bullet(p)$, then there is unique minimizing geodesic connecting p and q . Thus theorem will follow from lemma 3.1.6.

Note that it is enough to show that for all sufficiently small¹ $t > 0$ there is unique point z such that

$$t = |qz| = |pq| - |pz| \quad (*)$$

First let us show uniqueness. Assume z and z' both satisfy (*). Take a sequence $r_n \rightarrow q$ so that $(|pr_n| - |pq|)/|qr_n| \rightarrow 1$. From triangle inequality,

$$|zr_n| - |zq|, \quad |z'r_n| - |z'q| \geq |pr_n| - |pq|;$$

thus, as $n \rightarrow \infty$,

$$(|zr_n| - |zq|)/|qr_n|, \quad (|z'r_n| - |z'q|)/|qr_n| \rightarrow 1.$$

Therefore² $\tilde{\Delta}_\kappa(q_{r_n}^z), \tilde{\Delta}_\kappa(q_{r_n}^{z'}) \rightarrow \pi$. From 3-angles comparison, $\tilde{\Delta}_\kappa(q_{z'}^z) = 0$ and thus $z = z'$.

The proof of existence is similar. Choose sequence r_n as above. Since L is length space, there is a sequence $z_k \in L$ such that $|qz_k|, |pq| - |pz_k| \rightarrow t$ as $k \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \tilde{\Delta}_\kappa(q_{r_n}^{z_k}) = \pi.$$

Thus, for any $\varepsilon > 0$ and large enough n, k we have $\tilde{\Delta}_\kappa(q_{r_n}^{z_k}) > \pi - \varepsilon$. From 3-angles comparison, for all large k, j , we have $\tilde{\Delta}_\kappa(q_{z_j}^{z_k}) < 2\varepsilon$ and thus

$$|z_k z_j| < \varepsilon \cdot \text{Const}(\kappa, t);$$

i.e. $\{z_n\}$ is a Cauchy sequence, and one can take $z = \lim_n z_n$. \square

3.2 More angle comparisons.

Now we are ready to give a number of equivalent definitions of Alexandrov space:

3.2.1. Theorem. *Let L be complete length space, then*

(1) $L \in \text{Alex}[\kappa, \infty]$

if and only if L is almost geodesic and one of the following conditions holds for all $p, x, y \in L$, once $\tilde{\Delta}_\kappa(pxy)$ is well defined:

(2) (2-angles comparison) *for any minimizing geodesic $[xy]$ and $z \in [xy]$, we have*

$$\tilde{\Delta}_\kappa(z_x^p) + \tilde{\Delta}_\kappa(z_y^p) \leq \pi. \quad (\perp_\kappa)$$

(3) (angle monotonicity) *for any minimizing geodesic $[xy]$ and $z \in [xy]$, we have*

$$\tilde{\Delta}_\kappa(x_y^p) \leq \tilde{\Delta}_\kappa(x_z^p);$$

or, equivalently, the function

$$t \mapsto \tilde{\Delta}_\kappa(x_{\mathbb{Y}_{[xy]}^p(t)}^p)$$

is nonincreasing.

¹we need small t and $|qr|$ only to make sure that model triangle $\tilde{\Delta}_\kappa(zqr)$ is well defined

²Here we use that t is small, otherwise the angles might be undefined.

(4) (hinge comparison)³ for any hinge $[x_y^p]$, the angle $\angle[x_y^p]$ is well defined and

$$\angle[x_y^p] \geq \tilde{\Delta}_\kappa(x_y^p).$$

or equivalently

$$\tilde{\Upsilon}_\kappa[x_y^p] \geq |py|.$$

Moreover, if $z \in [xy]$ then for any two hinges $[z_y^p]$ and $[z_x^p]$ with common side $[zp]$

$$\angle[z_y^p] + \angle[z_x^p] \leq \pi.$$

Proof. By theorem 3.1.4, we have that (1) implies that L is almost geodesic.

(1) \Rightarrow (2). Since $z \in [xy]$, we have $\tilde{\Delta}_\kappa(z_y^x) = \pi$. Thus, 3-angles comparison

$$\tilde{\Delta}_\kappa(z_y^x) + \tilde{\Delta}_\kappa(z_x^p) + \tilde{\Delta}_\kappa(z_y^p) \leq 2\pi$$

implies

$$\tilde{\Delta}_\kappa(z_x^p) + \tilde{\Delta}_\kappa(z_y^p) \leq \pi.$$

(2) \Leftrightarrow (3). Follows directly from from Alexandrov's lemma (lemma 4.2.1).

(2) + (3) \Rightarrow (4). From (3), the angle $\angle[p_y^x]$ is well defined and $\angle[p_y^x] \geq \tilde{\Delta}_\kappa(p_y^x)$. From (2), we have $\angle[z_y^p] + \angle[z_x^p] \leq \pi$.

(4) \Rightarrow (1). Assume first that L is geodesic. Consider point $w \in [pz]$ close to p . From (4), it follows that

$$\angle[w_z^x] + \angle[w_p^x] \leq \pi \quad \text{and} \quad \angle[w_z^y] + \angle[w_p^y] \leq \pi.$$

Applying triangle inequality $\angle[w_y^x] \leq \angle[w_p^x] + \angle[w_z^x]$ (see 2.4.2), we get

$$\angle[w_z^x] + \angle[w_z^y] + \angle[w_y^x] \leq 2\pi.$$

Applying the first inequality of (4), we obtain

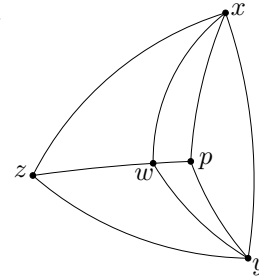
$$\tilde{\Delta}_\kappa(w_z^x) + \tilde{\Delta}_\kappa(w_z^y) + \tilde{\Delta}_\kappa(w_y^x) \leq 2\pi.$$

Passing to the limits $w \rightarrow p$, we obtain

$$\tilde{\Delta}_\kappa(p_z^x) + \tilde{\Delta}_\kappa(p_z^y) + \tilde{\Delta}_\kappa(p_y^x) \leq 2\pi.$$

Finally, if L is not geodesic, we can apply above arguments for sequences of points $p_n, w_n \rightarrow p, p_n \in L^\bullet(z), w_n \in [zp_n]$ and $x_n \rightarrow x, y_n \rightarrow y, x_n, y_n \in L^\bullet(w_n)$. \square

³The question whether the first part of (4) is enough to conclude that $L \in \text{Alex}[\kappa, \infty]$ is a long standing open problem.



3.3 Function comparison and development

In this section we will translate the angle comparison definitions (theorem 3.2.1) to a concavity-like property of the distance functions. Conceptually, this is quite important step; we reformulate global geometric condition into infinitesimal condition on distance functions. We will give two formulations of this comparison: analytical one — through function concavity, and geometrical — using development.

3.3.1. Theorem. *Let L be a complete length space, then following statements are equivalent:*

- (1) $L \in \text{Alex}[\kappa, \infty]$.
- (2) (function comparison) L is almost geodesic and for any $p \in L$, the function $f = \text{md}_\kappa \circ \text{dist}_p$ satisfies differential inequality

$$f'' \leq 1 - \kappa f.$$

in $\text{Ball}(\varpi_\kappa, p)$.

In particular, $L \in \text{Alex}[0, \infty]$ if and only if for any $p \in L$, the function $\text{dist}_p^2 : L \rightarrow \mathbb{R}$ is 2-concave in the sense of definition 7.1.1.

Proof. It is enough to show that angle monotonicity condition (3.2.1(3)) is equivalent to the fact that for any minimizing geodesic $[xy]$ in $\text{Ball}(\varpi_\kappa, p) \subset L$, the Jensen's inequality (7.1.2(3)) is satisfied for $y(t) = \text{md}_\kappa \circ \text{dist}_p \circ \mathbb{Y}_{[xy]}(t)$.

Consider model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(pxy)$ and set

$$r(t) = \text{dist}_{\tilde{p}} \circ \mathbb{Y}_{[\tilde{x}\tilde{y}]}(t).$$

Clearly $r(t) = \text{dist}_p \circ \mathbb{Y}_{[xy]}(t)$ for $t = 0$ and $|xy|$. The condition 3.2.1(3) is equivalent to the fact that $r(t) \leq \text{dist}_p \circ \mathbb{Y}_{[xy]}(t)$ for all $t \in [0, |xy|]$. From property (i) on page 5, $z(t) = \text{md}_\kappa \circ r(t)$ satisfies $z'' = 1 - \kappa z$. Since md_κ is increasing on $[0, \varpi_\kappa)$, we get $z \leq y$, which is 7.1.2(3) for $y(t) = \text{md}_\kappa \circ \text{dist}_p \circ \mathbb{Y}_{[xy]}(t)$. \square

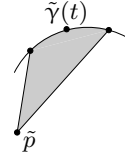
We will translate the above comparison into more geometric language, using the following definition of development. The definition is a bit long but, it defines a useful comparison object for a curve. It is easier to write proofs in terms of function comparison and, for geometers, it is easier to think in terms of κ -developments.

3.3.2. Definition of developing [Alexandrov 57]. *Let $\kappa \in \mathbb{R}$, X be a metric space, $\gamma : [a, b] \rightarrow X$ be a 1-Lipschitz curve and $p \in X$ such that $0 < |p\gamma(t)| < \varpi_\kappa$ for all $t \in [a, b]$.*

Then given $\tilde{p} \in \mathbb{I}_\kappa$, there exists a unique (up to rotation) curve $\tilde{\gamma} : [a, b] \rightarrow \mathbb{I}_\kappa$, parameterized by the arclength, and such that $|\tilde{p}\tilde{\gamma}(t)| = |p\gamma(t)|$ for all t and the direction of $[\tilde{p}\tilde{\gamma}(t)]$ turns counterclockwise as t increases (this is easy to prove).

The curve $\tilde{\gamma}$ is called the κ -development of γ with respect to p ; and point \tilde{p} — base of development.

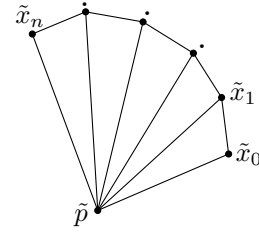
3.3.3. Definition. A curve $\tilde{\gamma} : [a, b] \rightarrow \mathbb{J}_\kappa$ is called convex with respect to $\tilde{p} \in \mathbb{J}_\kappa$ if for every $t \in (a, b)$, for sufficiently small $\tau > 0$ the curvilinear triangle, bounded by the segments $[\tilde{p}\tilde{\gamma}(t \pm \tau)]$ and the arc $\tilde{\gamma}([t - \tau, t + \tau])$, is convex.



3.3.4. Development comparison. $L \in \text{Alex}[\kappa, \infty]$ if and only if L is almost geodesic and for any point $p \in L$ and any geodesic γ in $\text{Ball}(\varpi_\kappa, p) \setminus p$, its κ -development with respect to p is convex with respect to its base.

Proof. (\Leftarrow). By theorem 3.1.4, L is almost geodesic.

Let $\gamma : [0, T] \rightarrow \text{Ball}(\varpi_\kappa, p)$ be a minimizing geodesic in L . Consider partition $0 = t_0 < t_1 < \dots < t_n = T$. Set $x_i = \gamma(t_i)$. Let us construct a chain of model triangles $[\tilde{p}\tilde{x}_{i-1}\tilde{x}_i] = \tilde{\Delta}(px_{i-1}x_i)$ on such a way that direction $[\tilde{p}\tilde{x}_i]$ turns counterclockwise as i grows. The condition (2) of theorem 3.2.1 implies that that



$$\angle[\tilde{x}_i \tilde{p} \tilde{x}_{i-1}] + \angle[\tilde{x}_i \tilde{p} \tilde{x}_{i+1}] \leq \pi;$$

i.e. broken line $\tilde{x}_0\tilde{x}_1 \dots \tilde{x}_n$ is convex with respect to \tilde{p}^4 .

Taking finer and finer partitions and passing to the limit, the broken line $\tilde{x}_0\tilde{x}_1 \dots \tilde{x}_n$ approaches development of γ with respect to p and the statement follows.

(\Rightarrow). Let $\tilde{\gamma} : [0, T] \rightarrow \mathbb{J}_\kappa$ be κ -development with base \tilde{p} of γ with respect to p . Take a partition $0 = t_0 < t_1 < \dots < t_n = T$ and set

$$\tilde{y}_i = \tilde{\gamma}(t_i) \quad \text{and} \quad \tau_i = |\tilde{y}_0\tilde{y}_1| + |\tilde{y}_1\tilde{y}_2| + \dots + |\tilde{y}_{i-1}\tilde{y}_i|.$$

Since $\tilde{\gamma}$ is convex, for fine partition we have that broken line $\tilde{y}_0\tilde{y}_1 \dots \tilde{y}_n$ is also convex. Applying Alexandrov's lemma (lemma 4.2.1) inductively to pairs of triangles $\tilde{\Delta}_\kappa\{\tau_{i-1}, |p\tilde{y}_0|, |p\tilde{y}_{i-1}|\}$ and $\tilde{\Delta}_\kappa\{|\tilde{y}_{i-1}\tilde{y}_i|, |p\tilde{y}_{i-1}|, |p\tilde{y}_i|\}$ we obtain that sequence $\angle_\kappa\{|p\tilde{y}_i|; |p\tilde{y}_0|, \tau_i\}$ is non increasing.

Taking finer and finer partitions and passing to the limit,

$$\max_i\{|\tau_i - t_i|\} \rightarrow 0,$$

thus, condition 3 in theorem 3.2.1 follows. \square

3.4 Short map extension definition.

Here is a very good definition, it requires no preliminary work, also the definition of spaces of curvature bounded above and below are completely analogous using this terms. We did not use this definition as the main one since it is not easy to apply it directly.

3.4.1. Theorem. Let L be complete length space, then the following statements are equivalent:

- (i) $L \in \text{Alex}[\kappa, \infty]$

⁴Moreover, since γ is minimizing geodesic, it follows that polygone bounded by broken line $\tilde{p}\tilde{x}_0\tilde{x}_1 \dots \tilde{x}_n\tilde{p}$ is convex

- (ii) (*Kirszbraun property*) For any 3-point set F_3 and any 4-point set $F_4 \supset F_3$ in L , any short map $f : F_3 \rightarrow \mathbb{J}_\kappa$ can be extended as a short map $\tilde{f} : F_4 \rightarrow \mathbb{J}_\kappa$ (so $f = \tilde{f}|_{F_3}$).

If one assumes in addition that $\text{diam } f(F_3) \leq \varpi_\kappa/2$ then above theorem follows from [Lang–Schroeder, Th. C], thus our statement is bit stronger in case $\kappa > 0$.

Proof. (ii) \Rightarrow (i). If L is geodesic, take $x_1, x_2, x_3 \in L$ so that model triangle $[\tilde{x}_1\tilde{x}_2\tilde{x}_3] = \tilde{\Delta}_\kappa(x_1x_2x_3)$ is well defined, take $p \in [x_1x_2]$; apply Kirszbraun property for $F_3 = \{x_1, x_2, x_3\}$ and $F_4 = \{x_1, x_2, x_3, p\}$ and the map $f(x_i) = \tilde{x}_i$. You obtain comparison definition 3.2.1(3).

In case L is not geodesic, pass to its ω -product L^ω and note that Kirszbraun property survives for L^ω . Thus, from above, $L^\omega \in \text{Alex}[\kappa, \infty]$ and therefore, from 3.1.2, $L \in \text{Alex}[\kappa, \infty]$.

(i) \Rightarrow (ii). Assume that for $x_1, x_2, x_3, p \in L$, $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in \mathbb{J}_\kappa$ are such that $|\tilde{x}_i\tilde{x}_j| \leq |x_ix_j|$ for all i, j and there is no point $\tilde{p} \in \mathbb{J}_\kappa$ such that $|\tilde{p}\tilde{x}_i| \leq |px_i|$ for all i .

Note that in this case all comparison triangles $\tilde{\Delta}_\kappa(px_ix_j)$ are well defined. That is always true if $\kappa \leq 0$. If $\kappa > 0$, and say $\tilde{\Delta}_\kappa(px_1x_2)$ is undfined then

$$|px_1| + |px_2| \geq 2\varpi - |x_1x_2| \geq 2\varpi - |\tilde{x}_1\tilde{x}_2| \geq |\tilde{x}_1\tilde{x}_3| + |\tilde{x}_2\tilde{x}_3|.$$

Thus one can take \tilde{p} on $[\tilde{x}_1\tilde{x}_3]$ or $[\tilde{x}_2\tilde{x}_3]$.

For each $i \in \{1, 2, 3\}$, consider point $\tilde{p}_i \in \mathbb{J}_\kappa$ such that $|\tilde{p}_i\tilde{x}_i|$ takes minimal amond points satisfying $|\tilde{p}_i\tilde{x}_j| \leq |px_j|$ for all $j \neq i$. Clearly every \tilde{p}_i is inside of triangle $[\tilde{x}_1\tilde{x}_2\tilde{x}_3]$ and $|\tilde{p}_i\tilde{x}_i| > |px_i|$ for each i . Since the function $x \mapsto \tilde{\Delta}_\kappa\{x; a, b\}$ is increasing, it follows that

- (i) $|\tilde{p}_i\tilde{x}_j| = |px_j|$ for $i \neq j$;
- (ii) all three trinagles $[\tilde{p}_1\tilde{x}_2\tilde{x}_3]$, $[\tilde{p}_2\tilde{x}_3\tilde{x}_1]$, $[\tilde{p}_3\tilde{x}_1\tilde{x}_2]$, do not overlap in $[\tilde{x}_1\tilde{x}_2\tilde{x}_3]$.

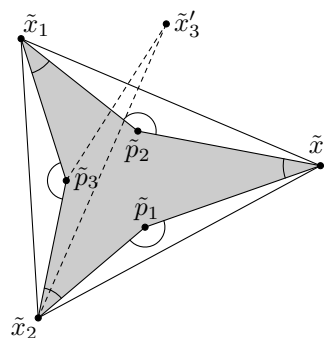
3.4.2. Claim. *In this case*

$$\angle[\tilde{p}_1 \frac{\tilde{x}_2}{\tilde{x}_3}] + \angle[\tilde{p}_2 \frac{\tilde{x}_3}{\tilde{x}_1}] + \angle[\tilde{p}_3 \frac{\tilde{x}_1}{\tilde{x}_2}] > 2\pi.$$

Once it is proved, we arrive to a contradiction since $|\tilde{x}_i\tilde{x}_j| \leq |x_ix_j|$ implies that for $k \neq i, j$

$$\angle[\tilde{p}_k \frac{\tilde{x}_i}{\tilde{x}_j}] \leq \tilde{\Delta}_\kappa(p \frac{x_i}{x_j}).$$

*Proof of the claim.*⁵ Rotate triangle $[\tilde{p}_1\tilde{x}_2\tilde{x}_3]$ around \tilde{x}_2 to make $[\tilde{x}_2\tilde{p}_1]$ coincide⁶ with $[\tilde{x}_2\tilde{p}_3]$.



⁵Note that the claim in the case $\kappa \leq 0$ follows immediatly from the formula which relates sum of angles for hexagone $\tilde{p}_1\tilde{x}_2\tilde{p}_3\tilde{x}_1\tilde{p}_2\tilde{x}_3$ and its area:

$$\angle\tilde{p}_1 - \angle\tilde{x}_2 + \angle\tilde{p}_3 - \angle\tilde{x}_1 + \angle\tilde{p}_2 - \angle\tilde{x}_3 = 2\pi - \kappa \cdot \text{area}.$$

⁶Remember $|\tilde{x}_2\tilde{p}_1| = |\tilde{x}_2\tilde{p}_3| = |px_2|$.

Denote by \tilde{x}'_3 the image of \tilde{x}_3 after rotating. Then $\angle[x_2 \frac{\tilde{x}'_1}{\tilde{x}'_3}] < \angle[x_2 \frac{\tilde{x}_1}{\tilde{x}_3}]$, thus $|\tilde{x}_1 \tilde{x}'_3| < |\tilde{x}_1 \tilde{x}_3|$ and therefore

$$\angle \tilde{p}_2 > \angle[\tilde{p}_3 \frac{\tilde{x}_1}{\tilde{x}'_3}] = \min\{\angle \tilde{p}_1 + \angle \tilde{p}_3, \pi - (\angle \tilde{p}_1 + \angle \tilde{p}_3)\}.$$

The same is true if one permutes indexes in this inequality. If sum of each pair of angles $\angle \tilde{p}_i$ is smaller than π , then summing these inequalities we get

$$\angle \tilde{p}_1 + \angle \tilde{p}_2 + \angle \tilde{p}_3 > 2(\angle \tilde{p}_1 + \angle \tilde{p}_2 + \angle \tilde{p}_3),$$

a contradiction. Therefore sum of at least one the pairs is $\geq \pi$ and thus the claim follows. \square

\square

3.5 Local definitions and globalization

The aim of this section is to give local analogs of the definitions given above and to prove the equivalence to the global definitions — the globalization theorem.

First we need a local analog of almost geodesic space (definition 3.1.3):

3.5.1. Definition. *Let X be complete length metric space, an open set $\Omega \subset X$ is called almost geodesic if for every point $p \in \Omega$ there is a dense in Ω G_δ -set E such that for any $q \in E$ there is unique geodesic $[pq]$ in X .*

3.5.2. Theorem. *Let X be a complete length-metric space and $p \in X$. Then the following conditions are equivalent:*

1. (local 3-angles comparison) there is $R_1 > 0$ such that comparison

$$\tilde{\Delta}_\kappa(p_{x_2}^{x_1}) + \tilde{\Delta}_\kappa(p_{x_3}^{x_2}) + \tilde{\Delta}_\kappa(p_{x_1}^{x_3}) \leq 2\pi$$

holds for any $p, x_1, x_2, x_3 \in \text{Ball}(R_1, p)$.

2. (local Kirszbraun property) there is $R_2 > 0$, such that for any 3-point subset F_3 and any 4-point subset $F_4 \supset F_3$ in $\text{Ball}(R_2, p)$, any short map $f : F_3 \rightarrow \mathbb{I}_\kappa$ can be extended as a short map $\tilde{f} : F_4 \rightarrow \mathbb{I}_\kappa$ (so $f = \tilde{f}|_{F_3}$).
3. (local function comparison) there is $R_3 > 0$, such that $\text{Ball}(R_3, p)$ is almost geodesic and for any $q \in \text{Ball}(R_3, p)$, the function $f = \text{md}_\kappa \circ \text{dist}_q$ satisfies $f'' \leq 1 - \kappa f$ in $\text{Ball}(R_3, p)$.

4. (local 2-angles comparison) there is $R_4 > 0$ such that $\text{Ball}(R_4, p)$ is almost geodesic and if q and geodesic $[xy]$ lie in $\text{Ball}(R_4, p)$ and $z \in [xy]$ then

$$\tilde{\Delta}_\kappa(z_x^q) + \tilde{\Delta}_\kappa(z_y^q) \leq \pi.$$

5. (local monotonicity) there is $R_5 > 0$, such that $\text{Ball}(R_5, p)$ is almost geodesic and if q and geodesic $[xy]$ lie in $\text{Ball}(R_5, p)$ and $z \in [xy]$, we have

$$\tilde{\Delta}_\kappa(x_y^q) \leq \tilde{\Delta}_\kappa(x_z^q);$$

or, equivalently, the function

$$t \mapsto \tilde{\Delta}_\kappa(x_{\mathbb{Y}_{[xy]}(t)}^q)$$

is nonincreasing.

6. (local hinge comparison) there is $R_6 > 0$, such that $\text{Ball}(R_6, p)$ is almost geodesic and if $x \in \text{Ball}(R_6, p)$ then (1) for any hinge $[x \frac{q}{y}]$, the angle $\angle[x \frac{q}{y}]$ is well defined and (2) if $x \in [yz]$ then⁷

$$\angle[x \frac{q}{y}] + \angle[x \frac{q}{z}] \leq \pi.$$

Moreover, if hinge $[x \frac{q}{y}]$ lies in $\text{Ball}(R_6, p)$ then

$$\angle[x \frac{q}{y}] \geq \tilde{\angle}_{\kappa}(x \frac{q}{y}).$$

or equivalently

$$\tilde{\gamma}_{\kappa}[x \frac{q}{y}] \geq |qy|.$$

7. (local development comparison) there is $R_7 > 0$, such that $\text{Ball}(R_7, p)$ is almost geodesic and if a geodesic γ lies in $\text{Ball}(R_7, p)$ and $q \in \text{Ball}(R_7, p) \setminus \gamma$ then κ -development $\tilde{\gamma}$ with respect to q is convex with respect to its base.

Moreover, for each pair $i, j \in [1..7]$ we can assume that $R_i > \frac{1}{9}R_j$.

Proof. The proof of each equivalence repeats the proof of correspondent equivalence in localized form, see proofs of theorems 3.2.1, 3.3.1, 3.3.2, 3.4.1. \square

Now, let us give a local definition of lower curvature bound (3.1.1).

3.5.3. Definition. Let X be a complete length-metric space and $p \in X$, we say that X has curvature $\geq \kappa_0$ at p (for shortness $\text{curv}_p X \geq \kappa_0$) if p satisfies any of the equivalent condition of theorem 3.5.2 for any $\kappa < \kappa_0$.

3.5.4. Globalization theorem. Let L be a complete length space such that $\text{curv}_p L \geq \kappa_0$ for any point $p \in L$ then $L \in \text{Alex}[\kappa_0, \infty]$.

In the two-dimensional case this theorem was proved in [Alexandrov 57], later, in [Toponogov] it was proved for Riemannian manifolds of all dimensions. In the above generality, theorem first appears in [BGP], simplicifications and modifications were given in [Plaut 96], [Shiohama], [BBI]. Our proof, is mostly based on presentations in [Plaut 96] and [BBI].

Applying this theorem, we are finally able to show that expression “space with curvature $\geq \kappa$ ” has sense:

3.5.5. Corollary. If L is a complete length space with $L \in \text{Alex}[\kappa_0, \infty]$ then $L \in \text{Alex}[\kappa, \infty]$ for any $\kappa < \kappa_0$.

Proof. Note that if $\kappa_0 \leq 0$, this statement follows directly from definition of Alexandrov space (3.1.1) and monotonicity of function $\kappa \mapsto \tilde{\angle}_{\kappa}(x \frac{y}{z})$ (iii, page 5).

For $\kappa_0 > 0$, the angle $\tilde{\angle}_{\kappa_0}(x \frac{y}{z})$ might be undefined, but, it is defined for sufficiently small triangles. Thus, if $\kappa < \kappa_0$ then

$$L \in \text{Alex}[\kappa_0, \infty] \implies \text{curv}_p L \geq \kappa \text{ for any } p \in L.$$

Applying globalization theorem (3.5.4), we get the result. \square

⁷Let us remind that $[x \frac{q}{y}]$ and $[x \frac{q}{z}]$ are short notations for pairs $([xq], [xy])$ and $([xq], [xz])$, thus these two hinges have common side $[xq]$

We will need a characterization of Alexandrov space which is slightly weaker than 3.2.1(4):

3.5.6. Short hinge lemma. *Let L be a complete almost geodesic space such that for any hinge $[x_y^p]$ in L the angle $\angle[x_y^p]$ is well defined and moreover, if $x \in [yz]$ then*

$$\angle[x_y^p] + \angle[x_z^p] \leq \pi.$$

Assume that for any hinge $[x_y^p]$ in L we have

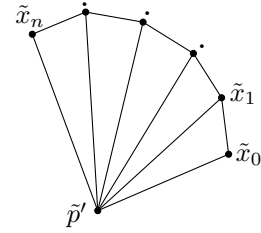
$$|px| + |xy| < \varpi_\kappa \Rightarrow \angle[x_y^p] \geq \tilde{\angle}_\kappa(x_y^p),$$

then $L \in \text{Alex}[\kappa, \infty]$.

Proof. If $\kappa \leq 0$, lemma follows directly from the hinge comparison (3.2.1(4)). For $\kappa > 0$ the proof is very similar to proof of equivalence of development comparison definition (3.3.4):

According to 3.3.4, it is enough to show that for any point $p \in L$ and any minimizing geodesic $\gamma : [0, T] \rightarrow \text{Ball}(\varpi_\kappa, p) \setminus p$, its κ -development with respect to p is convex with respect to its base.

Clearly $|p\gamma(t)| < \varpi_\kappa - \varepsilon$ for some $\varepsilon > 0$. Consider partition $0 = t_0 < t_1 < \dots < t_n = T$ such that $|t_i - t_{i-1}| < \varepsilon$. Set $x_i = \gamma(t_i)$ and take $p' \approx p$, $p' \in L^\bullet(x_0, x_1, \dots, x_n)$. Construct a chain of model triangles $[\tilde{p}'\tilde{x}_{i-1}\tilde{x}_i] = \tilde{\Delta}_\kappa(p'x_{i-1}x_i)$ on such a way that direction $[\tilde{p}'\tilde{x}_i]$ turns counterclockwise as i grows. From the assumption of lemma,



$$\begin{aligned} \angle[\tilde{x}_i \tilde{p}'^{i-1}] + \angle[\tilde{x}_i \tilde{p}'^{i+1}] &= \tilde{\angle}_\kappa(x_i p'^{i-1}) + \tilde{\angle}_\kappa(x_i p'^{i+1}) \leq \\ &\leq \angle[x_i p'^{i-1}] + \angle[x_i p'^{i+1}] \leq \pi; \end{aligned}$$

i.e. broken line $\tilde{x}_0\tilde{x}_1 \dots \tilde{x}_n$ is convex with respect to \tilde{p}' .

Taking finer and finer partitions and passing to the limit $p' \rightarrow p$, the broken line $\tilde{x}_0\tilde{x}_1 \dots \tilde{x}_n$ approaches development of γ with respect to p and the statement follows. \square

Proof of globalization theorem. Note that since for fixed points $x, y, z \in L$, the function $\kappa \mapsto \tilde{\angle}_\kappa(x_y^z)$ is continuous, it is enough to show that $L \in \text{Alex}[\kappa, \infty]$ for any $\kappa < \kappa_0$. For the rest of the proof we fix some $\kappa < \kappa_0$.

Exactly the same argument as in the proof of theorem 3.1.4 shows that L is almost geodesic. By theorem 3.5.2(6), for any hinge $[x_y^p]$ in L the angle $\angle[x_y^p]$ is well defined and moreover, if $x \in [yz]$ then

$$\angle[x_y^p] + \angle[x_z^p] \leq \pi.$$

Let us denote by $\text{Rad}(p)$ the maximal value (possibly ∞) such that the comparison

$$\angle[x_y^p] \geq \tilde{\angle}_\kappa(x_y^p)$$

holds for any hinge $[x_y^p]$ with $|px| + |xy| \leq \text{Rad}(p)$.

According to the short hinge lemma (3.5.6), that is enough to show that

$$\text{Rad}(p) \geq \varpi_\kappa \text{ for any } p \in L. \quad (**)$$

By theorem 3.5.2(3), the lemma on almost minimum (lemma 3.5.8) can be applied to the function $p \mapsto \text{Rad}(p)$: fix small⁸ $\varepsilon > 0$ and if $(\star\star)$ is not satisfied for $p \in L$, apply the lemma to find a point $p^* \in L$ such that $r^* = \text{Rad}(p^*) \leq \text{Rad}(p)$ and $\text{Rad}(q) > (1 - \varepsilon)r^*$ if $|pq| < \frac{1}{\varepsilon}r^*$. Then applying the key lemma (3.5.7) for ℓ slightly bigger⁹ than r^* we get that

$$\angle[x_q^{p^*}] \geq \tilde{\angle}_\kappa(x_q^{p^*})$$

for any hinge $[x_q^{p^*}]$ such that $|p^*x| + |xq| < \ell$. Thus $r^* \geq \ell$, a contradiction. \square

3.5.7. Key lemma. *Let $\kappa \in \mathbb{R}$, $0 < \ell \leq \varpi_\kappa$, X be a complete almost geodesic space and $p \in X$ be a point such that $\text{curv}_x X \geq \kappa$ for any $x \in B(2\ell, p)$.*

Assume that for any point $q \in B(\ell, p)$, comparison

$$\angle[x_q^y] \geq \tilde{\angle}_\kappa(x_q^y)$$

holds for any hinge $[x_q^y]$ with $|xy| + |xq| < \frac{2}{3}\ell$. Then comparison

$$\angle[x_q^p] \geq \tilde{\angle}_\kappa(x_q^p)$$

holds for any hinge $[x_q^p]$ with $|xp| + |xq| < \ell$.

Proof. We will prove an equivalent inequality:

$$\tilde{\nabla}_\kappa[x_q^p] \geq |pq|. \quad (*)$$

for any hinge $[x_q^p]$ with $|xp| + |xq| < \ell$.

Fix q . Given a hinge $[x_q^p]$ such that $\frac{2}{3}\ell \leq |px| + |xq| < \ell$, let us construct a new hinge $[x'_q^p]$ the following way:

Assume $|xq| \geq |xp|$, otherwise switch the roles of p and q in the following construction. Take $x' \in [xq]$ such that

$$|px| + 3|xx'| = \frac{2}{3}\ell \quad (**)$$

Choose a geodesic $[x'p]$ and form a hinge $[x'_q^p]$ from $[x'p]$ and $[x'q] \subset [xq]$ ¹⁰.

Note that we have $x, x' \in \text{Ball}(\ell, p) \cap \text{Ball}(\ell, q)$ and moreover $|px| + |xx'|, |px'| + |x'x| < \frac{2}{3}\ell$. In particular,

$$\angle[x_{x'}^p] \geq \tilde{\angle}_\kappa(x_{x'}^p) \quad \text{and} \quad \angle[x'_q^p] \geq \tilde{\angle}_\kappa(x'_q^p). \quad (***)$$

From the triangle inequality, we have

$$\ell > |px| + |xq| \geq |px'| + |x'q| \quad (***)$$

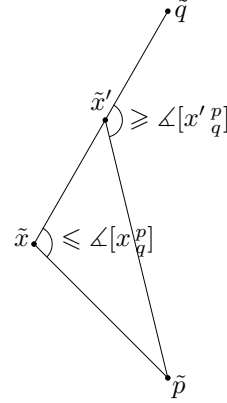
Let us show that in addition, we have

$$\tilde{\nabla}_\kappa[x_q^p] \geq \tilde{\nabla}_\kappa[x'_q^p]. \quad (***)$$

⁸ $\varepsilon = 0.01$ will do.

⁹we have to assume $\frac{2}{3}\ell \leq (1 - \varepsilon)r^*$ and $\ell \leq \varpi_\kappa$

¹⁰in fact, as follows from 3.6.1, $[x'q]$ is unique and $[x'q] \subset [xq]$.



Indeed, let $[\tilde{x}\tilde{x}'\tilde{p}] = \tilde{\Delta}_\kappa(xx'p)$. Take \tilde{q} on the extension of $[\tilde{x}\tilde{x}']$ beyond x' such that $|\tilde{x}\tilde{q}| = |xq|$ (and therefore $|\tilde{x}'\tilde{q}| = |x'q|$). From (**),

$$\angle[x_q^p] = \angle[x_{x'}^p] \geq \tilde{\Delta}_\kappa(x_0^p) \Rightarrow \tilde{\Upsilon}_\kappa[x_p^q] \geq |\tilde{p}\tilde{q}|.$$

From the assumptions of the lemma, we have $\angle[x_x^p] + \angle[x_q^p] \leq \pi$; thus (**) implies

$$\pi - \tilde{\Delta}_\kappa(x_x^p) \geq \pi - \angle[x_x^p] \geq \angle[x_q^p].$$

Therefore $|\tilde{p}\tilde{q}| \geq \tilde{\Upsilon}_\kappa[x_p^q]$ and (***) follows.

Given a hinge $[x_0^p]$ such that $|px_0| + |x_0q| < \ell$, let us apply inductively the above construction to get a sequence of hinges $[x_n^p]$ with $x_{n+1} = x'_n$.

The sequence might terminate at n only if $|px_n| + |x_nq| < \frac{2}{3}\ell$. In this case, from the assumptions of lemma $\tilde{\Upsilon}_\kappa[x_n^p] \geq |pq|$. From (***), we get monotonicity of sequence $s_n = \tilde{\Upsilon}_\kappa[x_n^p]$ which implies (*).

Otherwise, the sequence $r_n = |px_n| + |x_nq|$ must converge and its limit has to be at least $\frac{2}{3}\ell$. Pass to subsequence of (x_n) such that $x'_n = x_{n+1} \in [x_nq]$, we have $|px_n| + |x_nx'_n| - |px'_n| \rightarrow 0$. From (**), it follows that $|px_n| > \delta$ and $|x_nx'_n| > \delta$ for some fixed $\delta > 0$. Therefore, $\tilde{\Delta}_\kappa(x_n^p) \rightarrow \pi$. Since $\angle[x_n^p] \geq \tilde{\Delta}_\kappa(x_n^p)$, we have

$\angle[x_n^p] = \angle[x_n^p] \rightarrow \pi$. Therefore,

$$|px_n| + |x_nq| - \tilde{\Upsilon}_\kappa[x_n^p] \rightarrow 0$$

(that is the place where we need $\ell \leq \varpi_\kappa$). Hence

$$\lim_{n \rightarrow \infty} \tilde{\Upsilon}_\kappa[x_n^p] \geq |pq|.$$

Applying monotonicity of sequence $s_n = \tilde{\Upsilon}_\kappa[x_n^p]$ we again obtain (*) □

3.5.8. Lemma on almost minimum. Assume X is a complete metric space and $r : X \rightarrow \mathbb{R}$ is a strictly positive function such that $\underline{\lim}_n r(x_n) > 0$ for any convergent sequence (x_n) in X .

Then, given $\varepsilon > 0$, for any $p \in X$ there is a point $p^* \in X$ such that

- (i) $r(p^*) \leq r(p)$ and
- (ii) if $|xp^*| < \frac{1}{\varepsilon}r(p^*)$, then $r(x) > (1 - \varepsilon)r(p^*)$.

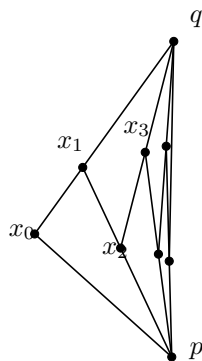
Proof. Assume the statement is wrong. Then for any $x \in X$ with $r(x) \leq r(p)$, there is a point $x' \in X$ such that

$$|xx'| < \frac{1}{\varepsilon}r(x) \quad \text{and} \quad r(x') \leq (1 - \varepsilon)r(x).$$

Take $x_0 = p$ and consider a sequence of points (x_n) such that $x_{n+1} = x'_n$. Clearly

$$|x_{n+1}x_n| \leq \frac{r(p)}{\varepsilon}(1 - \varepsilon)^n \quad \text{and} \quad r(x_n) \leq r(p)(1 - \varepsilon)^n.$$

Therefore (x_n) is a Cauchy sequence and $\lim_n r(x_n) = 0$, a contradiction. □



3.6 Properties of geodesics and angles

3.6.1. Geodesics do not split. *In Alexandrov space with curvature bounded below, geodesics do not bifurcate.*

More precicely: let $L \in \text{Alex}[\kappa, \infty]$ and $[px], [py]$ be two minimizing geodesics, then

(1) *If $\mathbb{Y}_{[px]}$ and $\mathbb{Y}_{[py]}$ coincide for all $t \in [0, \varepsilon)$, $\varepsilon > 0$ then $[px] \subset [py]$ or $[py] \subset [px]$.*

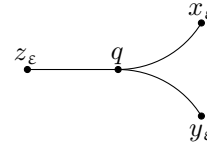
(2) *If $\angle[p_y^x] = 0$ then $[px] \subset [py]$ or $[py] \subset [px]$.*

3.6.2. Corollary. *The restriction of any minimal geodesic to a proper segment is the unique minimal geodesic joining its endpoints.*

In case $\kappa \leq 0$, the proof is easier, since the model triangles are always well defined. To deal with $\kappa > 0$ we have to do everything locally.

Proof (1). Let t_{\max} be the maximal number such that $\mathbb{Y}_{[px]}(t) = \mathbb{Y}_{[py]}(t)$ for all $t \in [0, t_{\max})$. Since geodesics are continuous $\mathbb{Y}_{[px]}(t_{\max}) = \mathbb{Y}_{[py]}(t_{\max})$, set

$$q = \mathbb{Y}_{[px]}(t_{\max}) = \mathbb{Y}_{[py]}(t_{\max}).$$



We have to show that $t_{\max} = \max\{|px|, |py|\}$.

If that is not true, choose sufficiently small $\varepsilon > 0$ such that points

$$x_\varepsilon = \mathbb{Y}_{[px]}(t_{\max} + \varepsilon) \quad \text{and} \quad y_\varepsilon = \mathbb{Y}_{[py]}(t_{\max} + \varepsilon)$$

are distinct. Set

$$z_\varepsilon = \mathbb{Y}_{[px]}(t_{\max} - \varepsilon) = \mathbb{Y}_{[py]}(t_{\max} - \varepsilon).$$

Clearly, $\tilde{\angle}_\kappa(q^{z_\varepsilon}) = \tilde{\angle}_\kappa(q^{y_\varepsilon}) = \pi$. Thus from 3-angle comparison (3.1.1), $\tilde{\angle}_\kappa(q^{x_\varepsilon}) = 0$ and thus $x_\varepsilon = y_\varepsilon$, a contradiction.

(2). From hinge comparison (3.2.1(4))

$$\angle[p_y^x] = 0 \quad \Rightarrow \quad \tilde{\angle}_\kappa \left(p \begin{matrix} \mathbb{Y}_{[px]}(t) \\ \mathbb{Y}_{[py]}(t) \end{matrix} \right) = 0$$

and thus $\mathbb{Y}_{[px]}(t) = \mathbb{Y}_{[py]}(t)$ for all small t . Therefore we can apply (1). □

3.6.3. First variation formula. *Let L be an Alexandrov space and for two points $p, q \in L$ there is there is unique geodesic $[pq]$ in L° . Then for any $[q_x^p]$ in L we have*

$$\text{dist}_q \circ \mathbb{Y}_{[px]}(t) = |pq| - t \cos \angle[p_x^q] + o(t). \quad (*)$$

In particular, () holds if $q \in L^\bullet(q)$ as well as if $p \in L^\bullet(p)$.*

The above lemma is also called lemma about strong angle, since it in particular says that if minimizing geodesic $[pq]$ is extendable (as a minimizing geodesic) behind q then $\angle[p_x^q]$ can be defined as $\lim_{t \rightarrow 0^+} \tilde{\angle}_\kappa(q^p_{\mathbb{Y}_{[qx]}(t)})$.

Proof. Let $L \in \text{Alex}[\kappa, \infty]$, set $\varphi = \angle [p_x^q]$. According to 2.4.3,

$$\text{dist}_q \circ \mathbb{Y}_{[px]}(t) \leq |pq| - t \cos \varphi + o(t).$$

Thus, it is enough to show that

$$\text{dist}_q \circ \mathbb{Y}_{[px]}(t) \geq |pq| - t \cos \varphi + o(t).$$

Assume contrary, i.e. for some $\varepsilon > 0$ there is a sequence $t_n \rightarrow 0+$ such that

$$\text{dist}_q \circ \mathbb{Y}_{[px]}(t_n) < |pq| - t_n \cos(\varphi - \varepsilon).$$

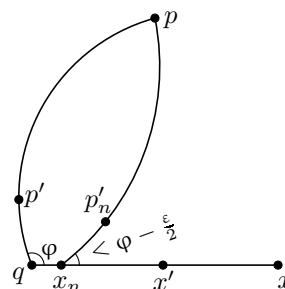
Set $x_n = \mathbb{Y}_{[px]}(t_n)$ and choose points $p' \in [qp]$ and $x' \in [qx]$ such that $\tilde{\Delta}_\kappa(q, p'_x) > \varphi - \frac{\varepsilon}{2}$, set $\ell = |p'q|$.

Assume L is geodesic. Choose sequence of minimizing geodesics $[x_n p]$ and set $p'_n = \mathbb{Y}_{[x_n p]}(\ell)$. Clearly $\liminf_n \tilde{\Delta}_\kappa(x_n, p'_n) \geq \pi - \varphi + \varepsilon$. From hinge comparison,

$$\tilde{\Delta}_\kappa(x_n, p'_n) \leq \angle [x_n p] \leq \pi - \angle [x_n q] \leq \pi - \tilde{\Delta}_\kappa(x_n, p'_n) < \varphi - \frac{\varepsilon}{2}$$

for all large n . Therefore p'_n does not converges to p' and thus two geodesics $[pq]$ and $\omega\text{-lim}_n [x_n p]$ are distinct in L° .

Finally, if L is not geodesic chose a sequence $p_n \rightarrow p$, such that $p_n \in L^\bullet(x_n)$, set $p'_n = \mathbb{Y}_{[x_n p_n]}(\ell)$ and the same argument as above show that $[xp] \neq \omega\text{-lim}_n [x_n p_n]$. \square



3.7 Remarks on positive lower bound

The following theorem states that if one ignores few exceptional spaces, then diameter of a space with positive lower curvature bound is bounded. Note that many authors (but not us) exclude these spaces in the definition of Alexandrov space with positive lower curvature bound.

3.7.1. On diameter of space. *Let $\kappa > 0$ and $L \in \text{Alex}[\kappa, \infty]$, then either*

- (i) $\text{diam } L \leq \varpi_\kappa$;
- (ii) or L is isometric to one of the following exceptional spaces:
 - (a) real line \mathbb{R} ,
 - (b) a ray \mathbb{R}_{\geq} ,
 - (c) a closed interval $[0, a] \in \mathbb{R}$, $a > \varpi_\kappa$,
 - (d) a circle \mathbb{S}_a^1 of length $a > 2\varpi_\kappa$.

Proof. Using rescaling, we can assume that $\kappa = 1$ and thus $\varpi_\kappa = \pi$.

Assume L is a geodesic space and $\text{diam } L > \pi$. Choose $x, y \in L$ so that $|xy| = \pi + \varepsilon$, $0 < \varepsilon < \pi/4$. Let z be mid the point of a geodesic $[xy]$. Assume there is a point $q \in \text{Ball}(\varepsilon, z) \setminus [xy]$. Applying 2-angles comparison (3.2.1(2)), we get $\angle_1(z; qx) + \angle_1(z; qy) \leq \pi$ and since $|zx|, |zy| > \frac{\pi}{2}$, we get

$$|xq| + |qy| < |xz| + |zy| = |xy|,$$

which contradicts triangle inequality.

That implies that $\text{Ball}(\varepsilon, z)$ only contain points of $[xy]$. Since geodesics in CBB-spaces do not bifurcate (3.6.1(1)), it follows that all L coincides with the maximal extension of $[xy]$ as a geodesic γ (but not as minimizing geodesic). From this, it is easy to see that L falls into one of exceptional spaces described in the theorem. \square

3.7.2. On perimeter of triangle. Let $\kappa > 0$, $L \in \text{Alex}[\kappa, \infty]$ and $\text{diam } L \leq \varpi_\kappa$ then for any three points $p, q, r \in L$ we have

$$|pq| + |qr| + |rp| \leq 2\varpi_\kappa.$$

Proof. We can assume that $\kappa = 1$, so $\varpi_\kappa = \pi$.

Since L is almost geodesic (3.1.4), it is sufficient to consider case when there is a geodesic $[qr]$.

First note that $y(t) = \text{md}_1 \circ \text{dist}_p \circ \mathbb{Y}_{[qr]}(t)$ satisfies differential inequality $y'' \leq 1 - y$. Indeed, given $\varepsilon > 0$ one can find a finite set of points $F_\varepsilon = \{p_i\} \subset \text{Ball}(\varepsilon, p)$ such that for the function $\text{dist}_{F_\varepsilon} = \min_i \text{dist}_{p_i}$ we have $\text{dist}_{F_\varepsilon}(z) < \pi$ for any $z \in [qr]$. Thus, from function comparison (3.3.1),

$$y_\varepsilon(t) = \text{md}_1 \circ \text{dist}_{F_\varepsilon} \circ \mathbb{Y}_{[qr]}(t) = \min_i \{ \text{md}_1 \circ \text{dist}_{p_i} \circ \mathbb{Y}_{[qr]}(t) \}$$

satisfies $y_\varepsilon'' \leq 1 - y_\varepsilon$. Clearly $\text{dist}_{F_\varepsilon} \rightrightarrows \text{dist}_p$ as $\varepsilon \rightarrow 0$, therefore $y_\varepsilon \rightrightarrows y$ and thus we have $y'' \leq 1 - y$.

Take $z_0 \in [qr]$ so that restriction $\text{dist}_p|_{[qr]}$ attains its maximum at z_0 , set $t_0 = |qz_0|$ so $z_0 = \mathbb{Y}_{[qr]}(t_0)$. Consider model configuration: two minimizing geodesics $[\tilde{p}\tilde{z}_0]$, $[\tilde{q}\tilde{r}]$ in $\mathbb{H}_1 = \mathbb{S}^2$ such that

$$|\tilde{p}\tilde{z}_0| = |pz_0|, \quad |\tilde{q}\tilde{r}| = |qr|, \quad |\tilde{z}_0\tilde{q}| = |z_0q|, \quad |\tilde{z}_0\tilde{r}| = |z_0q|$$

and

$$\angle[\tilde{z}_0\tilde{q}] = \angle[\tilde{z}_0\tilde{r}] = \pi/2.$$

Clearly, $\bar{y}(t) = \text{md}_1 \circ \text{dist}_{\tilde{p}} \circ \mathbb{Y}_{[\tilde{q}\tilde{r}]}(t)$ satisfies $\bar{y}'' = 1 - \bar{y}$ and $\bar{y}'(t_0) = 0$, $\bar{y}(t_0) = y(t_0)$. Since z_0 is a maximum point, $y(t) \leq y(t_0) + o(t - t_0)$; thus, $\bar{y}(t)$ is the barrier for $y(t) = \text{md}_1 \circ \text{dist}_p \circ \mathbb{Y}_{[qr]}(t)$ at t_0 . From barrier inequality (7.1.2(2)), we get

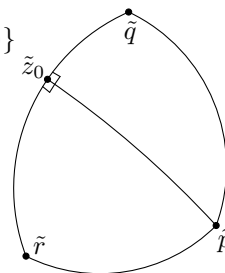
$$\text{dist}_{\tilde{p}} \circ \mathbb{Y}_{[\tilde{q}\tilde{r}]}(t) \geq \text{dist}_p \circ \mathbb{Y}_{[qr]}(t),$$

thus, $|\tilde{p}\tilde{q}| \geq |pq|$ and $|\tilde{p}\tilde{r}| \geq |pr|$. Thus, $|pq| + |qr| + |rp|$ can not exceed a perimeter of a spherical triangle $[\tilde{p}\tilde{q}\tilde{r}]$, therefore

$$|pq| + |qr| + |rp| \leq 2\pi.$$

\square

3.7.3. Definition of extended angle. Let $\kappa > 0$, consider the following extension $\tilde{\angle}_{\kappa+}(*_*)$ of model angle function $\tilde{\angle}_\kappa(*_*)$ defined the following way:



1. $\tilde{\Delta}_{\kappa+}(p_r^q) = \tilde{\Delta}_{\kappa}(p_r^q)$ once $\tilde{\Delta}_{\kappa}(p_r^q)$ is well defined;
2. $\tilde{\Delta}_{\kappa+}(p_r^q) = \tilde{\Delta}_{\kappa+}(p_q^r) = 0$ if $|pq| + |qr| = |pr|$ and $|pq| > 0$;
3. $\tilde{\Delta}_{\kappa+}(p_r^q) = \pi$ if non of above is aplicable.

3.7.4. Extended angle comparison. Let $\kappa > 0$ and $L \in \text{Alex}[\kappa, \infty]$, then for any hinge $[p_r^q]$ we have $\angle[p_r^q] \geq \tilde{\Delta}_{\kappa+}(p_r^q)$.

Proof. Note that definition of extended angle can be red the following way: $\tilde{\Delta}_{\kappa+}(p_r^q) = \sup \tilde{\Delta}_{\kappa'}(p_r^q)$ where we take supremum along all $\kappa' \leq \kappa$ such that $\tilde{\Delta}_{\kappa'}(p_r^q)$ is well defined. Thus, the extended angle comparison follows since if $\kappa' < \kappa$ then $\text{Alex}[\kappa', \infty] \supset \text{Alex}[\kappa, \infty]$ (3.5.5). \square

3.8 Exercises

1. (Wald's definition) Let L be complete length space and $\kappa \leq 0$, prove that $L \in \text{Alex}[\kappa, \infty]$ if and only if for any quadruple of points $p, q, r, s \in L$ can be isometrically embedded into some \mathbb{J}_\varkappa for $\varkappa \geq \kappa$.

Is the same true for $\kappa > 0$, what is the difference?

2. (a) Show that any space $L \in \text{Alex}[0, \infty]$ satisfies the following condition:
 - For any three points $p, q, r \in L$, if q' and r' are mid-points of geodesics $[pq]$ and $[pr]$ correspondently then $2|q'r'| \geq |qr|$.
 (b) Construct a geodesic space, which satisfies above condition, but which is not an Alexandrov space.
3. Assume X is a complete length space which satisfy the following condition, any 4-point subset can be isometrically embedded in Euclidean 3-space. Prove that X is isometric to a closed convex substet of a Hilbert space.

4. (a) Construct a geodesic-metric space X which is not an Alexandrov space, but which meets the following condition: for any 3 points $p, x, y \in X$ there is a geodesic $[xy]$ such that for any $z \in [xy]$

$$\tilde{\Delta}_0(z_x^p) + \tilde{\Delta}_0(z_y^p) \leq \pi.$$

Hint: Look at the construction of Uryson's space [Gromov, 3.11 $\frac{3}{2}$].

- (b) Let us change condition a bit: for any 2 points $x, y \in X$ there is a geodesic $[xy]$ such that for any $p \in X$ and $z \in [xy]$

$$\tilde{\Delta}_0(z_x^p) + \tilde{\Delta}_0(z_y^p) \leq \pi.$$

Show that $X \in \text{Alex}[0, \infty]$

5. **Schröder's globalization.** Assume X is a non-complete geodesic space for which the κ -comparison is satisfied in a neighborhood of every point. Prove that completeion of X is an Alexandrov's space with curvature $\geq \kappa$.
6. Assume $L \in \text{Alex}[\kappa, \infty]$, $\varkappa > \kappa$ and $\text{curv}_p L \geq \varkappa$ for all p in a dense G_δ -set of L . Prove that $L \in \text{Alex}[\varkappa, \infty]$