Comment. Math. Helvetici **59** (1984) 187–192 0010–2571/84/020187–06$01.50 + 0.20/0

A sharp four dimensional isoperimetric inequality

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Introduction

Let $(M, \partial M)$ be a compact n-dimensional Riemannian manifold (with boundary) of non positive sectional curvature. Assume further that every geodesic ray in M minimizes length up to the point it hits the boundary. In this paper we show:

THEOREM. *If* $(M, \partial M)$ *is as above* $(n \geq 3)$ *then* $\mathrm{Vol}(\partial M)^n \geq C(n)\,\mathrm{Vol}(M)^{n-1}$ *where*

$$C(n) = \frac{\alpha(n-1)^{n-1}}{\alpha(n-2)^{n-2}\left\{\int_0^{\pi/2} \cos(t)^{n/n-2} \sin(t)^{n-2}\, dt\right\}^{n-2}}$$

and $\alpha(n)$ *represents the volume of the unit n sphere. If* $n \neq 4$ *equality never holds. If* $n = 4$ *equality holds if and only if M is isometric to a flat ball.*

This answers a long standing conjecture in dimension 4. The conjecture states that for $(M, \partial M)$ a compact domain in a complete simply connected manifold of non-positive curvature (which implies the condition of the theorem) we have $\mathrm{Vol}(\partial M)^n \geq \bar{C}(n)\,\mathrm{Vol}(M)^{m-1}$ for $\bar{C}(n) = n^{n-1}\alpha(n-1)$, with equality holding if and only if M is isometric to a flat ball. It is an easy computation to see that $C(4) = \bar{C}(4)$. The conjecture was proved in dimension 2 by Beckenbach and Radó (see [B–R]) in 1933, and is open in all dimensions except 2, and now 4.

This conjecture is a special case of a more general conjecture (see [A]) where an upper bound K (not necessarily 0) on the curvature is assumed. The more general conjecture was proved in dimension 2 by Aubin (see [A]), and for constant curvature in all dimensions by Schmidt (see [Sc]).

The isoperimetric constants are related to Sobolev constants (see [A], [Bo], and [FF] among other). In particular we see that for a domain D in a simply

* Supported by NSF Grant #MCS-01780

connected n-dimensional Riemannian manifold of non-positive curvature and for any $g \in H_0^1(M)$ we have

$$\int_D \|dg\| \geq C(n)\left\{\int_D |g|^{n/n-1}\right\}^{(n-1)/n}$$

where $C(4)$ is the flat constant.

Such isoperimetric inequalities are interesting even with non-sharp constants. Previous non-sharp versions of the theorem are consequences of results in [H–S] and [C]. The constants $C(n)$ given here are the best known to the author in all dimensions (greater than 2). In particular $C(3) = 32\pi$ while $\bar{C}(3) = 36\pi$.

Notation and definitions

We will use the notation of [C]. Let $UM \xrightarrow{\pi} M$ represent the unit sphere bundle with the canonical (local product) measure. For $v \in UM$, let γ_v be the geodesic with $\gamma_v'(0) = v$ and let $\xi^t(v)$ represent the geodesic flow (i.e. $\xi^t(v) = \gamma_v'(t)$). For $v \in UM$ we let $l(v) = \max\{t \mid \gamma_v(t) \in M\}$. Note $\xi^t(v)$ is defined for $t \leq l(v)$ and $\gamma_v(l(v)) \in \partial M$.

For $p \in \partial M$ let N_p be the inwardly pointing unit normal vector to ∂M at p. Let $U^+\partial M \xrightarrow{\pi} \partial M$ be the bundle of inwardly pointing unit vectors (i.e. $U^+\partial M = \{u \in UM \mid \pi(u) \in \partial M$ and $\langle u, N_{\pi(u)}\rangle > 0\}$). We let $U_p^+\partial M$ represent $\pi^{-1}(p)$. For $u \in U^+\partial M$ we will use $\cos(u)$ to represent $\langle u, N_{\pi(u)}\rangle$. The measure on $U^+\partial M$ is the local product measure du where the measure of the fibre is that of the unit upper hemisphere.

The proof

The main tool in the proof is a formula due to Santalo:

$$\text{(i)} \quad \int_{UM} f(v)\, dv = \int_{U^+\partial M} \int_0^{l(u)} f(\xi^t(u)) \cos(u)\, dt\, du$$

for all integrable functions f. The formula takes this form in our case since all geodesics in M hit ∂M. For a proof see [Sa] pp. 336–338 or [B] p. 286.

From this we derive:

LEMMA 1. a) $\mathrm{Vol}\,(M) = 1/\alpha(n-1)\int_{U^+\partial M} l(u)\cos(u)\,du$
b) *For all integrable functions* g

$$\int_{U^+\partial M} g(u)\cos(u)\,du = \int_{U^+\partial M} g(\mathrm{ant}\,(u))\cos(u)\,du$$

where $\mathrm{ant}\,(u) = -\gamma'_u(l(u))$.

Proof. Part a) follows directly from (i) by letting $f(v) \equiv 1$ and integrating the t. That is

$$\alpha(n-1)\,\mathrm{Vol}\,(M) = \int_{UM} dv = \int_{U^+\partial M}\int_0^{l(u)} \cos(u)\,dt\,du = \int_{U^+\partial M} l(u)\cos(u)\,du.$$

To prove part b) we first note that (i) says that the geodesic flow ξ is a measure preserving map from Q to UM where $Q = \{(u, t) \mid u \in U^+\partial M \text{ and } 0 \le t \le l(u)\}$ is given the measure $\cos(u)\,dt\,du$. ξ has an inverse (smooth almost everywhere) ξ^{-1} which is also measure preserving, for $v \in UM$ $\xi^{-1}(v) = (-\gamma'_{-v}(l(-v)), l(-v))$. Since the antipodal map $-1: UM \to UM$ is also measure preserving we have $\xi^{-1}\circ(-1)\circ\xi: Q \to Q$ is measure preserving. Since $\xi^{-1}\circ(-1)\circ\xi(u, t) = (\mathrm{ant}\,(u), l(u)-t)$ we see that for every integrable $G: Q \to \mathbb{R}$ we have:

$$\int_{U^+\partial M}\int_0^{l(u)} G(u, t)\cos(u)\,dt\,du = \int_{U^+\partial M}\int_0^{l(u)} G(\mathrm{ant}\,(u), l(u)-t)\cos(u)\,dt\,du$$

To complete the proof of part b) simply take $G(u, t) = g(u)/l(u)$ and integrate the t (note: $l(\mathrm{ant}\,u) = l(u)$).

LEMMA 2. a) $\int_{U^+\partial M} l(u)^{n-1}/\cos(\mathrm{ant}\,u)\,du \le \mathrm{Vol}\,(\partial M)^2$ *with equality holding if and only if M is flat and convex.*
b) $\int_{U^+\partial M}(\cos(\mathrm{ant}\,u))^{1/n-2}(\cos(u))^{n-1/n-2}\,du \le \mathrm{Vol}\,(\partial M)\cdot C_2(n)$ *where* $C_2(n) = \alpha(n-2)\int_0^{\pi/2}\cos^{n/n-2}(t)\sin^{n-2}(t)\,dt$. *Equality holds if and only if* $\cos(u) = \cos(\mathrm{ant}\,u)$ *almost everywhere.*

Remark. "almost everywhere" above can be replaced with "everywhere" but it is not worth going into.

Proof. Let dx be the volume form on M and dp the volume form on ∂M. Let $q \in \partial M$. In normal polar coordinates (u, r) about q in the region $\operatorname{Exp}\{tu \mid u \in U_q^+ \partial M \text{ and } 0 \le t \le l(u)\}$ we have $dx = F(u, r)\, du\, dr$ for some function $F(u, r)$. Let $A = \operatorname{Exp}\{tu \mid t = l(u)\}$. Then $A \subset \partial M$ and dp on A is precisely $(F(u, l(u))/\cos(\operatorname{ant} u))\, du$. Thus we see

$$\int_{U_q^+\partial M} \frac{F(u, l(u))}{\cos(\operatorname{ant} u)}\, du = \operatorname{Vol}(A) \le \operatorname{Vol}(\partial M).$$

Equality holds if and only if $A = \partial M$. That is, M is (geodesically) star shaped from q.

Integrating over q we get

$$\int_{U^+\partial M} \frac{F(u, l(u))}{\cos(\operatorname{ant} u)}\, du \le \operatorname{Vol}(\partial M)^2$$

with equality holding if and only if M is convex. Part a) now follows since M having non-positive curvature implies $F(u, l(u)) \ge l(u)^{n-1}$ with equality if and only if the sectional curvatures of all sections containing $\gamma_u'(t)$ for some t, are 0 (see [B–C] Section 11.10).

To prove part b) we apply a Schwarz inequality and Lemma 1b.

$$\int_{U^+\partial M} (\cos(\operatorname{ant} u))^{1/n-2}(\cos(u))^{n-1/n-2}\, du$$

$$= \int_{U^+\partial M} (\cos(\operatorname{ant} u))^{1/n-2}(\cos(u))^{1/n-2}\cos(u)\, du$$

$$\le \left\{\int_{U^+\partial M} (\cos(\operatorname{ant} u))^{2/n-2}\cos(u) du\right\}^{1/2} \cdot \left\{\int_{U^+\partial M} (\cos(u)^{2/n-2}\cos(u)\, du\right\}^{1/2}$$

$$= \int_{U^+\partial M} (\cos(u))^{n/n-2}\, du = \int_{\partial M}\left(\int_{U^+\partial M} (\cos(u)^{n/n-2}\, du\right) dq$$

$$= \operatorname{Vol}(\partial M)\cdot C_2(n)$$

In order for equality to hold we need to have equality in the inequality, i.e. $\cos(\operatorname{ant} u) = K\cos(u)$ almost everywhere for some constant K. Since the maximum values of both $\cos(\operatorname{ant} u)$ and $\cos(u)$ are 1 it is clear that K must be 1.

Proof of the theorem. By Lemma 1a and a Hölder inequality we have

$$\begin{aligned}\mathrm{Vol}\,(M)&=\frac{1}{\alpha(n-1)}\int_{U^+\partial M} l(u)\cos(u)\,du\\&=\frac{1}{\alpha(n-1)}\int_{U^+\partial M}\frac{l(u)}{(\cos(\mathrm{ant}\,u))^{1/n-1}}(\cos(\mathrm{ant}\,u))^{1/n-1}\cos(u)\,du\\&\leq\frac{1}{a(n-1)}\left\{\int_{U^+\partial M}\frac{l(u)^{n-1}}{\cos(\mathrm{ant}\,u)}\,du\right\}^{1/n-1}\\&\quad\cdot\left\{\int_{U^+\partial M}(\cos(\mathrm{ant}\,u))^{1/n-2}(\cos(u))^{n-1/n-2}\,du\right\}^{n-2/n-1}.\end{aligned}$$

Applying Lemmas 2a and 2b we get

$$\mathrm{Vol}\,(M)\leq\frac{1}{\alpha(n-1)}(\mathrm{Vol}\,(\partial M))^{2/n-1}\cdot(\mathrm{Vol}\,(\partial M))^{n-2/n-1}C_2(n)^{n-2/n-1}.$$

hence

$$C(n)\,\mathrm{Vol}\,(M)^{n-1}=\frac{\alpha(n-1)^{n-1}}{C_2(n)^{n-2}}(\mathrm{Vol}\,(M))^{n-1}\leq(\mathrm{Vol}\,(\partial M))^n.$$

In order for equality to hold we must have equality in Lemmas 2a and 2b as well as the above Hölder inequality. By Lemma 2a we see that M must be flat and hence the theorem follows from the classical result in $\mathbb{R}^n$, since $C(4)$ is sharp and $C(n)$ for $n\neq 4$ is not. One can also see the $n=4$ case directly. Equality in the Hölder inequality gives

$$\frac{l^3(u)}{\cos(\mathrm{ant}\,u)}=K\cos(\mathrm{ant}\,u)^{1/2}\cos(u)^{3/2}$$

almost everywhere for some constant K. By the equality condition in Lemma 2b we see $l(u)=2r\cos(u)$ for some constant r. It is now easy to see (since M is flat) that M is a ball of radius r.

Remark. A similar equality analysis for $n\neq 4$ would require:

1) M flat and convex
2) $\cos(\mathrm{ant}\,u)=\cos(u)$
3) $l(u)=K\cos^{2/n-2}(u)$.

No such M exists.

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Received March 28, 1983