Toponogov's Theorem and Applications

by

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These notes have been prepared for a series of lectures given at the College on Differential Geometry at Trieste in the Fall of 1989. The lectures center around Toponogov's triangle comparison theorem, critical point theory and applications. In the short amount of time available not all the aspects can be covered. We focus on those applications which seem to be most important and at the same time most suitable for an exposition. Some basic knowledge in geometry will be assumed. It has been provided by K. Grove in the first series of these lectures. Nevertheless we try to keep the lectures selfcontained and independent as much as possible. For the result about the sum of Betti numbers in section 3.5 a lemma from algebraic topology is needed. A proof for this result has been provided in the appendix.

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Contents

1 Review of notation and some tools

1.1 Covariant derivatives

We consider a complete Riemannian manifold M with tangent bundle TM and Riemannian metric $\langle\ ,\ \rangle$ and corresponding covariant derivative ∇ of Levi Civita, which is the unique torsion free connection for which $\langle\ ,\ \rangle$ is parallel, i.e. for any vector fields X, Y, Z on M we have

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{1}$$

and

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle . \tag{2}$$

The last two equations are equivalent to the Levi Civita equation

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &\quad + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \end{aligned} \tag{3}$$

If $\tilde{M}$ is an arbitrary manifold and $f : \tilde{M} \to M$ a differentiable map, $f_* : T\tilde{M} \to TM$ denotes the differential of f. ∇ naturally extends to a covariant derivative for vector fields along f. For any vector field A on $\tilde{M}$ and any vector field Y along f, i.e. $Y : \tilde{M} \to TM$ satisfies $\pi \circ Y = f$ where $\pi : TM \to M$ denotes the projection, the covariant derivative $\nabla_A Y$ is well defined. Due to the fact that $(\nabla_A Y)_p$ depends only on A_p and the values of Y in a neighbourhood of the point p, this extension is uniquely determined by requiring the chain rule $\nabla_v(X_f) = \nabla_{f_* v} X$ for any tangentvector $v \in T\tilde{M}$ and any vector field X on M.

In a similar way the corresponding covariant derivative for tensor fields carries over to a covariant derivative for tensorfields along a map. As a consequence one obtains for example the Cartan structural equations for the Levi Civita connection:

$$\nabla_A f_* B - \nabla_B f_* A - f_*[A,B] = 0 \tag{4}$$

$$R(f_* A, f_* B)Y = \nabla_A \nabla_B Y - \nabla_B \nabla_A Y - \nabla_{[A,B]} Y, \tag{5}$$

where R is the curvature tensor of ∇, A, B are vector fields on $\tilde{M}$ and Y is a vector field along the map f.

For a curve $c : I \to M$ the parameter vector field on I with respect to the parameter t will be denoted by $\frac{\partial}{\partial t}$ or D_t, $\dot{c}(t) = c_* \frac{\partial}{\partial t}|_t$ is the the tangent vector of c at t. The covariant derivative $\nabla_{D_t} Y$ for a vector field Y along c is abbreviated by Y'. A parallel vector field Y along c is characterized by the linear differential equation $Y' = 0$, a geodesic curve by the non-linear second order equation $\dot{c}' = 0$. For consistency reasons we avoid the often found notation $\nabla_{\dot{c}}\dot{c}$ resp. $\nabla_{\dot{c}} Y$ for the expressions $\nabla_{D_t}\dot{c}$ resp. $\nabla_{D_t} Y$ when Y is a vector field along c. The inconsistency of such notation becomes apparent when c is a singular curve for example a constant curve and Y a non-constant vector field along c. If X is a vector field on M, $\nabla_{\dot{c}} X = \nabla_{D_t} X_c$ (chain rule) is well defined.

The exponential map $\exp : TM \to M$ is determined by the initial value problem for geodesics. If $v \in T_p M$, then $\exp(v) = c(1)$ where c is the geodesic with initial condition $c(0) = p$ and $\dot{c} = v$. The restriction of exp to the tangent space $T_p M$ at p is denoted by $\exp_p$. Notice that for complete manifolds the exponential map is defined on all of TM by the Hopf-Rinow theorem.

For a function $f : M \to \mathbb{R}$ and a vector field X on M, Xf denotes the derivative of f in direction X. The gradient of f is defined via the equation

$$\langle \operatorname{grad} f, X \rangle = Xf \tag{6}$$

and the Hessian $\operatorname{Hess} f$ of f by

$$\operatorname{Hess} f(X) = \nabla_X \operatorname{grad} f . \tag{7}$$

$\operatorname{Hess} f$ is a selfadjoint endomorphism field, i.e. $\langle \nabla_X \operatorname{grad} f, Y \rangle = \langle \nabla_Y \operatorname{grad} f, X \rangle$.

Important functions on a Riemannian manifold are distance functions or local distance functions from some point in M or from a submanifold of M. A local distance function is a function in an open subset U of M considered as a Riemannian submanifold. If $p \in U \subset M$ and $r(q) = dist_M(p,q)$, $r_U(q) = dist_U(q,p)$ then $r_U(q) \geq r(q)$. r_U may be differentiable in points where r fails to be differentiable. A typical example arises as follows: Let $c : [\alpha, \beta] \to M$ be an injective geodesic segment with initial point $p = c(\alpha)$ and without conjugate points. Then there is a neighborhood U of $c(]\alpha, \beta])$ where r_U is differentiable. However r is not differentiable in any point of the cut locus of p. For explicit examples look at geodesics on a cylinder.

On the set of points where a (local) distance function is differentiable it satisfies $\|\operatorname{grad} f\| = 1$. The gradient lines of any function with this property are geodesics parametrized by arc length, since $\left\langle \nabla_{\operatorname{grad} f} \operatorname{grad} f, X \right\rangle = \langle \operatorname{Hess} f \operatorname{grad} f, X \rangle = \langle \operatorname{Hess} f X, \operatorname{grad} f \rangle = \langle \nabla_X \operatorname{grad} f, \operatorname{grad} f \rangle = \frac{1}{2} X \langle \operatorname{grad} f, \operatorname{grad} f \rangle = 0$ for any vector field X on M and hence $\nabla_{\operatorname{grad} f} \operatorname{grad} f = 0$. Therefore the level surfaces of such a function are equidistant. They are referred to as a family of parallel surfaces.

1.2 Jacobi fields

Jacobi fields J along a geodesic arise naturally as variational vector fields in one parameter families of geodesic lines and are characterized by the linear second order differential equation

$$J'' + R(J, \dot{c})\dot{c} = 0. \tag{8}$$

If V is a geodesic variation of c, i.e. $V : I \times (-\varepsilon, \varepsilon) \to M$ is differentiable and $V(t,0) = c(t)$ and $t \mapsto V(t,s)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$, then $J(t) = V_* \frac{\partial}{\partial s}|_{t,0}$ is a Jacobi field along c: