On the almost negatively curved 3-sphere

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1. Introduction. By Hadamard's Theorem every simply connected Riemannian manifold with non positive curvature is diffeormorphic to $\mathbb{R}^n$. In view of the many pinching theorems of qualitative Riemannian geometry one might expect a similar theorem to hold if small amounts of positive curvature are permitted, e.g. if the upper bound of the sectional curvature is positive but small as compared to the maximal rank radius of the exponential map. Without additional assumptions this is not possible. In fact, Gromov points out in [3] that almost negatively curved metrics exist on the 3-sphere in the following sense.

Theorem. *For all* $\varepsilon > 0$ *there exists a Riemannian metric on* S^3 *with diameter* d *and upper sectional curvature bound* K *satisfying* $Kd^2 \le \varepsilon$.

It follows among others that for a given point $p \in S^3$ the exponential map $\exp_p$: $T_pS^3 \to S^3$ has maximal rank within a ball in T_pS^3 whose radius is much larger than the diameter of S^3. We may therefore lift the interior of the cut locus of p from S^3 to T_pS^3 via $\exp_p^{-1}$ and obtain a tesselation of the ball with fundamental domains very much the same way as Hadamard manifolds are tesselated with fundamental domains of compact quotients.

Gromov's example has been generalized to all compact 3-manifolds by Bavard. [1] He uses open books which yield, in addition, control of the volume. In [2] Gao and Yau used a cutting and pasting technique similar to the one we are going to explain below, to prove that S^3 admits metrics with strictly negative Ricci curvature. Both papers are quite technical. The aim of this note is to give a simplified version of Gromov's original construction. Although we obtain no new results we hope that the note makes this interesting example more accessible.

2. Surgery in dimension 2. Let us first explain the idea in dimension 2 although we know in advance that it cannot work. The idea is to use surgery which produces many short cuts with the effect that the diameter shrinks. It is easy to do this without affecting the upper curvature bound.

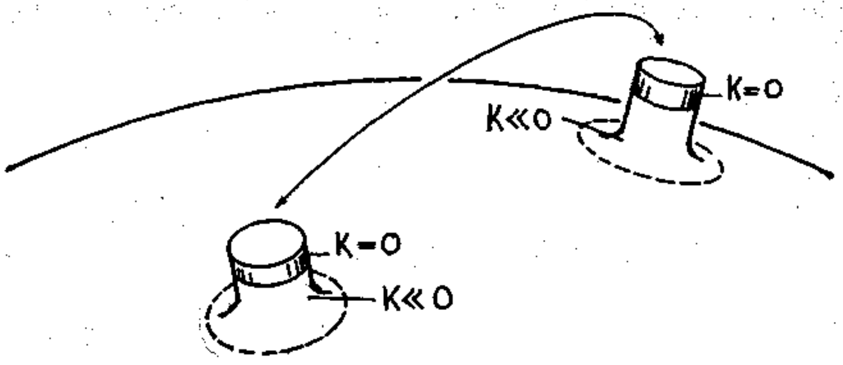

Fig. 1

From S^2 cut out two arbitrarily small discs which are at some distance apart. In the 2-holed sphere which remains we change the metric near each hole such that a neighbourhood of the hole becomes a flat cylinder. It takes highly negative curvature to adapt this metric smoothly to the standard metric outside some small annular region of the hole, but it is possible and we can do it such that the curvature remains bounded above by 1. Now identify the two boundaries (they are assumed to have the same length). This reduces the diameter. Unfortunately it produces a handle.

Of course we know that in dimension two we cannot reduce the diameter keeping the upper curvature bound and not change the topology: If K is an upper curvature bound and d the diameter then we have

Every Riemannian metric on S^2 *satisfies* $Kd^2 \geq \pi^2$

In fact, if $Kd^2 < \pi^2$ on some manifold M, then for $p \in M$, $\exp_p : T_pM \to M$ is a local diffeomorphism in a ball of radius greater than the diameter of M. We find therefore a fundamental domain for $\exp_p$, i.e. a simply connected compact domain whose boundary ∂ is mapped onto the cut locus of M and whose connected interior is mapped onto the interior of the cut locus. Since $\exp_p$ is a local diffeomorphism, $\exp_p : \partial \to M$ is an *immersion*. But if $M = S^2$ there is a topological obstruction: ∂ is an S^1, and any smoothly immersed image of S^1 in S^2 disconnects its complement. Hence we cannot have $Kd^2 < \pi^2$ on S^2.

In contrast to this we have no such obstruction in dimension 3.

3. Surgery in dimension 3. Let $\gamma : S^1 \to S^3$ and $\gamma_1 : S^1 \to S^2 \times S^1$ be closed geodesics of length

$$L = \ell(\gamma) = \ell(\gamma_1)$$

with respect to some given Riemannian metrics, and let γ_1 be homotopic to the S^1-factor of $S^2 \times S^1$. Assume that for small $r > 0$ the tubular neighbourhoods

$$\gamma^{3r} = \left\{ p \in S^3 \mid \mathrm{dist}(p,\gamma) < 3r \right\}$$

and

$$\gamma_1{}^{3r} = \left\{ p \in S^2 \times S^1 \mid \mathrm{dist}(p,\gamma_1) < 3r \right\}$$

are isometric to the flat tube

$$B^{3r} \times \mathbb{R} / [t \mapsto t + L]$$

where B^{3r} is the open disc of radius $3r$ in $\mathbb{R}^2$ and the metric is the product metric. In Fermi coordinates $p = p(\rho,\theta,t)$, the metric tensors are

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dt^2.$$

($0 < \rho < 3r$ is the radial distance, θ is the angle coordinate and t is from $\mathbb{R} / [t \mapsto t + L]$)

The point is that *the interior* of $\gamma_1{}^r$ is *diffeomorphic* to the exterior. (This is true because γ_1 is homotopic to the S^1-factor of $S^2 \times S^1$ and the interior of a disc in S^2 is diffeomorphic to the exterior). Thus if we remove γ^r from S^3 and replace it by the exterior of $\gamma_1{}^r$ the topology does not change . We shall see that using the exterior of $\gamma_1{}^r$ will permit short cuts. But let us first see how to control smoothness and curvature. For $r \le \rho \le 3r$ we replace the factor ρ^2 of the metric tensor by a factor $\phi^2(\rho)$ where $\phi(\rho)$ is a smooth convex function satisfying $\phi(\rho) = 2.5\, r$ for $2r \le \rho \le 2.4r$, and $\phi(\rho) = \rho$ for $2.6r \le \rho \le 3r$. In $\gamma^{3r} - \gamma^r$ we have non positive curvature. $\gamma^{2r} - \gamma^r$ and $\gamma_1{}^{2r} - \gamma_1^r$ are isometric to the Riemannian product $[0,r] \times T$ where T is a flat torus.

Hence we can glue together $S^3 - \gamma^r$ and $S^2 \times S^1 - \gamma_1{}^r$ along their boundaries $T\gamma$ resp. $T\gamma_1$ (which are both isometric to T) using an arbitrary identifying isometry $T\gamma \to T\gamma_1$. The metric remains smooth in this pasting. By abuse of notation we shall write $T\gamma = T\gamma_1$ on the new manifold.

4. Short cuts. Short cuts may occur as follows. Let $p,q \in S^3 - \gamma^r$ be near $T\gamma$. Connect p, q with nearby points $p', q' \in T\gamma = T\gamma_1$. Assume that in $S^2 \times S^1 - \gamma_1^r$ points p' and q' have a short connecting curve. The three curves together then yield a connection from p to q which may be much shorter than the original distance of p and q in S^3.

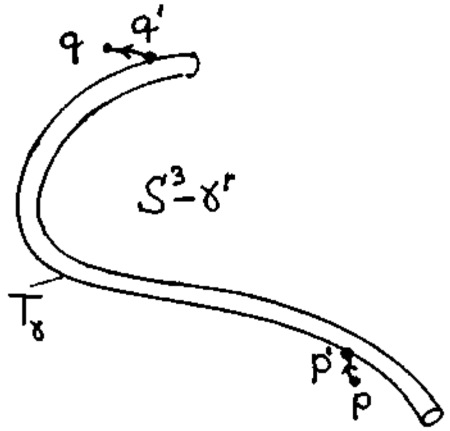

Fig. 2a

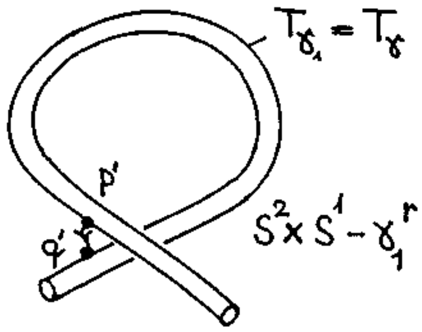

Fig. 2b

Now let us produce such short cuts. Let $\varepsilon > 0$ be arbitrarily small. Choose S^2 to be a rotational ellipsoid with curvature almost equal to 1 so that S^2 contains a closed geodesic γ_2 satisfying $\gamma_2^\varepsilon = S^2$. i.e every point of S^2 lies within distance ε from γ_2. Geodesic γ_2 consists of a number of segments, say m, of length approximately equal to 2π whose endpoints are within a distance $\leq \varepsilon$ from the north pole of S^2. Parametrize γ_2 in the form $t \mapsto \gamma_2(t)$, $0 \leq t \leq 1$ with $\gamma_2(0) = \gamma_2(1)$. Then take $S^1 = \mathbb{R}/[y \mapsto y + \varepsilon]$ and introduce the Riemannian product structure on the product $S^2 \times S^1$ of the ellipsoid and the small circle. The curve

$$t \mapsto \gamma_1(t) := (\gamma_2(t), \varepsilon t), \quad 0 \leq t \leq 1$$

is a simple closed geodesic on $S^2 \times S^1$ which is homotopic to the S^1 factor and satisfies $\gamma_1^\varepsilon = S^2 \times S^1$, i.e every point of $S^2 \times S^1$ lies within a distance ε from γ_1. Moreover, γ_1 has zero holonomy. By the lemma of section 5 below we can, for arbitrarily small $r > 0$, modify the Riemannian metric in γ_1^{4r} without affecting the radial distances (i.e. the distances from γ_1) such that the upper curvature bound does not exceed some universal constant K and such that γ_1^{3r} is isometric to the flat tube $B^{3r} \times \mathbb{R}/[t \mapsto t + L]$, L being the length of γ_1.

Now take S^3 with constant curvature such that the great circles have length L (we start with a big diameter to produce a small diameter). Let γ be a great circle and modify the Riemann metric in γ^{4r} as we did for γ_1^{4r}. Finally, perform the surgery of section 3.

Due to the choice of γ_2 on the ellipsoid we find points $p'_1,\ldots,p'_m \in T\gamma = T\gamma_1$ whose distances *with respect to the metric of* S^3 are $\mathrm{dist}(p'_i,p'_{i+1}) = L/m$, $(i=1,\ldots,m-1)$ but such that in $S^2\times S^1 - \gamma_1{}^r$ they all lie within distance $\leq \varepsilon$ from the north pole of S^2. This is the situation of fig. 2b and we have produced short cuts, so far for the points $p'_1,\ldots,p'_m \in T\gamma$. Choosing the identifying isometry $T\gamma \rightarrow T\gamma_1$ properly (cf section 3), we may prescribe $p'_1 \in T\gamma$ arbitrarily.

To reduce the diameter of S^3 fix some "north pole" $N \in S^3$ and distribute a finite set $\mathcal{R}$ of points on S^3 such that balls of radius ε around these points cover S^3. For any $p \in \mathcal{R}$ there exists a point $P \in S^3$ whose distance from p and from N is an integer multiple of L/m. We obtain a short cut from p to N as follows. Take great circles γ, η such that $\gamma \cap \eta = \emptyset$ with γ ε-close to p and P but $p, P \notin \gamma$ and with η ε-close to P,N and $P,N \notin \eta$. This is easy to do in S^3. Then take positive $r < \varepsilon$ so small that $\gamma^{4r} \cap \eta^{4r} = \emptyset$ and $p,P,N \notin \gamma^{4r} \cup \eta^{4r}$. With the above surgery we find P'_1, $P'_k \in T\gamma$ and P''_1, $P''_\ell \in T\eta$ whose respective distances from p, P, resp. P,N are $\leq \varepsilon$ with short cuts from P'_1 to P'_k and from P''_1 to P''_ℓ The two surgeries together yield a short cut from p via P to N (fig.3).

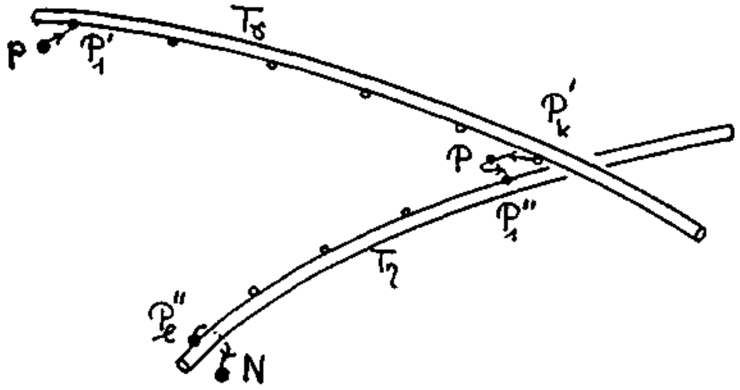

Fig. 3

Finally, we produce these short cuts for all p in the finite set $\mathcal{R}$ simultaneously, using pairwise disjoint great circles and sufficiently small r. On the new manifold, every point which comes from S^3 has distance $\leq 8\varepsilon$ from N, and every point which comes from one of the glued-in $S^2\times S^1 - \gamma_1{}^r$ has distance $\leq \varepsilon$ from some point of S^3. Hence, the diameter is now $\leq 18\varepsilon$, the upper curvature bound is the universal constant K of the lemma below and the topology is still that of S^3.

5. The flattening of tubes. It remains to prove the following technical lemma.

Lemma. *Let γ be a simple closed geodesic on a Riemannian manifold* (M^n, g) *and let* $r>0$ *be arbitrarily small. There exists a new metric* $\underline{g}$ *on* M *with the following properties :*

(i) *γ is again a geodesic and length and holonomy have not changed.*

(ii) *The distances to γ are the same under this new metric.*

(iii) $g = \underline{g}$ *in* $M - \gamma^{4r}$.

(iv) γ^{3r} *is isometric to the Riemannian product* $B^{3r} \times \gamma$.

(v) *In $\gamma^{4r} - \gamma^{3r}$ the upper bound of the absolute curvature has increased at most by a factor* K *where* K *depends only on the dimension.*

Proof. Let $t \to V_1(t),\dots,V_n(t)$, with $V_n(t) = \dot\gamma(t)$ be parallel orthonormal vector fields along the unit speed geodesic γ and define

$$\phi(x_1,\dots,x_n) := \exp_{\gamma(x_n)}(x_1V_1(x_n) + \dots + x_{n-1}V_{n-1}(x_n))$$

for $x^2_1+\dots+x^2_{n-1} \le 4r$, $0 \le x_n = L = \ell(\gamma)$. If r is sufficiently small we may use ϕ^{-1} as a coordinate map for γ^{4r}. The components of g then are

$$g_{ij} = g(d\phi(\frac{\partial}{\partial x_i}), d\phi(\frac{\partial}{\partial x_j})), \quad i,j = 1,\dots,n.$$

Now consider a unit vector X at $(0,\dots,0,x_n) \in \mathbf{R}^n$, perpendicular to $\frac{\partial}{\partial x_n}$. The curve

$$\rho \to c(\rho) = \phi(\rho X)$$

is a geodesic with $\dot c(0) \perp \dot\gamma(x_n)$. The vectorfields

$$Y_i := d\phi_{\rho X}(\rho\frac{\partial}{\partial x_i}), \qquad i = 1,\dots,n-1; \qquad Y_n := d\phi_{\rho X}\frac{\partial}{\partial x_n}$$

are Jacobi fields along c with the initial conditions

$$Y_i(0) = 0, \qquad Y'_i(0) = \frac{\partial}{\partial x_i}, \qquad i = 1,\dots,n-1$$

$$Y_n(0) = \frac{\partial}{\partial x_n}, \qquad Y'_n(0) = 0$$

(' is the covariant derivative). From the Jacobi equation $Y_k'' + R(Y_k,\dot{c})\dot{c} = 0$, $k=1,\dots,n$, where R is the curvature tensor, we find the Taylor series expansion

$$Y_i = \rho A_i - \frac{1}{6}\rho^3 B_i + \dots$$

$$Y_n = A_n - \frac{1}{2}\rho^2 B_n + \dots$$

where A_k, B_k are parallel vectorfields along c, satisfying

$$A_k(0) = V_k, \qquad B_k(0) = R(V_k,X)X, \qquad\qquad k = 1,\dots,n.$$

Looking at $g(Y_i,Y_j)$ we find

$$\frac{\partial}{\partial\rho} g_{ij}(0) = 0, \qquad \left| \frac{\partial^2 g_{ij}}{\partial\rho^2}(0) \right| \leq 2\,\|R\|$$

independently of X. Since $g_{ij}(0,\dots,0,x_n) = \delta_{ij}$ we find

$$g_{ij} = \delta_{ij} + h_{ij}$$

where

$$|h_{ij}| \leq \kappa\rho^2 + o(\rho^2), \qquad \left| \frac{\partial}{\partial x_k} h_{ij} \right| \leq 4n\kappa\rho + 0(\rho^2)$$

$$\left| \frac{\partial^2}{\partial x_k \partial x_\ell} h_{ij} \right| \leq 4\kappa + 0(\rho),$$

$i,j,k,\ell = 1,\dots,n$, where κ is a *positive* upper bound of the norm $\|R\|$ of the curvature tensor along γ. To define $\tilde{g}$ we use a monotone smooth function $\varphi\colon [0,4] \to [0,1]$ such that $\varphi(t) = 0$ for $0 \leq t \leq 3$, $\varphi(t) = 1$ for $3.5 \leq t \leq 4$, $\varphi' \leq 10$, $|\varphi''| \leq 100$ and define

$$\tilde{g}_{ij} = \delta_{ij} + h_{ij}\,\varphi(\rho/r).$$

Here we have to assume that r is sufficiently small such that the above o- and 0-terms are negligible. Points (i) - (iv) of the Lemma are clear. For the curvature bound in (v) we first get from a direct computation

$$|g_{ij} - \delta_{ij}| \leq \text{const } \kappa\rho^2, \qquad \left|\frac{\partial}{\partial x_k} g_{ij}\right| \leq \text{const } \kappa\rho, \qquad \left|\frac{\partial^2}{\partial x_k \partial x_\ell} g_{ij}\right| \leq \text{const } \kappa$$

$$|g^{ij} - \delta_{ij}| \leq \text{const } \kappa\rho^2, \qquad \left|\frac{\partial}{\partial x_k} g^{ij}\right| \leq \text{const } \kappa\rho ,$$

where κ is the upper bound of $\|R\|$ from above, and "const" are various dimension constants.

Thus, we have $|\tilde{R}^m_{ijk}| <$ const κ for the curvature symbols of $\tilde{g}$ in γ^{4r}. This proves the Lemma.

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