1 Metric spaces

We start with minimalistic introduction to metric geometry. A more comprehencive introduction is given in the book of Burago–Burago–Ivanov [2].

1.1. Definition. *A metric space is a pair* (X, dist) *where* X *is a set and* dist *is a function*

$$\operatorname{dist}\colon X \times X \to [0, \infty)$$

such that

a) $\operatorname{dist}(x, y) = 0$ *if and only if* $x = y$*;*
b) $\operatorname{dist}(x, y) = \operatorname{dist}(y, x)$ *for any* $x, y \in X$*;*
c) $\operatorname{dist}(x, z) \leqslant \operatorname{dist}(x, y) + \operatorname{dist}(y, z)$ *for any* $x, y, z \in X$*.*

The function $\operatorname{dist}\colon X\times X \to [0,\infty)$ is called a *metric*, the value $\operatorname{dist}(x, y) \geqslant 0$ is called the *distance* from x to y and the set X is called *underlying set* of the metric space.

Examples:

- ⋄ Discrete metric. For any set X, the discrete metric on X is defined by

$$\operatorname{dist}(x, y) = \left[\begin{array}{lll} 1 & \text{if} & x \ne y \\ 0 & \text{if} & x = y \end{array}\right.$$

- ⋄ Euclidean space. The set is formed by arrays of n real numbers

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

and the distance function defined as

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) \stackrel{\text{def}}{=\!=} |\boldsymbol{x} - \boldsymbol{y}|,$$

where

$$|\boldsymbol{x}| \stackrel{\text{def}}{=\!=} \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

1.2. Exercise. *Show that Euclidean space is a metric space.*

- ⋄ Manhattan metric on the plane. The set is formed by all pairs $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ and the metric is defined as

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) \stackrel{\text{def}}{=\!=} |\boldsymbol{x} - \boldsymbol{y}|,$$

where

$$|\boldsymbol{x}| \stackrel{\text{def}}{=\!=} |x_1| + |x_2|.$$

- ⋄ Space of functions with sup-norm. Given a set X, consider the set $\mathcal{F}_X$ of all bounded functions $f\colon X \to \mathbb{R}$ with the metric defined as

$$\operatorname{dist}(f, g) \stackrel{\text{def}}{=\!=} |f - g|,$$

where

$$|f| \stackrel{\text{def}}{=\!=} \sup_{x\in X} |f(x)|.$$

◇ Subspaces. Given an arbitrary subset $A \subset X$ of a metric space (X, dist), one can give A the metric defined by restriction of dist to $A \times A \subset X \times X$. In this situation, A is called subspace of (X, dist).

Notation for distance

The distance $\operatorname{dist}(x, y)$ between x and y in a metric space X will be further also denoted as

$$|x - y| = |x - y|_X = \operatorname{dist}_x y = \operatorname{dist}_y x = \operatorname{dist}(x, y).$$

We will write $|x-y|_X$ to emphasize that the points x and y belong to the metric space X and the notation dist_x is used when we need to consider the distance to the point x as a function $\operatorname{dist}_x \colon X \to \mathbb{R}$.

It should be noted that the expression $x - y$ for two points in a metric space makes no sense and $|x - y|$ should be read as *distance from x to y*.

Isometries

1.3. Definition. *Let X and Y be metric spaces.*

a) A map $f \colon X \to Y$ is called distance preserving *if*

$$|f(x) - f(x')|_Y = |x - x'|_X$$

for any $x, x' \in X$.

b) A distance preserving bijection $f \colon X \to Y$ is called an isometry.

c) The spaces X and Y are called isometric *(briefly $X \stackrel{iso}{=\!=} Y$) if there is an isometry $f \colon X \to Y$. (Note that this defines an equivalence relation on the class of metric spaces.)*

Note that a distance preserving map is necessarily injective. An existence of distance preserving map $X \to Y$ is equivalent to existence of subset of Y which is isometric to X.

Kuratowski embedding

1.4. Exercise. *Let X be a metric space and fix an element $x \in X$. Then the map*

$$K_x \colon X \to \mathcal{F}_X$$

defined by

$$K_x(z) = \operatorname{dist}_z - \operatorname{dist}_x$$

is distance preserving; i.e.

$$|K_x(y) - K_x(z)| = |y - z|$$

for any $y, z \in X$.

The map $K_x\colon X \to \mathcal{F}_X$ is called the *Kuratowski embedding with base x.*

If X has bounded diameter, i.e., if there is a constant $D < \infty$, such that $|z - y| \leqslant D$ for any $z, y \in X$, then dist_x is bounded for any $x \in X$ and the map

$$K\colon X \to \mathcal{F}_X$$

given by

$$K(x) = \mathrm{dist}_x$$

is also distance preserving. The latter map is also called the *Kuratowski embedding.*

Calculus in metric spaces

1.5. Definition. *Let X be a metric space. A sequence of points $x_1, x_2, \dots$ in X is called* convergent *if there is $x_\infty \in X$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$, we have*

$$|x - x_n| < \varepsilon.$$

In this case we say that x_n converges to x_∞ and write

$$x_\infty = \lim_{n\to\infty} x_n$$

or $x_n \to x_\infty$ as $n \to \infty$.

1.6. Definition. *Let X and Y be metric spaces. A map $f\colon X \to Y$ is called continuous if for any convergent sequence $x_n \to x_\infty$ in X, the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$ in Y.*

Equivalently, $f\colon X \to Y$ is continuous if for any $x \in X$ and any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x - x'|_X < \delta \;\Rightarrow\; |f(x) - f(x')|_Y < \varepsilon.$$

1.7. Exercise. *Prove that the two definitions of continuity given above are equivalent.*

1.8. Definition. *A subset A of a metric space X is called* closed *if whenever a sequence (x_n) of points from A converges in X, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset X$ is called open *if the complement $X \setminus \Omega$ is a closed set. Equivalently, $\Omega \subset X$ is open if for any $z \in \Omega$ there is $\varepsilon > 0$ such that the ball*

$$B_\varepsilon(z) = \{\, x \in X \mid |x - z| < \varepsilon \,\}$$

is contained in Ω.

1.9. Exercise. *Prove the equivalence of the two definitions of an open set given above.*

1.10. Exercise. *Give an example of a subset of a metric space that is both open and closed. Also, give an example of a subset that is neither open nor closed.*

Note that intersection of arbitrary number of closed set is closed. It follows that for any set Q in a metric space there is minimal closed set which contains Q; this set is called *closure of* Q. The closure of Q can be obtained as the intersection of all closed sets $A \supset Q$ in the metric space or by taking the set of all limit points for all sequences in Q.

Complete spaces

1.11. Definition. *Let X be a metric space. A sequence of points $x_1, x_2, x_3, \dots$ in X is called* Cauchy*, if for every $\varepsilon > 0$, there is an integer N such that*

$$|x_m - x_n| < \varepsilon,$$

for any $m, n > N$.

1.12. Definition. *A metric space X is called* complete *if any Cauchy sequence in X is convergent.*

Examples:

- ⋄ The real line as well as Euclidean space are complete.
- ⋄ The subspace of real line formed by the rational numbers is not complete.

1.13. Exercise. *Let X be a complete metric space and $A \subset X$, then the subspace formed by A is complete if and only if A is* closed.

Completion

For any metric space X, there is a canonical construction of a complete metric space $\bar{X}$, which contains X. The space $\bar{X}$ is called the *completion* of X and it is constructed as a set of equivalence classes of Cauchy sequences in X.

For any two Cauchy sequences $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ in X, we may define their distance as

$$|\boldsymbol{x} - \boldsymbol{y}| = \lim_{n \to \infty} |x_n - y_n|.$$

This limit exists because the real numbers are complete.

This defines only a pseudometric; i.e., it does not satisfy $1.1a$ — two different Cauchy sequences may have the distance 0. But "having distance 0" is an equivalence relation on the set of all Cauchy sequences, and the set of equivalence classes, which we denote $\bar{X}$, is a metric space called the completion of X.

The original space is embedded in this space via the identification of an element x of X with the equivalence class of the sequence with constant value x. This defines an distance preserving map $X \to \bar{X}$, as required.

1.14. Exercise. *Verify all claims made about $\bar{X}$ and prove that the distance function given is well-defined and is indeed a metric on $\bar{X}$.*

1.15. Exercise. *Show that completion of any metric space is isometric to the closure of its image under Kuratowski embedding.*

Compact metric spaces

1.16. Definition. *A metric space X is* compact *if any sequence of points in X contains a convergent subsequence.*

Properties:

- (Heine–Borel theorem.) A subset of Euclidean space is compact if and only if it is both closed and bounded.
- Any closed subset of a compact space is compact.
- Any compact subset of a metric space is closed.
- Any compact metric space is complete.
- The Cartesian product of two compact spaces $X \times Y$ equipped with the metric
$$|(x_0, y_0) - (x_1, y_1)| \stackrel{def}{=\!=} \max\{|x_0 - x_1|, |y_0 - y_1|\}$$
is compact.

1.17. Definition. *A metric space X is called* proper *if any bounded*[1]*, closed set in X is compact.*

1.18. Exercise. *Prove that if $f\colon X \to Y$ is continuous map between metric spaces and X is compact, then the image $f(X)$ is a compact subset of Y.*

1.19. Exercise. *Prove that if $f\colon X \to Y$ is a continuous bijection between metric spaces and X is compact, then the inverse map $f^{-1}: Y \to X$ is continuous.*

1.20. Exercise. *(Extreme Value Theorem) Prove that if X is a compact metric space, then any continuous function*[2] *$f : X \to \mathbb{R}$ attains a global maximum value at some point of X. That is, there exists $x \in X$ such that $f(y) \leq f(x)$ for all $y \in X$.*

[1] A subset $A \subset X$ is called *bounded* if for one (and therefore any) point x there is a constant $D < \infty$ such that $|x - a| \leqslant D$ for any $a \in A$.

[2] Here, it is understood that the metric we are considering on $\mathbb{R}$ is the usual Euclidean metric.

Nets and maximal packing.

1.21. Definition. *Let X be a metric space and let $\varepsilon > 0$. A subset A is called an ε-net if for any point $x \in X$ there is a point $a \in A$ such that $|x - a| \leqslant \varepsilon$.*

1.22. Definition. *Let X be a metric space and let $\varepsilon > 0$. The supremum of all integers n for which there is an array of points $x_1, x_2, \dots, x_n \in X$ with $|x_i - x_j| > \varepsilon$ for all $i \neq j$ is denoted as* $\operatorname{pack}_\varepsilon X$. *($\operatorname{pack}_\varepsilon$ takes integer value or ∞.)*

If $n = \operatorname{pack}_\varepsilon X$ is finite, then an array $x_1, x_2, \dots, x_n \in X$ such that $|x_i - x_j| > \varepsilon$ for all $i \neq j$ is called a maximal ε-packing.

1.23. Exercise. *Any maximal ε-packing is an ε-net.*

1.24. Exercise. *Let $\{a_1, a_2, \dots, a_n\}$ be a finite ε-net in a metric space X. Show that* $\operatorname{pack}_{2\cdot\varepsilon} X \leqslant n$.

1.25. Theorem. *Let X be a complete metric space. Then the following conditions are equivalent:*

a) X is compact;

b) $\operatorname{pack}_\varepsilon X$ is finite for any $\varepsilon > 0$;

c) X is totally bounded*; i.e., for any $\varepsilon > 0$ there is a finite ε-net in X.*

Proof; (a)⇒(c). Assume X has no finite ε-net. Then for any point array $z_1, z_2, \dots, z_{n-1}$ in X there is a $z_n \in X$ such that $|z_i - z_n| > \varepsilon$ for any $i < n$.

Applying the above statement inductively, we can construct an infinite sequence (z_n) in X such that $|z_i - z_j| > \varepsilon$ for all $i \neq j$. Therefore (z_n) has no convergent subsequence, a contradiction.

(c)⇔(b). Follows from exercises 1.24 and 1.23.

(c)⇒(a). Set $\varepsilon_k = \frac{1}{2^k}$. For each k, let $\{z_{1,k}, z_{2,k}, \dots, z_{n_k,k}\}$ be an ε_k-net. Note that for each k, the collection of balls

$$\bar{B}_{\varepsilon_k}(z_{i,k}) = \{\, x \in X \mid |x - z_i| \leqslant \varepsilon_k \,\}$$

cover all of X.

To show X is compact, given an infinite sequence $x_1, x_2, \dots$ in X, we must find a convergent subsequence. We shall apply a diagonal process to choose a subsequence $x_{n_1}, x_{n_2}, \dots$ with the following property: *for each k there is i_k such that $x_{n_m} \in \bar{B}_{\varepsilon_k}(z_{i_k,k})$ for all $m \geqslant k$.*

For the first step, note that since the finite collection of balls $\bar{B}_{\varepsilon_1}(z_{i,1})$ cover X, there must be an index i_1 such that $\bar{B}_{\varepsilon_1}(z_{i_1,1})$ contains infinitely many terms of our sequence (x_n). Let $F_1 = \bar{B}_{\varepsilon_1}(z_{i_1,1})$. Choose an integer n_1 such that $x_{n_1} \in F_1$.

For the second step, since the balls $\bar{B}_{\varepsilon_2}(z_{i,2})$ cover X, in particular they cover F_1. Since F_1 contains infinitely many points of our sequence (x_n), there must be an index i_2 such that $F_1 \cap \bar{B}_{\varepsilon_2}(z_{i_2,2})$ also contains infinitely many of the x_n. Let $F_2 = F_1 \cap \bar{B}_{\varepsilon_2}(z_{i_2,2})$ and choose $n_2 > n_1$ such that $x_{n_2} \in F_2$.

Proceeding inductively at the k-th step, we know that the balls $\bar{B}_{\varepsilon_k}(z_{i,k})$ cover X and hence F_{k-1}, which contains infinitely many terms of our sequence (x_n). So there must exist i_k so that $F_{k-1} \cap \bar{B}_{\varepsilon_k}(z_{i_k,k})$ also contains infinitely many of the x_n. Let $F_k = F_{k-1} \cap \bar{B}_{\varepsilon_k}(z_{i_k,k})$ and choose $n_k > n_{k-1}$ with $x_{n_k} \in F_k$.

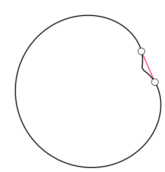
Note that $F_{k+1} \subseteq F_k$, so that for all $m \geq k$, $x_{n_m} \in F_k \subseteq B_{\varepsilon_k}(z_{i_k,k})$ as desired. It follows that (x_{n_k}) is a Cauchy sequence. Since X is complete, (x_{n_k}) converges, and we have proven that X is compact. □

Lyrical digression: Steiner's 4-joint method

1.26. Theorem. *Let F be a plane figure bounded by a curve of length 1 which has maximal area. Then F is a round disc.*

Note that if we would know that such F exists, then as a corollary we would obtain the isoperimetric inequality in the plane:

1.27. Theorem. *Any closed simple curve in the plane of length L bounds area at most $\frac{1}{4\cdot\pi}\cdot L^2$. Moreover, in case of equality, the curve is a circle.*

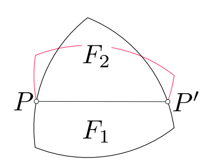


The question of existence of F will be considered later; we use it as a motivation for considering Hausdorff distance.

We give an argument which was found by Jacob Steiner in 1842.

Proof of Theorem 1.26. Let F be a maximal such figure. We shall show that F is a disc. First note that F is convex (i.e. the straight light segment connecting any two points in F is also contained in F); otherwise one could make the perimeter of F smaller and the area larger.

Take any point P in the bounding curve of F (further denoted by ∂F). Consider a point $P' \in \partial F$ so that both arcs ∂F of from P to P' have the same length (which has to be $\frac{1}{2}$).



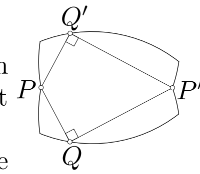
Divide F by the segment $[PP']$ into two parts F_1 and F_2. Without loss of generality we can assume that area $F_1 \geqslant$ area F_2. Let F_1' be the reflection of F_1 in the line (PP'). Set $F' = F_1 \cup F_1'$

Note that

$$\operatorname{area} F' = 2\cdot \operatorname{area} F_1 \geqslant \operatorname{area} F_1 + \operatorname{area} F_2 = \operatorname{area} F$$

and the perimeter of F' is the same as perimeter of F. Hence F' also has maximal possible area and area $F_1 =$ area F_2.

Note that if F is not a round disc then points P and P' can be chosen so that F' is not a round disc. Chose arbitrary point $Q \in \partial F'$ and let Q' be the reflection of Q in (PP').



One can think of F' as quadrilateral $PQP'Q'$ with a lune attached at each side. Think about these lunes as being made of rigid material (say cut it from cardboard) and imagine that at each vertex P, Q, P', Q' we have a joint; so the quadrilateral $PQP'Q'$ can be moved continuously keeping its sides fixed.

Note that if $\angle PQP' \neq \frac{\pi}{2}$ then we can move this construction slightly and increase the area of the obtained figure.

Hence $\angle PQP' = \frac{\pi}{2}$ for any $Q \in \partial F'$. It follows that F' is a disc; hence F is also a disc. □

As it was mentioned above, in order to prove isoperimetric inequality (Theorem 1.27) one only has to show existence of an extremal object (the figure F in Theorem 1.26) and then apply Steiner's argument. One possible approach is to cook up a compact metric space out of plane figures and show that volume and perimeter depend continuously (or semicontinuously) on the figure. Then existence of a maximal F would follow, as a continuous function on a compact metric space attains a maximum.

The first step is to define a metric on the set of figures in the plane; this is done in the next section.

Hausdorff metric

Let X be a metric space. Given a subset $A \subset X$, consider the distance function to A

$$\operatorname{dist}_A : X \to [0, \infty)$$

defined as

$$\operatorname{dist}_A x \overset{def}{=\!=} \inf_{a \in A} \{\operatorname{dist}_a x\}.$$

1.28. Definition. *Let A and B be two compact subsets of a metric space X. Then the Hausdorff distance between A and B is defined as*

$$d_H(A, B) = d_H^X(A, B) \overset{def}{=\!=} \sup_{x \in X} |\operatorname{dist}_A x - \operatorname{dist}_B x|.$$

The notation $d_H^X(A, B)$ will be used only in the case we need to emphasise that A and B are subsets in the metric space X.

1.29. Exercise. *Show that $d_H(A, B) \leqslant R$ if and only if $\operatorname{dist}_A b \leqslant R$ for any $b \in B$ and $\operatorname{dist}_B a \leqslant R$ for any $a \in A$.*

1.30. Exercise. *Show that the set of all nonempty compact subsets of a metric space X equipped with the Hausdorff metric forms a metric space.*

This new metric space will be denoted as $\mathcal{H}_X$.

HWA 1

1.A. Give an example of a metric space X with a distance preserving map $f\colon X \to X$ which is not a bijection.

1.B. Let K be a compact metric space and $f\colon K \to K$ be a non-contracting map; i.e.,

$$|f(x) - f(y)|_K \geqslant |x - y|_K$$

for any $x, y \in K$. Prove that f is an isometry.

1.C. Show that any compact space is isometric to a subset of $\mathcal{F}_{\mathbb{N}}$; i.e., the space of bounded sequences with the metric induced by sup-norm.

1.D. Consider the set $A = \{1, 2, 4, 8\} \subset \mathbb{R}$. Find a 3-point set B in $\mathbb{R}$ such that the Hausdorff distance $d_H(A, B)$ is minimal. Describe all such sets.

1.E. Do all the exercises in the lecture notes. Write down the proof of one of your choice.

2 Hausdorff metric continued

2.1. Blaschke's theorem. *Let X be a metric space. Then the space $\mathcal{H}_X$ is compact if and only if X is compact.*

2.2. Exercise. *Let X be a metric space. Given a subset $A \subset X$ define its diameter as*

$$\operatorname{diam} A \stackrel{def}{=\!=} \sup_{a,b \in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \mathcal{H}_X \to \mathbb{R}$$

is a continuous function.

Proof of Theorem 2.1; "only if" part. Note that the map $\iota\colon X \to \mathcal{H}_X$, defined as $\iota\colon x \mapsto \{x\}$ (i.e., point x mapped to the one-point subset $\{x\}$ of X) is distance preserving. Thus X is isometric to the subset $\iota(X)$ of $\mathcal{H}_X$.

Note that $\iota(X)$ is closed in X; this follows from Exercise 2.2 since for a nonempty subset $A \subset X$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Hence $\iota(X)$ is compact, as it is a closed subset of a compact space. So the "only if" part follows because if two spaces Y and Z are isometric, then Y is compact if and only if Z is compact. □

To prove "if" part we will need the following two lemmas.

2.3. Lemma. *If K_n is a decreasing sequence of nonempty compact sets in a metric space X (that is $K_{n+1} \subset K_n$) then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; i.e., $d_H(K_\infty, K_n) \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact and nonempty. If the assertion were false, then

$$K_n \backslash B_\varepsilon(K_\infty) \neq \varnothing$$

for all n and some fixed $\varepsilon > 0$.[3]

Choose $x_n \in K_n \backslash B_\varepsilon(K_\infty)$. Since K_1 is compact, there is a partial limit[4] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty} x_\infty \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty} x_\infty = 0$, a contradiction. □

2.4. Exercise. *Prove the following statement which was used in the previous proof: If K_n is a decreasing sequence of nonempty compact subsets of a metric*

[3] Here $B_\varepsilon(K_\infty)$ denotes ε-neighborhood of K_∞; i.e.,

$$B_\varepsilon(K_\infty) = \{\, x \in X \mid \operatorname{dist}_{K_\infty} x < \varepsilon \,\}.$$

[4] Partial limit is a limit of a subsequence.

space, then $K_\infty = \bigcap_n K_n$ is nonempty. Give an example that shows that this statement is false if we replace "compact" with "closed".

2.5. Exercise. *A complete metric space Q is compact if for any $\varepsilon > 0$ there is a compact ε-net in Q.*

2.6. Lemma. *If X is a complete metric space then $\mathcal{H}_X$ is complete.*

Proof. Let (K_n) be a Cauchy sequence in $\mathcal{H}_X$. Set Q_n to be closure of $\bigcup_{m\geqslant n} K_m$. Note that Q_n is a decreasing sequence of nonempty compact sets. Indeed, since (K_n) is a Cauchy sequence, given $\varepsilon > 0$ there is $N < \infty$ such that $d_H(K_N, K_i) \leqslant \varepsilon$ for all $i > n$. Hence $Q_{n,N} \stackrel{def}{=\!=} \bigcup_{N\geqslant m\geqslant n} K_m$ forms an ε-net in Q_n. Clearly $Q_{n,N}$ is compact, as it is a finite union of compact sets. Therefore, by applying Exercise 2.5, we get that Q_n is compact.

Hence, by Lemma 2.3, Q_n converges in the Hausdorff metric to $Q_\infty = \bigcap_n Q_n$. Hence, for given $\varepsilon > 0$, there exists N such that

❶ $$B_\varepsilon(Q_\infty) \supset K_n$$

for all $n \geqslant N$. Since (K_n) is a Cauchy sequence, there exists $N' \geqslant N$ such that $B_\varepsilon(K_n) \supset K_m$ for all $m, n \geqslant N'$. It follows that $B_\varepsilon(K_n) \supset Q_m$ for all $m, n \geqslant N'$; in particular

❷ $$B_\varepsilon(K_n) \supset Q_\infty$$

for all $n \geqslant N'$.

The inequalities ❶ and ❷ imply that for each $\varepsilon > 0$ there is N' such that $d_H(Q_\infty, K_n) < \varepsilon$ for all $n \geqslant N'$; i.e. Q_∞ is the Hausdorff limit of (K_n). □

Proof of Theorem 2.1; "if" part. According to Lemma 2.6, $\mathcal{H}_X$ is complete. So to show that $\mathcal{H}_X$ is compact, it only remains to show that $\mathcal{H}_X$ is totally bounded; i.e., given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}_X$.

Choose a finite ε-net A in X. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}_X$. We shall show that $\mathcal{A}$ is an ε-net in $\mathcal{H}_X$.

For each compact set $K \subset X$, consider the subset K' of all points $a \in A$ such that $\operatorname{dist}_K a \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $d_H(K, K') \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}_X$. □

Isoperimetric inequality

Let Q be a figure in the plane which is bounded by a closed curve. Note that if Q is not convex then the *convex hull*[5] of Q has bigger area and smaller perimeter. Therefore it is sufficient to prove the isoperimetric inequality only for convex figures. In other words, to prove Theorem 1.27, it is sufficient to prove the following.

[5] i.e., the minimal convex set which contains the given set

2.7. Theorem. *Any convex plane figure wth area A is bounded by a curve of length at least $\sqrt{4\cdot\pi\cdot A}$. Moreover, in case of equality, the figure is a disc.*

Note that by Steiner's argument (1.26), Theorem 2.7 follows from the following.

2.8. Proposition. *There is a convex figure F in the plane, which maximize the area among all convex figures with perimeter* 1.

2.9. Exercise. *Show that any convex figure with perimeter* 1 *is congruent to a figure in a given 1 by 1 square.*

It follows from that it is sufficient to show existence of F which maximizes area among convex figures with perimeter 1 in a given 1 by 1 square, which will be denoted as $\square$.

Note that $\square$ is compact. According to Theorem 2.1, $\mathcal{H}_\square$ is compact. Let us denote by $\mathcal{C}$ the subset of $\mathcal{H}_\square$ formed by all convex sets.

2.10. Exercise. *Show that $\mathcal{C}$ is a closed subset of $\mathcal{H}_\square$.*

Since $\mathcal{H}_\square$ is compact, the above exercise implies that $\mathcal{C}$ is also compact.

The area and perimeter define two functions on $\mathcal{C}$.

$$\operatorname{area}\colon \mathcal{C} \to \mathbb{R} \quad \text{and} \quad \operatorname{perim}\colon \mathcal{C} \to \mathbb{R}.$$

In order to ensure existence of F it remains to prove the following claim:

2.11. Claim. *Both functions*

$$\operatorname{area}\colon \mathcal{C} \to \mathbb{R} \quad \textit{and} \quad \operatorname{perim}\colon \mathcal{C} \to \mathbb{R}$$

are continuous.

(In other words, for any Hausdorff converging sequence $K_n \to K_\infty$ of convex compact sets in $\square$, we have

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad \textit{and} \quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.)$$

Indeed, since $\operatorname{perim}\colon \mathcal{C} \to \mathbb{R}$ is continuous, we have that the figures in $\mathcal{C}$ with perimeter 1 form a closed set, say $\mathcal{C}_1$. Since $\mathcal{C}$ is compact, so is $\mathcal{C}_1$. Then the restriction of $\operatorname{area}\colon \mathcal{C} \to \mathbb{R}$ to $\mathcal{C}_1$ admits a maximal value by Exercise 1.20; hence Proposition 2.8 follows.

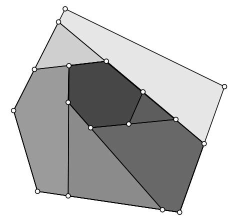
In the proof of Claim 2.11, we will use the following two lemmas.

2.12. Lemma. *Let P and Q be two convex compact figures in the plane. Then*

$$P \subset Q \quad \textit{and} \quad \operatorname{perim} P \leqslant \operatorname{perim} Q.$$

Proof. First note that it is sufficient to prove the lemma only for convex polygons P and Q. Note that if P is obtained from Q by intersecting Q with a half-plane then the inequality $\operatorname{perim} P \leqslant \operatorname{perim} Q$ follows from the triangle inequality.

Now note that there is an increasing sequence of polygons



$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} is intersection of P_i with a half-plane. Therefore

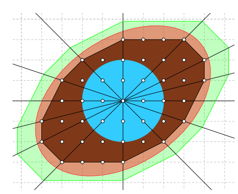
$$\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots$$
$$\dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q$$

and the lemma follows. □

2.13. Lemma. *Let Q be a compact convex plane figure which contains a disc of radius $R > 0$ centered at the origin of the plane. Given $\varepsilon > 0$ there is a convex polygon $P \subset Q$ such that rescaled polygon*

$$(1+\varepsilon)\cdot P = \left\{ (1+\varepsilon)\cdot x \in \mathbb{R}^2 \,\middle|\, x \in P \right\}$$

contains Q.



Proof. Fix small $\delta > 0$ and consider the set Z all points $(\delta\cdot m, \delta\cdot n) \in Q$ such that m and n are integers. Set P be the convex hull of Z. Since Q is convex, we have $P \subset Q$.

On the other hand, it is a straightforward calculation to show that if $\delta < \frac{R}{10}$ then

$$(1 + 10\cdot\tfrac{D}{R}\cdot\delta)\cdot P \supset Q,$$

where D is diameter of Q. I.e., the condition of lemma holds for P for any sufficiently small $\delta > 0$. □

2.14. Exercise. *Perform the "straightforward calculation" in the above proof.*

Proof of Claim 2.11. Assume K_∞ is nondegenerate; i.e. K_∞ contains a disc, say $B_R(z)$. Without loss of generality, we may assume that z is the origin in $\mathbb{R}^2$; so we can apply Lemma 2.13 to K_∞.

Choose small $\varepsilon > 0$ and choose P as in the Lemma 2.13. Note that there is N such that

$$(1 - 2\cdot\varepsilon)\cdot P \subset K_n \subset (1 + 2\cdot\varepsilon)\cdot P$$

for all $n \geqslant N$. It follows that[6]

$$\operatorname{area} K_n \lesssim (1 \pm 2\cdot\varepsilon)^2 \cdot \operatorname{area} P \lesssim (1 \pm 2\cdot\varepsilon)^4 \cdot \operatorname{area} K_\infty.$$

[6]Here

$$a \lesssim (1 \pm \varepsilon)\cdot b$$

means two inequalities:

$$a < (1+\varepsilon)\cdot b \quad \text{and} \quad a > (1-\varepsilon)\cdot b.$$

From Lemma 2.12, we get

$$\operatorname{perim} K_n \lesssim (1 \pm 2{\cdot}\varepsilon){\cdot} \operatorname{perim} P \lesssim (1 \pm 2{\cdot}\varepsilon)^2{\cdot} \operatorname{perim} K_\infty.$$

Hence the we get

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad\text{and}\quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.$$

The following exercise finishes the proof. □

2.15. Exercise. *Prove that*

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad\textit{and}\quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.$$

in case K_∞ is degenerate K_∞; i.e., if K_∞ is either one-point set or a segment. (If ℓ is the length of a segment then its perimeter is $2{\cdot}\ell$.)

Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of isometry classes[7] of compact metric spaces.

Given two metric spaces, the Gromov–Hausdorff distance from the isometry class of X to the isometry class of Y will be denoted as $d_{GH}(X,Y)$; but we will often say (not quite correctly) "$d_{GH}(X,Y)$ is the Gromov–Hausdorff distance from X to Y". (As you will see further, $d_{GH}(X,Y) = 0$ if and only if X is isometric to Y.)

Let us describe the idea behind the definition. First, the Gromov–Hausdorff distance between subspaces in the same metric space has to be no greater than the Hausdorff distance between them. In other words, if two subspaces of the same space are close to each other in the sense of Hausdorff distance in the ambient space, they must be close to each other as abstract metric spaces. Second, one definitely wants the distance between isometric spaces to be zero. The Gromov–Hausdorff distance is in fact the maximum distance satisfying these two requirements.

2.16. Definition. *Let X and Y be compact metric spaces. The Gromov–Hausdorff distance between them is defined by the following relation. For an $r > 0$, $d_{GH}(X,Y) < r$ if and only if there exist a metric space Z and subspaces X' and Y' of it which are isometric to X and Y respectively and such that $d_H(X,Y) < r$. Here d_H denotes the Hausdorff distance between subsets of Z.*

In other words, $d_{GH}(X,Y)$ is the infimum of all $r > 0$ for which the above Z, X' and Y' exist.

[7] Being isometric is an equivalence relation on the collection of metric spaces, and an isometry class is an equivalence class with respect to this equivalence relation.

We say that a sequence X_n of compact metric spaces converges in the sense of Gromov–Hausdorff *to a compact metric space X_∞ if $d_{GH}(X_n, X_\infty) \to 0$ as $n \to \infty$.*

2.17. Theorem. *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space.*

This metric space will be denoted further as $\mathcal{M}$; named for "metric space".

Before proving this theorem, we will discuss the Gromov–Hausdorff distance.

Definition 2.16 deals with a huge class of metric spaces, namely, all metric spaces Z that contain subspaces isometric to X and Y. It is possible to reduce this class to metrics on the disjoint unions of X and Y. More precisely,

2.18. Proposition. *The Gromov–Hausdorff distance between two compact metric spaces X and Y is the infimum of $r > 0$ such that there exists a metric $|{*}-{*}|_W$ on the disjoint union $W = X \sqcup Y$ such that the restrictions of $|{*}-{*}|_W$ to X and Y coincide with $|{*}-{*}|_X$ and $|{*}-{*}|_Y$ and $d_H(X, Y) < r$ in the space W.*

Proof. Identify $X \sqcup Y$ with $X' \cup Y' \subset Z$ (the notation is from Definition 2.16).

More formally, fix isometries $f\colon X \to X'$ and $g\colon Y \to Y'$, then define the distance between $x \in X$ and $y \in Y$ by $|x - y|_W = |f(x) - g(y)|_Z + \varepsilon$ for small enough $\varepsilon > 0$.[8] This yields a metric on $W = X \sqcup Y$ for which $d_H(X, Y) < r$. □

[8]We add ε to ensure that $d(x, y) > 0$ for any $x \in X$ and $y \in Y$; so $|x - y|_W$ is indeed a metric.

HWA 2

2.A. Let K be convex compact set in the plane. Given a point x it the plane consider a point $\bar{x} \in K$ which minimize the distance $|x - \bar{x}|$. Show that $\bar{x}$ is uniquely defined and the map $x \mapsto \bar{x}$ is distance non-increasing; i.e.,

$$|\bar{x} - \bar{y}| \leqslant |x - y|$$

for any points x and y in the plane.

2.B. Show that there is a plane figure F of least area which is capable of covering any plane figure of unit diameter.

Try to guess what is F.

2.C. Let $X = \{x, y, z\}$ be a three point subset of Euclidean plane with distances

$$|x - y| = |y - z| = |z - x| = 1.$$

(i) Find the minimal Hausdorff distance from X to a one-point subset of the plane.
(ii) Find the Gromov–Hausdorff distance from X to the one-point metric space.

2.D. Let X and Y be a compact metric spaces which have isometric ε-nets. Show that

$$d_{GH}(X, Y) \leqslant 2{\cdot}\varepsilon.$$

2.E. Do all the exercises in the lecture notes. Write down the proof of one of your choice.

3 Gromov–Hausdorff metric continued

3.1. Exercise. *Let P be a one-point metric space. Prove that*

$$d_{GH}(X,P) = \frac{\operatorname{diam} X}{2}$$

for any compact metric space X.

3.2. Exercise. *Let X and Y be two compact metric spaces. Prove that*

$$|\operatorname{diam} X - \operatorname{diam} Y| \leqslant 2\cdot d_{GH}(X,Y).$$

In other words, diam *is a* 2*-Lipschitz function on $\mathcal{M}$.*

3.3. Exercise. *Assume X_n be a sequence of compact metric spaces which converges to a compact metric space X_∞ in the sense of Gromov–Hausdorff. Show that for any $\varepsilon > 0$*

$$\operatorname{pack}_\varepsilon X_n \geqslant \operatorname{pack}_\varepsilon X_\infty$$

for all large enough n. In particular, $\operatorname{pack}_\varepsilon$ *is a lower semicontinuous function on $\mathcal{M}$.*

A definition with fixed Z

3.4. Proposition. *In the Definition 2.16, one can fix the space Z once for all, by taking $Z = \mathcal{F}_\mathbb{N}$[9]. That is,*

$$d_{GH}(X,Y) = \inf d_H^{\mathcal{F}_\mathbb{N}}(X,Y)$$

where the infimum is taken over all pairs of distance preserving maps of X and Y into $\mathcal{F}_\mathbb{N}$. Again, here we are identifying X and Y with their images in $\mathcal{F}_\mathbb{N}$ under the distance preserving maps, and we are using the Hausdorff distance between compact subsets of $\mathcal{F}_\mathbb{N}$.

Proof. It is clear that $d_{GH}(X,Y) \leq \inf d_H^{\mathcal{F}_\mathbb{N}}(X,Y)$.Let W be an arbitrary metric space with the underlying set $X \sqcup Y$ as in the proof of Proposition 2.18. Note W is compact since it is union of two compact subsets $X, Y \subset W$. According to Problem 1.C, W admits a distance preserving map to $Z = \mathcal{F}_\mathbb{N}$. So $\inf d_H^{\mathcal{F}_\mathbb{N}}(X,Y) \leq d_H^W(X,Y)$, and taking the infimum over all such W gives $\inf d_H^{\mathcal{F}_\mathbb{N}}(X,Y) \leq d_{GH}(X,Y)$. □

3.5. Exercise. *Let X, Y be two compact sets in the Euclidean plane $\mathbb{R}^2$. Show that X is isometric to Y if and only if there is an isometry $\iota\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends X to Y.*

3.6. Exercise. *Find two isometric subsets X, Y of $\mathcal{F}_\mathbb{N}$ such that there is no isometry $\iota\colon \mathcal{F}_\mathbb{N} \to \mathcal{F}_\mathbb{N}$ which sends X to Y.*

Hint: First do the same for two two-point sets in the plane equiped with Manhattan metric; it is defined on page 1.

[9] i.e., the space of bounded infinite sequences.

Gromov–Hausdorff convegence

In order to determine that a given sequence of metric spaces (X_n) converges in the Gromov–Hausdorff sense to X_∞, it is sufficient to estimate distances $d_{GH}(X_n, X_\infty)$ and check if $d_{GH}(X_n, X_\infty) \to 0$. This problem turns to be simpler than finding Gromov–Hausdorff distance between a particular pair of spaces. Propositions 3.7 and 3.14 describe ways to do this.

Via partial order

Given two metric spaces X and Y, we will write $X \preccurlyeq Y$ if there is a noncontracting map $f\colon X \to Y$; i.e., if

$$|x - x'|_X \leqslant |f(x) - f(x')|_Y$$

for any $x, x' \in X$.

Further, given $\varepsilon > 0$, we will write $X \preccurlyeq Y + \varepsilon$ if there is a map $f\colon X \to Y$ such that

$$|x - x'|_X \leqslant |f(x) - f(x')|_Y + \varepsilon$$

for any $x, x' \in X$.

Define

$$d'_{GH}(X, Y) = \inf \{\, \varepsilon \mid X \preccurlyeq Y + \varepsilon \ \text{ and } \ Y \preccurlyeq X + \varepsilon \,\}$$

It turns out that d'_{GH} is a different metric on the set of isometry classes of compact metric spaces; i.e., $d'_{GH} \neq d_{GH}$. However, these two metrics define the same notion of convergence on $\mathcal{M}$. More precicely:

3.7. Proposition. *For any sequence of compact metric spaces (X_n) and a compact metric space X_∞, we have*

$$d_{GH}(X_n, X_\infty) \to 0 \quad \Leftrightarrow \quad d'_{GH}(X_n, X_\infty) \to 0$$

as $n \to \infty$.

We will not give a proof of this proposition. Likely, we will not use it further in the lectures, but it might help you to build intuition for Gromov–Hausdorff convergence. If you want to prove it yourself look in the proof of Theorem 2.17 and try to modify it using ideas from the proof of Problem 1.B.

Via amost isometries

3.8. Definition. *Let X and Y be metric spaces and $\varepsilon > 0$. A map*[10] *$f\colon X \to Y$ is called an ε-isometry if $f(X)$ is an ε-net in Y and if*[11]

$$|f(x) - f(x')|_Y \lesseqgtr |x - x'|_X \pm \varepsilon$$

[10] possibly noncontinuous

[11] We write $a \lesseqgtr b \pm \varepsilon$ for two inequalities: $a \leqslant a + \varepsilon$ and $a \geqslant b - \varepsilon$

for any $x, x' \in X$.

3.9. Exercise. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two ε-isometries. Show that $g \circ f\colon X \to Z$ is a $(3{\cdot}\varepsilon)$-isometry.*

3.10. Exercise. *Assume $f\colon X \to Y$ is an ε-isometry. Show that there is a $(3{\cdot}\varepsilon)$-isometry $g\colon Y \to X$.*

3.11. Exercise. *Assume $d_{GH}(X, Y) < \varepsilon$, show that there is a $(2{\cdot}\varepsilon)$-isometry $f\colon X \to Y$.*

3.12. Proposition. *Let X and Y be metric spaces and let $f\colon X \to Y$ be an ε-isometry. Then $d_{GH}(X, Y) \leqslant 2{\cdot}\varepsilon$.*

Proof. First do the following exercise.

3.13. Exercise. *Show that the following defines a metric on $W = X \sqcup Y$*

1. *For any $x, x' \in X$*
$$|x - x'|_W = |x - x'|_X;$$
2. *For any $y, y' \in Y$,*
$$|y - y'|_W = |y - y'|_Y$$
3. *For any $x \in X$ and $y \in Y$,*
$$|x - y|_W = \varepsilon + \inf_{x' \in X} \{|x - x'|_X + |f(x') - y|_Y\}.$$

Since $f(X)$ is an ε-net in Y, for any $y \in Y$ there is $x \in X$ such that $|f(x) - y|_Y \leqslant \varepsilon$; therefore $|x - y|_W \leqslant 2{\cdot}\varepsilon$. On the other hand for any $x \in X$, we have $|x - y|_W \leqslant \varepsilon$ for $y = f(x) \in Y$.

It follows that the Hausdorff distance $d_H^W(X, Y)$ between X and Y in W is at most $2{\cdot}\varepsilon$. Therefore

$$d_{GH}(X, Y) \leqslant d_H^W(X, Y) \leqslant 2{\cdot}\varepsilon.$$ □

3.14. Proposition. *A sequence of compact metric spaces (X_n) converges to X_∞ in the sense of Gromov–Hausdorff if and only if there is a sequence $\varepsilon_n \to 0^+$ and an ε_n-isometry $f_n\colon X_n \to X_\infty$ for each n.*

Proof. Follows from Propsition 3.12 and Exercise 3.11 □

Proof of Theorem 2.17

The conditions in Definition 1.1 can be reformulated the following way.

Let X, Y and Z be arbitrary compact metric spaces. Then we need to check

(i) $d_{GH}(X,Y) \geqslant 0$;

(ii) $d_{GH}(X,Y) = 0$ if and only if X is isometric to Y;

(iii) $d_{GH}(X,Y) = d_{GH}(Y,X)$;

(iv) $d_{GH}(X,Y) + d_{GH}(Y,Z) \geqslant d_{GH}(X,Z)$.

Note that *(i)*, *(iii)* and "if"-part of *(ii)* follow directly from the definition of Gromov–Hausdorff metric (2.16).

Proof of (iv). Choose arbitrary $a, b \in \mathbb{R}$ such that

$$a > d_{GH}(X,Y) \quad \text{and} \quad b > d_{GH}(Y,Z).$$

Choose two metrics on $U = X \sqcup Y$ and $V = Y \sqcup Z$ so that $d_H^U(X,Y) < a$ and $d_H^V(Y,Z) < b$ and the inclusions $X \to U$, $Y \to U$, $Y \to V$ and $Z \to V$ are distance preserving.

Consider the metric on $W = X \sqcup Z$ so that inclusions $X \to W$ and $Z \to W$ are distance preserving and

$$|x - z|_W = \inf_{y \in Y} \{|x - y|_U + |y - z|_V\}.$$

Note that $|{*} - {*}|_W$ is a metric and

$$d_H^W(X,Z) < a + b.$$

The last inequality holds for any $a > d_{GH}(X,Y)$ and $b > d_{GH}(Y,Z)$; hence *(iv)*. □

Proof of "only if"-part of (ii). According to Exercise 3.11, for any sequence of $\varepsilon_n \to 0^+$ there is a sequence of ε_n-isometries $f_n \colon X \to Y$.

Fix a countable dense set[12] $S \subset X$ and use a diagonal procedure if necessary, to pass to a subsequence of (f_n) such that for every $x \in S$ the sequence $(f_n(x))$ converges in Y. Consider the pointwise limit map $f_\infty \colon S \to Y$ defined by

$$f_\infty(x) = \lim_{n \to \infty} f_n(x)$$

for every $x \in S$. Since

$$|f_n(x) - f_n(x')|_Y \lesseqgtr |x - x'|_X \pm \varepsilon_n,$$

we have

$$|f_\infty(x) - f_\infty(x')|_Y = \lim_{n \to \infty} |f_n(x) - f_n(x')|_Y = |x - x'|_X$$

for all $x, x' \in S$; i.e., $f_\infty \colon S \to Y$ is a distance-preserving map. Then f_∞ can be extended to a distance-preserving map from all of X to Y by setting

[12] Exercise: Show that there is one.

$f_\infty(x) = \lim_{n\to\infty} f_\infty(x_n)$ for some (and therefore any) sequence of points x_n in S which converges to x in X.[13]

This way we obtain a distance preserving map $f_\infty\colon X \to Y$. It remains to show that f_∞ is surjective; i.e. $f_\infty(X) = Y$.

Note that in the same way we can obtain a distance preserving map $g_\infty\colon Y \to X$. If f_∞ is not surjective, then neither is $f_\infty \circ g_\infty\colon Y \to Y$. So $f_\infty \circ g_\infty$ is a distance preserving map from a compact space to itself which is not an isometry, but this contradicts Problem 1.B. □

[13]Note that if $x_n \to x$ then (x_n) is Cauchy. Since f_∞ is distance preserving, $y_n = f_\infty(x_n)$ is also a Cauchy sequence in Y; therefore it converges.

HWA 3; due Fri, Sep 16

3.A.[14] Let $\mathcal{Q}$ be a closed subset of $\mathcal{M}$. Show that $\mathcal{Q}$ is compact if and only if there is a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \dots$ such that $\varepsilon_n \to 0$ and

$$\text{❶} \qquad \operatorname{pack}_{\varepsilon_n} X \leqslant n$$

for any space X in $\mathcal{Q}$.

Is it still true if the inequality holds only for $n \geqslant 2$?

3.B. Define the *radius* of a metric space X as

$$\operatorname{rad} X = \inf_x \sup_y \{|x - y|_X\}.$$

Equivalently,

$$\operatorname{rad} X = \inf \{\, R > 0 \mid \text{there is } x \in X \text{ such that } B_R(x) \supset X \,\}.$$

(i) Show that for any compact metric space X we have

$$\tfrac{1}{2} \cdot \operatorname{diam} X \leqslant \operatorname{rad} X \leqslant \operatorname{diam} X.$$

(ii) Give an example of a compact metric space X such that

$$\operatorname{rad} X = \operatorname{diam} X.$$

(iii) Show that for any compact metric spaces X, Y we have

$$|\operatorname{rad} X - \operatorname{rad} Y| \leqslant 2 \cdot d_{GH}(X, Y).$$

The next problem covers ideas in the solution for Problem 2.B (this is existance of a solution for the Lebesgue Minimal Problem. Make sure you can do the first three parts and that you can reconstruct the solution of Problem 2.B using all four parts. Write down a solution of one part of your choice.

3.C. Recall that a figure of constant width a is a compact convex set in the plane such that the length of the orthogonal projection to any line is equal to a.

(i) Show that any set K with diameter 1 is a subset of a figure constant width 1.

(ii) Show that the area of any figure of constant width 1 is at least 1/100.

(iii) Let F be a compact set in the plane which can be presented as a union of figures of constant width 1. Assume $\operatorname{diam} F > 10000$. Show that either F can be presented as a union of two disjoint compact sets or $\operatorname{area} F > 1$.

---

[14] Hint: Note that one can assume that $\mathcal{Q}$ contains all compact spaces which satisfy ❶ for all n. Note that $\operatorname{diam} X \leqslant \varepsilon_1$. Show that the set $\mathcal{W}_n$ of (isometry classes of) all k-point metric spaces with $k \leqslant n$ and diameter $\leqslant \varepsilon_1$ is compact in $\mathcal{M}$. Show that $\mathcal{W}_n \cap \mathcal{Q}$ is an ε_n-net in $\mathcal{Q}$. Use Exercise 2.5.

(iv) Let $\mathcal{Q}$ be the set of all compact sets in the plane, which can be presented as a union of figures of constant width 1. Try to prove that area is a continuous function on $\mathcal{Q}$ (with respect to the Hausdorff metric).

3.D. Read the subsection about almost isometries (page 18). Do all the exercises in the lecture notes. Write down the proof of one of your choice.

Problem 3.A

3.15. Lemma. *$\mathcal{M}$ is complete.*

Proof. Let (X_n) be a Cauchy sequence in $\mathcal{M}$. Passing to a subsequence if necessary, we can assume that there is an $\frac{1}{2^n}$-isometry $\iota_n\colon X_n \to X_{n+1}$ for each n. Let us call a sequence of points $x_n \in X_n$ "nice" if

$$|\iota_n(x_n) - x_{n+1}|_{X_{n+1}} \leqslant \tfrac{3}{2^n}$$

for each n.

Note that if $\boldsymbol{x} = (x_1, x_2, \dots)$ and $\boldsymbol{y} = (y_1, y_2, \dots)$ are nice then

$$\ell_n = |x_n - y_n|_{X_n}$$

is a Cauchy sequence of real numbers. Therefore the following limit is well defined:

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) \overset{def}{=\!=} \lim_{n\to\infty} |x_n - y_n|_{X_n}.$$

Consider the equivalence relation $\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$ on the set of nice sequences. Then dist induces a metric on the set of $\sim$-equivalence classes of nice sequences. The obtained metric space will be denoted as X_∞.

Clearly for each $x_n \in X_n$ there is a nice sequence with n-th element x_n. Choosing one such a sequence for each $x_n \in X_n$ defines a map $X_n \to X_\infty$ which is $\frac{10}{2^n}$-isometry. I.e., X_n converges to X_∞ in the sense of Gromov–Hausdorff. □

Proof of 3.A; "only if" part. First note that any compact set has bounded diameter. Further $\operatorname{pack}_\varepsilon X \leqslant 1$ if and only if $\operatorname{diam} X \leqslant \varepsilon$, Therefore there is finite value ε_1 such that $\operatorname{pack}_{\varepsilon_1} X \leqslant 1$ for any $X \in \mathcal{Q}$.

If there is no sequence $\varepsilon_n \to 0$ as described in the problem, then for a fixed fixed $\delta > 0$ there is a sequence of spaces $X_n \in \mathcal{Q}$ such that

$$\operatorname{pack}_\delta X_n \to \infty \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{Q}$ is compact, this sequence has a partial limit say $X_\infty \in \mathcal{Q}$. It is easy to see that $\operatorname{pack}_{\delta/10} X_\infty = \infty$; the later contradicts Theorem 1.25.

"If" part. Let us fix the sequence $\varepsilon_n \to 0$ as in the problem and consider the set $\hat{\mathcal{Q}}$ of all (isometry classes of all) metric spaces X such that $\operatorname{pack}_{\varepsilon_n} X \leqslant n$ for any n. According to Exercise 3.3, $\hat{\mathcal{Q}}$ is closed in $\mathcal{M}$. Clearly $\mathcal{Q} \subset \hat{\mathcal{Q}}$. Therefore it is sufficient to prove that $\hat{\mathcal{Q}}$ is compact.

Given positive integer n consider set of all metric spaces $\mathcal{W}_n$ with number of points at most n and diameter $\leqslant \varepsilon_1$. Note that $\mathcal{W}_n$ is compact for each n. Further a maximal ε_n-packing of any $X \in \hat{\mathcal{Q}}$ forms a subspace from $\mathcal{W}_n \cap \hat{\mathcal{Q}}$. Therefor $\mathcal{W}_n$ forms an ε_n-net in $\hat{\mathcal{Q}}$. Exercise 2.5 implies that $\hat{\mathcal{Q}}$ is compact. $\square$

4 Length of curves

4.1. Definition. *A* curve *is a continuous mapping* $\alpha\colon \mathbb{I} \to X$*, where* $\mathbb{I}$ *is a real interval and* X *is a metric space.*

If $\mathbb{I} = [a, b]$ *and*

$$\alpha(a) = p, \quad \alpha(b) = q,$$

we say that α *is a* curve from p to q.

4.2. Definition. *Let* $\alpha\colon \mathbb{I} \to X$ *be a curve. Define* length *of* α *as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.\end{aligned}$$

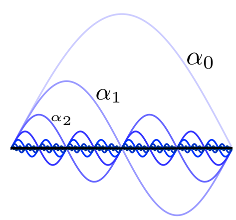
where the supremum is taken over all k *and all sequences* $t_0 < t_1 < \dots < t_k$ *in* $\mathbb{I}$.

Note that the length might be infinite. A curve is called *rectifiable* if its length is finite.

4.3. Semicontinuity of length. *Length is a lower semi-continuous functional on the space of curves* $\alpha\colon \mathbb{I} \to X$ *with respect to point-wise convergence.*

In other words: assume that a sequence of curves $\alpha_n\colon \mathbb{I} \to X$ *converges point-wise to a curve* $\alpha_\infty\colon \mathbb{I} \to X$*; i.e., for any fixed* $t \in \mathbb{I}$*, we have* $\alpha_n(t) \to \alpha_\infty(t)$ *as* $n \to \infty$*. Then*

➊ $$\liminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty.$$



Note that the inequality ➊ might be strict. For example, take a sequence of plane curves $\alpha_n\colon [0, 1] \to \mathbb{R}^2$

$$\alpha_n(t) = (t, \tfrac{1}{2^n} \cdot \sin(2^n \cdot \pi \cdot t)).$$

Clearly α_n converges (uniformly) to $\alpha_\infty(t) = (t, 0)$ and it is easy to see that

$$\operatorname{length}\alpha_n = \mathrm{Const} > 1 = \operatorname{length}\alpha_\infty.$$

Proof. Fix $\varepsilon > 0$ and choose a sequence $t_0 < t_1 < \dots < t_k$ in $\mathbb{I}$ such that

$$\begin{aligned}\operatorname{length}\alpha_\infty - (|\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \dots \\ \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|) < \varepsilon\end{aligned}$$

Set

$$\begin{aligned}\Sigma_n &\stackrel{def}{=\!=} |\alpha_n(t_0) - \alpha_n(t_1)| + |\alpha_n(t_1) - \alpha_n(t_2)| + \dots \\ &\qquad \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|. \\ \Sigma_\infty &\stackrel{def}{=\!=} |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \dots \\ &\qquad \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length} \alpha_n$ for each n. Hence

$$\liminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❶. □

Intrinsic spaces

4.4. Definition. *A metric space X is called* intrinsic *if for any two points $x, y \in X$ and any $\varepsilon > 0$, there is an ε-midpoint; i.e., a point $z \in X$ such that*

$$|x - z|, |y - z| < \tfrac{1}{2}{\cdot}|x - y| + \varepsilon.$$

Examples. The Real line as well as higher dimensional Euclidean spaces are intrinsic. A discrete space (with at least two points) is not intrinsic. Also, a circle in the plane forms a subspace of an intrinsic space which is not intrinsic.

4.5. Exercise. *Show that the set of (isometry classes of) intrinsic spaces in $\mathcal{M}$ is closed.*

4.6. Proposition. *Let X be a complete intrinsic space. Then for any pair of points $x, y \in X$ we have*

$$|x - y| = \inf\{\operatorname{length} \alpha\}$$

where the infimum is taken over all curves α from x to y.

Proof. It is sufficient to construct a curve $\alpha\colon [0,1] \to X$ from x to y such that the length of α is arbitrary close to $|x - y|$, say

❷ $$\operatorname{length} \alpha \leqslant (1 + \varepsilon_0){\cdot}|x - y|$$

for a given $\varepsilon_0 > 0$.

Let us choose a sequence of small positive numbers ε_n which converges to zero "very fast"; i.e., such that inequality ❹ holds. Set

- ⋄ $\alpha(0) = x$ and $\alpha(1) = y$.

Further set

- ⋄ $\alpha(\frac{1}{2})$ to be an $(\varepsilon_1{\cdot}|\alpha(0) - \alpha(1)|)$-midpoint of $\alpha(0)$ and $\alpha(1)$.

Further set

- ⋄ $\alpha(\frac{1}{4})$ to be an $(\varepsilon_2{\cdot}|\alpha(0) - \alpha(\frac{1}{2})|)$-midpoint of $\alpha(0)$ and $\alpha(\frac{1}{2})$
- ⋄ $\alpha(\frac{3}{4})$ to be an $(\varepsilon_2{\cdot}|\alpha(\frac{1}{2}) - \alpha(1)|)$-midpoint of $\alpha(\frac{1}{2})$ and $\alpha(1)$

Continuing this process define α at all rational numbers of the form $\frac{m}{2^n}$ in $[0,1]$.

By construction, we get

❸ $$|\alpha(t_0) - \alpha(t_1)| \leqslant (1 + \varepsilon_0){\cdot}|x - y|{\cdot}|t_1 - t_0|,$$

where t_0 and t_1 are arbitrary numbers in the domain of definition of α and

❹ $$1 + \varepsilon_0 \geqslant (1 + \varepsilon_1)\cdot(1 + \varepsilon_2)\cdot\ldots.$$

In particular, α is uniformly continuous on the domain of definition.

Since X is complete we can extend α to a continuous map on whole interval $[0, 1]$. Note that inequality ❸ still holds for any $t_0, t_1 \in [0, 1]$. Further, from the definition of length (4.2) and ❸ we get ❷. □

Geodesics

4.7. Definition. *A curve $\alpha\colon \mathbb{I} \to X$ is called a* geodesic[15] *if it is a distance preserving map; i.e., if*

$$|\alpha(t_0) - \alpha(t_1)|_X = |t_0 - t_1|$$

for any $t_0, t_1 \in \mathbb{I}$.

The metric space X is called geodesic *if any two points in X can be joined by a geodesic.*

4.8. Exercise. *Show that any proper intrinsic space is geodesic.*

4.9. Definition. *A metric space X is called* locally compact *if for any $x \in X$ there is an $\varepsilon > 0$ such that the closed ball*

$$\bar{B}_\varepsilon(x) = \{\, y \in X \mid |x - y|_X \leqslant \varepsilon \,\}$$

is compact.

Note that any proper metric space is locally compact (see definition 1.17 on page 5). The converse does not hold in general. For example, any infinite set equipped with the discrete metric is locally compact, but not proper.

4.10. Exercise. *Give an example of metric space which is locally compact, but not complete.*

4.11. Hopf–Rinow theorem. *Any complete, locally compact, intrinsic space is proper.*

Proof. Let X be a complete, locally compact, intrinsic space. Given $x \in X$, denote by $\varrho(x)$ the supremum of all $R > 0$ such that the closed ball $\bar{B}_R(x)$ is compact. Since X is locally compact

❺ $$\varrho(x) > 0 \quad \text{for any} \quad x \in X.$$

[15] formally our "geodesic" should be called "unit-speed minimizing geodesic", and the term "geodesic" is reserved for curves which *locally* satisfy the identity

$$|\alpha(t_0) - \alpha(t_1)|_X = \mathrm{Const}\cdot|t_0 - t_1|$$

for some $\mathrm{Const} \geqslant 0$.

It is sufficient to show that $\varrho(x) = \infty$ for some (and therefore any) point $x \in X$.

Assume contrary; i.e. $\varrho(x) < \infty$. Let us show that the closed ball $W = \bar{B}_{\varrho(x)}(x)$ is compact. To prove this claim, notice that $\bar{B}_{\varrho(x)}(x)$ forms a complete subspace and for any $\varepsilon > 0$, the set $\bar{B}_{\varrho(x)-\varepsilon}(x)$ is a compact ε-net in $\bar{B}_{\varrho(x)}(x)$.[16]

Note that if $\varrho(x)+|x-y| < \varrho(y)$, then $\bar{B}_{\varrho(x)+\varepsilon}(x)$ is a closed subset of $\bar{B}_{\varrho(y)}(y)$ for some $\varepsilon > 0$. In this case compactness of $\bar{B}_{\varrho(y)}(y)$ implies compactness of $\bar{B}_{\varrho(x)+\varepsilon}(x)$, a contradiction. Applying the same observation again switching x and y, we get

$$|\varrho(x) - \varrho(y)| \leqslant |x - y|_X;$$

in particular ϱ is a continuous function. Set $\varepsilon = \min_{y \in W}\{\varrho(y)\}$; the minimum is defined since W is compact. From ❺, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in W, The union U of the closed balls $\bar{B}_\varepsilon(a_i)$ is compact. Clearly $\bar{B}_{\varrho(x)+\frac{\varepsilon}{10}}(x) \subset U$. Therefore $\bar{B}_{\varrho(x)+\frac{\varepsilon}{10}}(x)$ is compact or $\varrho(x) > \varrho(x)$, a contradiction. □

Lebesgue's number

In the proof of Hopf–Rinow theorem (4.11), we implicitly used one important characterization of compact metric spaces.

Let X be a metric space, a collection of open subsets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is called *open cover* of X if

$$X = \bigcup_{\alpha \in A} U_\alpha.$$

4.12. Lebesgue's number lemma. *Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of a compact metric space X. Then there is an $\varepsilon > 0$ (it is called a* Lebesgue number *of the cover) such that for any $x \in X$ the ball $B_\varepsilon(x) \subset U_\alpha$ for some $\alpha \in \mathcal{A}$.*

Proof. Given $x \in X$, denote by $\varrho(x)$ the maximal value $R > 0$ such that $B_R(x) \subset U_\alpha$ for some $\alpha \in \mathcal{A}$. Clearly $\varrho(x) > 0$ for any $x \in X$.

Without loss of generality, we may assume $\varrho(x) < \infty$ for one (and therefore any) $x \in X$. Otherwise the conclusion of the lemma holds for arbitrary $\varepsilon > 0$.

The same argument as in the proof of the Hopf–Rinow theorem (4.11) shows that ϱ is continuous. Then the conclusion of the lemma holds for

$$\varepsilon = \tfrac{1}{2} \cdot \min_{x \in X}\{\varrho(x)\}.$$

□

As a corollary of Lebesgue's number lemma, we obtain an alternative definition of compact metric space using open coverings. This definition is the standard definition of compact spaces. Since it use only the notion of open sets, it can be generalized to so called topological spaces.

4.13. Theorem. *A metric space X is compact if and only if any open cover of X contains a finite subcover.*

[16] To prove this statement one has to use that space is intrinsic.

I.e., for any open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ *of* X *there is a finite set* $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathcal{A}$ *such that*

$$X = \bigcup_{i=1}^{n} U_\alpha.$$

Proof; "if"-part. First let us show that X is complete. Assume contrary; i.e., there is a Cauchy sequence (x_n) which is not converging. Set $r_n = \sup_{m\geqslant n}\{|x_n - x_m|_X\}$ and $U_n = X\backslash\bar{B}_{r_n}(x_n)$. Since x_n does not converge, we have

$$\bigcap_{n=1} \bar{B}_{r_n}(x_n) = \varnothing,$$

or equivalently $\{U_n\}_{n=1}^{\infty}$ is a cover of X. On the other hand it is easy to see that any finite sub-collection of $\{U_n\}_{n=1}^{\infty}$ does not contain x_n for all large n, a contradiction.

Fix $\varepsilon > 0$ and consider cover of X by open balls $\{B_\varepsilon(x)\}_{x\in X}$. Note that if $\{B_\varepsilon(x_i)\}_{i=1}^{n}$ is a finite subcover then $\{x_1, x_2, \dots, x_n\}$ forms an ε-net in X. Apply Theorem 1.25.

"only if"-part. Let $\varepsilon > 0$ be a Lebesgue's number of the covering. Choose a finite $\frac{\varepsilon}{2}$-net $\{x_1, x_2, \dots, x_n\}$ of X. Clearly

❻
$$\bigcup_{i=1}^{n} B_\varepsilon(x_i) = X.$$

For each x_i choose U_{α_i} such that $U_{\alpha_i} \supset B_\varepsilon(x_n)$. From ❻,

$$\bigcup_{i=1}^{n} U_{\alpha_i} = X.$$

□

Homeomorphism and embedding

4.14. Definition. *Let* X *and* Y *be metric spaces. A map* $f\colon X \to Y$ *is called a* homeomorphism *if it is a continuous bijection and the inverse* $f^{-1}\colon Y \to X$ *is also continuous.*

A map $f\colon X \to Y$ *is called an* embedding *if the map* $f\colon X \to f(X)$ *obtained by restricting the codomain is a homeomorphism.*

Two metric spaces X *and* Y *are called* homeomorphic *(briefly* $X \overset{hom}{=\!=} Y$*) if there is homeomorphism* $f\colon X \to Y$*.*

It is straightforward to check that $\overset{hom}{=\!=}$ is an equivalence relation. The homeomorphism class of metric space is much larger that its isometry class. To check that a given map $f\colon X \to Y$ is a homeomorphism, one only needs to know which sequences in both spaces X and Y are converging. To know the latter it is sufficient to know which sets in X and Y are open.

Given a metric space (X, ϱ), we may say "consider a metric d on (X, ϱ)" meaning that d is another metric on the set X and the identity map $X \to X$ is a homeomorphism $(X, \varrho) \to (X, d)$. For example,

$$d_1(x, y) = \sqrt{|x - y|}$$

is a metric on $[0, 1)$, but

$$d_2(x, y) = \min\{|x - y|, 1 - |x - y|\}$$

is a "metric on the set $[0, 1)$", but not a "metric on the space $[0, 1)$", as the identity map $([0, 1), d_1) \to ([0, 1), d_2)$ is not a homeomorphism.

4.15. Exercise. *Give an example of a continuous bijection $f : X \to Y$ between metric spaces that is not a homeomorphism.*

Induced intrinsic metric

Given a complete metric space (X, d). One can consider a function $\hat{d}\colon X \times X \to \mathbb{R}$ defined as

$$\hat{d}(x, y) \stackrel{def}{=\!=} \inf\{\text{length}\,\alpha\}$$

where the infimum is taken along all the curves α from x to y.

It is straightforward to see that $\hat{d}\colon X \times X \to \mathbb{R}$ satisfies all conditions of the metric if $\hat{d}(x, y) < \infty$ for all $x, y \in X$. In this case, the metric $\hat{d}$ will be called the *induced intrinsic metric* of d.

4.16. Exercise. *Construct a metric space (X, d) such that the induced intrinsic metric $\hat{d}$ is finite but (X, d) is not homeomorphic to $(X, \hat{d})$.*

HWA 4; due Fri, Sep 23

Solve all the problems, write a solution of one problem of your choice.

4.A. Show that $\mathcal{H}_{\mathbb{R}^2}$ is an intrinsic space.

4.B.[17] Show that $\mathcal{M}$ is an intrinsic space.

4.C. Let X be a complete metric space and $\varrho\colon X \to \mathbb{R}$ be a continuous positive function.

Show that there is $x \in X$ such that

$$\varrho(y) > \tfrac{99}{100}\cdot\varrho(x)$$

for any $y \in B_{\varrho(x)}(x)$.

4.D.[18] Let $K_n \to K_\infty$ be a sequence of compact convex bodies in $\mathbb{R}^3$ which converges in the sense of Hausdorff. Assume K_∞ is nondegenerate (i.e., its interior is not empty). Denote by ∂K_n the boundary[19] of K_n equipped with induced intrinsic metric. Show that $\partial K_n \to \partial K_\infty$ in the sense of Gromov–Hausdorff.

What happens if K_∞ degenerates to a plane figure or an interval?

[17] Hint: Use Proposition 3.4.

[18] Hint: Construct a disatance nonexpanding map as in the Problem 2.A.

[19] point x belongs to the boundary of set S of a metric space if for any $\varepsilon > 0$, the ball $B_\varepsilon(x)$ contains a point in S as well as a point in the complement of S.

5 Simplicial complexes

Simplex. Let $\{v_0, v_1, \dots, v_m\}$ be a nondegenerate set of points in $\mathbb{R}^N$ for $N \geqslant m$; i.e., the m vectors

$$v_1 - v_0, v_2 - v_0, \dots, v_m - v_0$$

are linearly independent. The convex hull[20] of $\{v_0, v_1, \dots, v_m\}$ is called an *m-dimensional simplex.*

So, a 0-dimensional simplex is a one-point set; a 1-dimensional simplex is a line segment; a 2-dimensional simplex is a triangle; a 3-dimensional simplex is a tetrahedron.

Let $\Delta^m = \mathrm{Conv}(v_0, v_1, \dots, v_m)$ be an m-dimensional simplex. Note that $x \in \Delta^m$ if and only if

$$x = \lambda_0 \cdot v_0 + \lambda_1 \cdot v_1 + \dots + \lambda_n \cdot v_m$$

for some nonnegative real numbers $\lambda_0, \lambda_1, \dots, \lambda_m$ such that

$$\sum_{i=0}^{m} \lambda_i = 1.$$

In this case, the real array $(\lambda_0, \lambda_1, \dots, \lambda_m)$ will be called the *barycentric coordinates* of the point x.

5.1. Exercise. *Verify the claim that*

$$\mathrm{Conv}(v_0, v_1, \dots, v_m) = \left\{ \sum_{i=0}^{m} \lambda_i \cdot v_i \,\middle|\, \lambda_i \geq 0 \textit{ and } \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

Faces. Note that the convex hull of any $(k+1)$-point subset of $\{v_0, v_1, \dots, v_m\}$ also forms a k-dimensional simplex which will be called *face* of Δ^m.

Finite simplicial complex. A *finite simplicial complex* can be defined as a finite collection $\mathcal{K}$ of simplices in $\mathbb{R}^N$ that satisfies the following conditions:

- ⋄ Any face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.
- ⋄ The intersection of any two simplices Δ_1 and $\Delta_2 \in \mathcal{K}$ is a face of both Δ_1 and Δ_2.

Let us denote by $\underline{\mathcal{K}}$ the union of all simpexes in $\mathcal{K}$; this is a subset of $\mathbb{R}^N$. Therefore $\underline{\mathcal{K}}$ can be equipped with the induced metric in inherits as a subspace of Euclidean space. In particular, this defines a notion of convergence of points in $\underline{\mathcal{K}}$.

Abstract finite simplicial complex. To describe a simplicial complex $\mathcal{K}$, it is sufficient to list the set of its vertexes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and list all the subsets $\mathcal{S}$ of $\mathcal{V}$ which appear as vertexes of single simplex in $\mathcal{K}$. I.e.,

[20] which will be further denoted as $\mathrm{Conv}(v_0, v_1, \dots, v_m)$

$W = \{w_0, w_1, \dots, w_k\} \in \mathcal{S}$ if and only if there is a simplex in $\mathcal{K}$ with vertexes $\{w_0, w_1, \dots, w_k\}$. The above condition imply that if $W \in \mathcal{S}$ then any subset $W' \subset W$ is also in S.

Any point x in $\underline{\mathcal{K}}$ can be described using barycentric coordinates $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in a way such the set

$$\{ v_i \in \mathcal{V} \mid \lambda_i > 0 \}$$

belongs to $\mathcal{S}$.

The latter gives a metric on $\underline{\mathcal{K}}$ which is completely determined by the combinatorial structure described above. If $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\mu_1, \mu_2, \dots, \mu_n)$ are barycentric coordinates of x and y in $\underline{\mathcal{K}}$ we set

$$|x - y|_{\underline{\mathcal{K}}} = \max_{i \in \{1,2,\dots,n\}} \{|\lambda_i - \mu_i|\}.$$

This metric differ from the one described above, but it defines the same convergence.

Locally finite simplicial complex. A few times we will need notion of *locally finite simplicial complex*. Instead of giving the definition, let us describe the necessary modification in the definition of abstract simplicial complex.

In the above construction, one can take $\mathcal{V}$ to be an infinite set and $\mathcal{S}$ to be any collection of finite subsets such that

- $\diamond$ if $W \in \mathcal{S}$ then any subset $W' \subset W$ is also in $\mathcal{S}$;
- $\diamond$ For any $v \in \mathcal{V}$, there are only finitely many $W \in \mathcal{S}$ such that $v \in W$.

Then one can identify a point of simplicial complex with a nonnegative function $\lambda\colon \mathcal{V} \to \mathbb{R}$ such that the set

$$\{ v \in \mathcal{V} \mid \lambda(v) > 0 \}$$

belongs to $\mathcal{S}$ and

$$\sum_{v \in \mathcal{V}} \lambda(v) = 1.$$

(Formally this sum has an infinite number of terms, but only finitely many of these terms differ from 0.) The obtained set of points can be then equipped with the metric

$$|\lambda - \mu| = \sup_{v \in \mathcal{V}} \{|\lambda(v) - \mu(v)|\}.$$

In particular we may talk about convergence of points in a locally finite simplicial complex.

5.2. Exercise. *Show that a compact locally finite simplicial complex has to be finite.*

Triangulations. Let $\mathcal{K}$ be a (finite or locally finite) simplicial complex and X be a metric space. A homeomorphism $\mathcal{K} \to X$ is called triangulation of X.

By choosing a triangulation of a metric space X, we identify the set of points of X and the set of points of the simplicial complex. This way we can talk about simplices in X.

5.3. Exercise. *Find a triangulation of sphere*[21] $\mathbb{S}^2$ *with minimal number of triangles.*

You will need at least 14 triangles to triangulate the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Finding such triangulation might be interesting, but proving its minimality is not fun.

Lyrical digression: Nerves and partition of unity

Here we describe one source of examples of simplicial complexes which appear in many branches of mathematics. We will not need it further, but understanding these constructions might help you to understand idea behind the notion of simplicial complex, and it will help you in the future.

Nerve. Let $\mathcal{V}$ be a collection of subsets of some set. Consider the abstract simplicial complex, where $\mathcal{V}$ is the set of vertexes and $\mathcal{S}$ is all collections of subsets in $\mathcal{V}$ which have non-empty intersection. We obtain a simplicial complex called the *nerve of* $\mathcal{V}$.

If $\mathcal{V}$ is finite then so is its nerve. If any set in $\mathcal{V}$ intersects only finitely many other sets in $\mathcal{V}$, then its nerve is locally finite.

5.4. Definition. *Given* $L \ge 0$*, a map* $f\colon X \to Y$ *between metric spaces is* L-Lipschitz *if*

$$|f(x) - f(x')|_Y \le L{\cdot}|x - x'|_X$$

for all $x, x' \in X$*. Note that this implies* f *is continuous.*

A map $f\colon X \to Y$ *is called* Lipschitz *if it is* L*-Lipschitz for some real* L*.*

A map $f\colon X \to Y$ *is called* locally Lipschitz *if for any point* $x \in X$ *there is* $\varepsilon > 0$ *such that the restriction* $f|_{B_\varepsilon(x)}$ *is Lipschitz.*

5.5. Partition of unity. *Let* $\mathcal{V} = \{V_1, V_2, \dots, V_n\}$ *is a finite open covering of a metric space* X*. Then there are locally Lipschitz functions* $\psi_i\colon X \to [0,1]$ *such that if* $\psi_i(x) > 0$ *then* $x \in V_i$ *and*

$$\sum_i \psi_i(x) = 1$$

for any $x \in X$*.*

A collection of functions ψ_i with above properies is called a *partition of unity subordinate to the open cover* $\{V_1, V_2, \dots, V_n\}$.

Proof. Consider the functions $\varphi_i\colon X \to \mathbb{R}$ defined as following:

$$\varphi_i(x) = \operatorname{dist}_{X \setminus V_i} x.$$

[21]The unit n-sphere is $\mathbb{S}^n = \{\, (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1 \,\}$.

Note φ_i is 1-Lipschitz for any i (see the definition below) and $\varphi_i(x) > 0$ if and only if $x \in V_i$. In particular,

$$\sum_i \varphi_i(x) > 0 \quad \text{for any} \quad x \in X.$$

Set

$$\psi_k(x) = \frac{\varphi_k(x)}{\sum_i \varphi_i(x)}.$$

Note that ψ_k are *locally Lipschitz*; $\varphi_i(x) \geqslant 0$ and if $\psi_i(x) > 0$ then $x \in V_i$; further

$$\sum_i \psi_i(x) = 1 \quad \text{for any} \quad x \in X.$$

□

Note that in the above proof for any point $x \in X$, the set

$$\{\, V_i \mid \psi_i(x) > 0 \,\}$$

corresponds to one of the simplices in the nerve. Therefore

$$\boldsymbol{\psi}\colon x \mapsto (\psi_1(x), \psi_2(x), \dots, \psi_n(x))$$

can be thought of as a Lipschitz map from X to the nerve of $\{V_1, V_2, \dots, V_n\}$; where the point x is mapped to the point with barycentric coordinates $\psi_i(x)$.

Polyhedral spaces

5.6. Definition. *A complete intrinsic space P is called a* polyhedral space *(or* Euclidean polyhedral space*) if it admits a triangulation such that each simplex in P is isometric to a simplex in Euclidean space.*

Further by a *triangulation of a polyhedral space* we will understand the triangulation as in the definition. We will also assume that the triangulation is linear; i.e. the metric is induced by a linear map into Euclidean space which is linear with respect to barycentric coordinates. In particular, if

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \quad \text{and} \quad (\mu_1, \mu_2, \dots, \mu_n)$$

are baricentric coordinates of two points x and y in one simplex of the triangulation then the point with coordinates

$$\left(\tfrac{\lambda_1+\mu_1}{2}, \tfrac{\lambda_2+\mu_2}{2}, \dots, \tfrac{\lambda_n+\mu_n}{2}\right)$$

is a midpoint of x and y.

The supremum of the dimensions of all simplices in such a triangulation is called *dimension* of P and denoted as $\dim P$.

5.7. Exercise. *Show that any convex nondegenerate polyhedron*[22] *in* $\mathbb{R}^m$ *is an* m*-dimensional polyhedral space.*

5.8. Exercise. *Show that boundary any convex nondegenerate polyhedron*[22] *in* $\mathbb{R}^m$ *equipped with its intrinsic metric is an* $(m-1)$*-dimensional polyhedral space.*

5.9. Exercise. *The dimension of a polyhedral space does not depend on the choice of triangulation.*

Zalgaller's theorem

5.10. Zalgaller's theorem. *Let* P *be an* m*-dimensional polyhedral space. Then there is* piecewise distance preserving *map* $f\colon P\to\mathbb{R}^m$*; i.e. there is a triangulation of* P *such that for any simplex* $\Delta\subset P$ *in the triangulation, the restriction* $f|_\Delta$ *is distance preserving.*

5.11. Exercise. *Show that piecewise distance preserving map is continuous and* length-preserving*; i.e., for any curve* $\alpha\colon\mathbb{I}\to P$*, the length of the curve* $f\circ\alpha$ *coincides with the length of* α*.*

5.12. Exercise. *Construct an* m*-dimensional polyhedral space* P *with a fixed triangulation which does not admit a map* $f\colon P\to\mathbb{R}^m$ *which is distance preserving on each simplex.*

Proof; P is compact and $\dim P=1$. Choose a triangulation of P; in addition to the vertexes (the 0-dimensional simplices), it contains a finite number of edges (1-dimensional simplices). Let $\{v_1,v_2,\dots,v_n\}$ be the set of vertexes of this triangulation. Consider the map $f\colon P\to\mathbb{R}$

$$f(x)\stackrel{def}{=\!=}\min_i\{|x-v_i|_P\}.$$

According to the definition of polyhedral space, each edge Δ^1_j in the triangulation admits an isometry $\iota\colon[0,a]\to\Delta^1_j$ for some $a\in\mathbb{R}$. Then

$$f\circ\iota(x)=\min\{|x|,|a-x|\}.$$

Therefore f is distance preserving on each half of Δ^1_j.

Bisecting each simplex Δ^1_j, we get a new triangulation of P and f is distance preserving on each simplex of this triangulation. □

Proof; P is compact, $\dim P=2$ *and it admits a triangulation with only acute triangles.* Denote by $\{v_1,v_2,\dots,v_n\}$ the set of its vertexes. For each v_i, consider

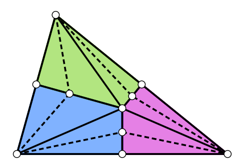
[22] i.e., a convex hull of finite number of points which has nonempty interior.

its *Voronoi domain*; this is the subset V_i of P of all points which are closer to v_i than to any other vertex. More precisely,

$$V_i = \{\, x \in P \mid |x - v_i|_P \leqslant |x - v_j|_P \text{ for any } j \,\}.$$

Note that since all triangles are acute, V_i is contained in the star[23] of v_i. In particular, any point $x \in V_i$ can be connected to v_i by unique line segment which lies in some of the simplices; this segment will be denoted as $[v_i x]$.

Note that in each triangle of the triangulation, we have one point where three Voronoi domains meet and three points on the edges of triangle where pairs of Voronoi domains meet. Let us subdivide each triangle in 6 triangles as it is done on the picture (solid lines only). In the new triangulation, each Voronoi domain V_i (from the original triangulation) is a union of all triangles which share the vertex v_i.



On the picture, the different Voronoi domains within one triangle are marked by different colors. Note that each triangle in the new triangulation has one vertex from the old triangulation (one of v_i's), one vertex is a mid-point of an old edge and one point is circumcenter of an old triangle. Further, the 6 triangles come in 3 congruent pairs.

Let us denote by $\alpha(x)$ the minimal angle between $[v_i x]$ and any edges[24] coming from v_i; note that $\alpha(x)$ is defined and continuous on $P\backslash\{v_1, v_2, \dots, v_n\}$. Further define a function $\varrho\colon P \to \mathbb{R}$ as

$$\varrho(x) = \min_i \{|v_i - x|_P\}.$$

Let us describe the map $f\colon P \to \mathbb{R}^2$ in polar coordinates the following way: $f(v_i) = 0$ for any vertex v_i and $f(x) = (\varrho(x), \alpha(x))$.

Subdividing each triangle by an angle bisector (see the dashed lines on the picture) at the old vertex gives a new triangulation which satisfies the conditions of the theorem. Indeed, the functions ϱ and α can be thought as polar coordinates on each such triangle. $\square$

5.13. Exercise. *Show that any triangle admits a triangulation into acute triangles.*

Comments about general case. In the above construction, one can add to the set of vertexes an arbitrary finite set of points. Then instead of acuteness of all triangles in the triangulation, one can use the following condition: If V is a Voronoi domain with center v (not necessary a vertex) then V contained in the star of v.

5.14. Exercise. *Show that one can add a finite number of points on the edges of P triangulation so that the above condition holds.*

[23] A star of a point v in a simplicial complex is the subcomplex formed by all simplices which contain v and all faces of such simplices.

[24] i.e., 1-dimensional simplices

This exercise would finish the proof of two dimensional case.

The general case can be proved by induction on m. First, one has to choose a finite set of points $\{v_1, v_2, \dots, v_n\}$ so that the above condition holds. (This set should include all the vertexes of original triangulation.)

Now let us glue any pair of points x and $y \in P$ if for some point $z \in V_i \cap V_j$, x lies on the geodesic $[v_i z]$, y lies on the geodesic $[v_j z]$ and $|v_i - x|_P = |v_j - y|_P$.

It is not hard to check that these gluing maps are piecewise distance presev-ing. Thus we get a piecewise distance preseving map from P to a new polyhedral space, say P'.

5.15. Definition. *Let Σ be a metric space with diameter $\leqslant \pi$. A metric space K is called a Euclidean cone over Σ if its underlying set is the quotient $\Sigma \times [0, \infty)/{\sim}$, where $(x, 0) \sim (y, 0)$ for any $x, y \in \Sigma$ and the metric is defined by the cosine rule; i.e.*

$$|(x, a) - (y, b)|_K^2 = a^2 + b^2 - 2{\cdot}a{\cdot}b{\cdot} \cos |x - y|_\Sigma.$$

Note that P' can be viewed as a star shaped subset of a Euclidean cone, say K.

Thus, it is sufficient to construct a piecewise distance preseving map $K \to \to \mathbb{R}^m$. The latter is equivalent to finding a piecewise distance preserving map from any $(m-1)$-dimensional spherical polyhedral space[25] to $\mathbb{S}^{m-1}$.

For a spherical polyhedral space, one can repeat the same construction once the cone is exchanged to so called *spherical suspension* defined below. On each step we get a dimension reduction, and the $\dim P = 1$ case can be proved the same way for spherical polyhedral space

5.16. Definition. *Let Σ be a metric space with diameter $\leqslant \pi$. A metric space Λ is called a spherical suspension over Σ if its underlying set is the quotient $\Sigma \times [0, \pi]/{\sim}$, where $(x, 0) \sim (y, 0)$ and $(x, \pi) \sim (y, \pi)$ for any $x, y \in \Sigma$ and the metric is defined by the spherical cosine rule; i.e.*

$$\cos |(x, a) - (y, b)|_\Lambda = \cos a{\cdot} \cos b + \sin a{\cdot} \sin b{\cdot} \cos |x - y|_\Sigma.$$

[25] It is defined the same way as Euclidean polyhedral space, but using spherical simplices.

HWA 5; due Fri, Sep 30

Solve all the problems, write a solution of one problem of your choice.

5.A. Let P be a (possibly nonconvex) polygon eqipped with the induced intrinsic metric. Show that P admits a triangulation[26] such that the set of vertexes of the triangulation is the set of vertexes of P.

5.B.[27] Show that the analogous statement for a polyhedron 3-dimensional space does not hold.

5.C. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulations of a polyhedral space[26]. Show that there is a triangulation $\mathcal{C}$ such that each triangle of $\mathcal{A}$ and $\mathcal{B}$ is a union of triangles in $\mathcal{C}$.

5.D.[28] Show that there is a triangulation of $\mathbb{S}^3$ with 1000 vertexes such that each pair of vertexes is connected by an edge.

[26] Recall that triangulations of polyhedral space are always assumed to be linear; see Definition 5.6 and the discussion right after that.

[27] Hint: look at the Figure 42.6, page 387 *Lectures on Discrete and Polyhedral Geometry* by Igor Pak

[28] Hint: Look at the natural triangulation of the boundary of convex polyhedra with all the vertexes on the curve $t \mapsto (t, t^2, t^3, t^4)$ in $\mathbb{R}^4$.

6 Length-preserving maps

Let X and Y be metric spaces. Recall that a continuous map $f\colon X \to Y$ is called *length-preserving* if for any curve $\alpha\colon \mathbb{I} \to X$, we have

$$\operatorname{length}\alpha = \operatorname{length}(f \circ \alpha).$$

(Note that since f is continuous, the composition $f \circ \alpha$ is a curve in Y.)

6.1. Exercise. *Show that if X is a complete intrinsic space then any length-preserving map $X \to Y$ is also a distance nonexpanding.*

Our next aim is to demonstrate that for a "reasonable pair of intrinsic spaces" X and Y, any distance nonexpanding map $f\colon X \to Y$ can be approximated by a length-preserving map. The "reasonable pairs" include

- ◇ pairs of smooth surfaces with the induced instrinsic metric from $\mathbb{R}^3$.
- ◇ pairs of *Riemannian manifolds*[29] of the same dimension.
- ◇ a polyhedral space P of dimension m and $\mathbb{R}^m$; see Theorem 6.10.

We will only show this for the pair $\mathbb{S}^2$ and $\mathbb{R}^2$; but our proof can be modified easily for much more general case.

6.2. Gromov's theorem (a very partial case). *Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ equipped with the induced intrinsic metric. Then any distance nonexpanding map*

$$f\colon \mathbb{S}^2 \to \mathbb{R}^2$$

can be approximated by length-preserving maps $\mathbb{S}^2 \to \mathbb{R}^2$.

More precisely, given $\varepsilon > 0$ there is a length-preserving map $f_\varepsilon\colon \mathbb{S}^2 \to \mathbb{R}^2$ such that

$$|f_\varepsilon(x) - f(x)| < \varepsilon$$

for any $x \in \mathbb{S}^2$.

6.3. Corollary. *There is a length-preserving map $\mathbb{S}^2 \to \mathbb{R}^2$.*

Note that the existence of a length-preserving map $\mathbb{S}^2 \to \mathbb{R}^2$ might look counterintuitive. The following exercise shows that such a map f has to crease on an everywhere dense set in $\mathbb{S}^2$.

6.4. Exercise. *Let $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ be a length-preserving map. Show that the restriction of f to any open subset of $\mathbb{S}^2$ cannot be injective.*

The plan of the proof. We will use Problem 4.D to construct a sequence of 2-dimensional polyhedral spaces P_n which converge to $\mathbb{S}^2$ in the sense of Gromov–Hausdorff. The sequence P_n will be chosen in such a way that there are surjective distance nonexpanding maps $\varphi_n\colon P_n \to P_{n-1}$.

The map f_ε will be constructed as the limit[30] of a sequence of piecewise distance preserving maps $f_n\colon P_n \to \mathbb{R}^2$ (which are automatically length-preserving).

[29] Its OK if you do not know what this is.

[30] We will eventually have to make rigorous what we mean by limit here. Note that one cannot talk about the limit of the maps f_n in the usual sense because each f_n has a different domain.

The maps f_n will be constructed from f_{n-1} recursively; we will apply Theorem 6.10 to the composition $f_{n-1} \circ \varphi_n \colon P_n \to \mathbb{R}^2$. Some extra care will be needed to ensure that the limit of f_n will be a length-preserving map that is ε-close to f.

Remarks. From Exercise 6.1, it follows that if we have a converging sequence of length-preserving map $f_n \colon X \to Y$ then their limit $f_\infty \colon X \to Y$ is distance nonexpanding. Gromov's theorem states that if the pair X and Y is "reasonable" then any distance nonexpanding map $X \to Y$ can be obtained as a limit of length-preserving maps $X \to Y$.

The existence of some piecewise distance preserving maps $P_n \to \mathbb{R}^2$ is provided by Zalgaller's theorem (5.10). But these maps cannot be used to prove Gromov's theorem (6.2) since the maps constructed in the proof of Zalgaller's Theorem converge to the map which sends all of $\mathbb{S}^2$ to one point. The latter follows from the observation below.

6.5. Observation. *Let P be a 2-dimensional polyhedral space and $f \colon P \to \mathbb{R}^2$ be the map constructed in the proof of Zalgaller's theorem for the collection of vertexes $\{v_1, v_2, \dots, v_n\} \in P$ then*

$$\operatorname{diam} f(P) \leqslant \max_{x \in P} \min_{i \in [1..n]} \{|x - v_i|_P\}.$$

Brehm's theorem

The following theorem will be used further in the proof of Gromov's theorem (6.2).

6.6. Brehm's theorem. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Then there is a piecewise distance preserving map $f \colon \mathbb{R}^m \to \mathbb{R}^m$ such that $f(a_i) = b_i$ for all i.

We prove only the case $m = 2$, although almost the same proof works for arbitrary dimension; the two-dimensional case is only slightly simpler in the extension of f into the "blind zone" which appears in the very end of the proof.

Proof; case $m = 2$. We will prove slightly more general statement, namely we will show that in additiion

$(*)$ *One can divide $\mathbb{R}^2$ into finite number of convex domains*[31] *so that the restriction of f to each domain is diatance preserving.*

The proof is by induction on n. The base case $n = 1$ is trivial.

[31]each domain is a possibly unbounded inetersection of finite number of halfplanes.

By applying the induction hypothesis to the first $n-1$ points, there is a piecewise distance preserving map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(a_i) = b_i$ for all $i < n$. We will use h to construct the desired map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Without loss of generality we can assume that $a_n = b_n$. Consider the set

$$\Omega = \left\{\, x \in \mathbb{R}^2 \;\middle|\; |a_n - x| < |a_n - h(x)| \,\right\}.$$

Notice that if $a_n \notin \Omega$, then $h(a_n) = a_n$ and we are done if we take $f = h$. So it remains to prove the case where $a_n \in \Omega$. First let us prove the following claim:

❶ *The set Ω is star-shaped with respect to a_n.*
I.e., if $x \in \Omega$ then the line segment $[a_n x]$ lies in Ω.

Indeed, if $y \in [a_n x]$ then

$$|a_n - y| + |y - x| = |a_n - x|.$$

Since $x \in \Omega$, we have

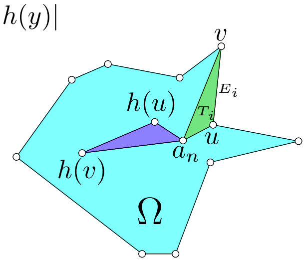
$$|a_n - h(x)| > |a_n - x|.$$

Since h is distance nonexpanding (see Exercise 6.1) we have

$$|h(x) - h(y)| \leqslant |x - y|.$$

Combining the above with the triangle inequality, we see

$$\begin{aligned}|a_n - y| &= |a_n - x| - |x - y| \\ &< |a_n - h(x)| - |h(x) - h(y)| \\ &\le |a_n - h(y)|;\end{aligned}$$



i.e., $y \in \Omega$. This proves Claim ❶.

According to Exercise 5.11, any piecewise distance preserving map is continuous. Therefore, from the definition of Ω is is clear that

❷ $\quad |a_n - x| = |a_n - h(x)|$ for any $x \in \partial\Omega$

(Recall that $\partial\Omega$ denotes the boundary of Ω.) To show this, one could consider a sequence of points in Ω that converge to x and another sequence of points in the complement of Ω that converge to x, and then use the continuity of h.

Further, note that

❸ *$\partial\Omega$ can be presented as a broken line, such that h is a distance preserving on each edge.*

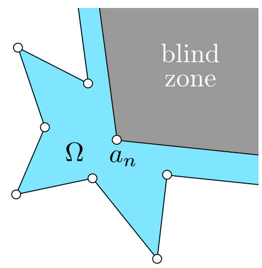
Indeed, fix a triangulation $\mathcal{T}$ for h; let Δ be an arbitrary triangle in $\mathcal{T}$. Since h is piecewise distance preserving with respect to $\mathcal{T}$, the restriction $h|_\Delta$ coincides with an isometry $\iota\colon \mathbb{R}^2 \to \mathbb{R}^2$. If $\iota(a_n) \ne a_n$ then $\partial\Omega \cap \Delta$ coincides with the intersection of Δ and the perpendicular bisector for a_n and $\iota^{-1}(a_n)$. The remaining case $\iota(a_n) = a_n$ is left to the reader.

For each edge E_i of $\partial\Omega$ consider the triangle T_i with vertex a_n and base E_i. Condition ❷ implies that there is an isometry ι_i of $\mathbb{R}^2$ which has a_n as a fixed point[32] and such that $\iota_i(x) = h(x)$ for any $x \in E_i$.

[32]So, ι_i is either a rotation around a_n or a reflection in a line passing through a_n

Let us define $f(x) = h(x)$ for any $x \notin \Omega$ and $f(x) = \iota_i(x)$ for any $x \in T_i$. Note that this completely defines f in the case that Ω is bounded.

If Ω is unbounded then using property $(*)$ for h, one can divide $\partial\Omega$ into finite number of rays and segments so that the restriction of h to each is distance preserving. We can apply the construction above to define f on the part of Ω which is mareked blue on the picture. But in addition we will have a "blind zone" formed by the angle (or few angles) with vertex at a_n which lie completely in Ω. Note that (the continuous extension of) f maps each of the boundary rays by rotations around a_n.



Since f is distance nonexpanding, the angle between the images of these rays is smaller than the corresponding angle of the blind zone. It is then easy to extend f as a piecewise distance preserving map into the blind zone by folding the angle along one ray. □

6.7. Exercise. *Look at the construction of Ω in the above proof and show that $a_i \notin \Omega$ for all $i < n$.*

6.8. Kirszbraun's theorem. *Let $Q \subset \mathbb{R}^m$ be an arbitrary subset and $f\colon Q \to \mathbb{R}^k$ be a distance nonexpanding map. Then there is a distance nonexpanding map $F\colon \mathbb{R}^m \to \mathbb{R}^k$ which extends f; i.e., $F(q) = f(q)$ for any $q \in Q$.*

6.9. Exercise. *Use Brehm's theorem to prove the Kirszbraun's theorem.*

Hint: First prove the case $k \geqslant m$. Note that the case $k < m$ can be reduced to the $k = m$ by applying a coordinate projection $\mathbb{R}^m \to \mathbb{R}^k$.

Piecewise distance preserving approximations

Here we prove yet another statement needed for Gromov's theorem.

6.10. Akopyan's Theorem. *Let P be an m-dimensional polyhedral space. Then any distance nonexpanding map $f\colon P \to \mathbb{R}^m$ can be approximated by piecewise distance preserving maps.*

More precisely, given $\varepsilon > 0$ there is a piecewise distance preserving map $f_\varepsilon\colon P \to \mathbb{R}^m$ such that

$$|f_\varepsilon(x) - f(x)| < \varepsilon$$

for any $x \in P$.

Proof in the case P is a nondegenerate convex polyhedron in $\mathbb{R}^m$. Choose a sufficiently fine triangulation of P, say the diameter of each triangle is less than ε. If $\{a_1, a_2, \dots, a_n\}$ is the set of vertexes for this triangulation, take $b_i = f(a_i)$ and apply Brehm's theorem. In this way, we obtain a map $f_\varepsilon\colon P \to \mathbb{R}^m$ which coincides with f on an arbitrary finite set of points.

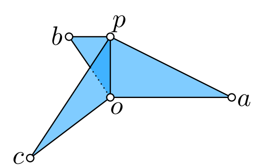
Then the statement follows since the triangulation is fine and both f and f_ε are distance nonexpanding. $\square$

As you can see from the following exercise, this approach is no longer possible for a general polyhedral space.

6.11. Exercise. *Consider the following 5 points in* $\mathbb{R}^3$*:*

$$o = (0,0,0), \quad p = (0,0,1), \quad a = (2,0,0), \quad b = (-1,2,0), \quad c = (-1,-2,0)$$

Let P be the "tripod" which is the polyhedral space consisting of the three triangles $\triangle opa$, $\triangle opb$ and $\triangle opc$ in $\mathbb{R}^3$, and equipped with the induced intrinsic metric.



Viewing $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$, note that the coordinate projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi(x,y,z) = (x,y)$ is distance nonexpanding. We have that

$$\pi(o) = \pi(p) = o, \quad \pi(a) = a, \quad \pi(b) = b, \quad \pi(c) = c.$$

Show that there is no piecewise distance preserving map $f\colon P \to \mathbb{R}^2$ such that $f(a) = a$, $f(b) = b$ and $f(c) = c$.

Theorem 6.10 will only be proved in the case that $m = 2$, P is compact, and f is piecewise linear. This will be enough to prove our partial case of Gromov's theorem (6.2). Even in this case the proof is quite technical.

In fact, any distance nonexpanding map from a polyhedral space to $\mathbb{R}^m$ can be approximated by piecewise linear distance nonexpanding maps. The proof of the latter statement is not hard, but technical.

Proof; case $m = 2$, P is compact and f is piecewise linear. For a fixed fine[33] triangulation $\mathcal{T}$ for f, we will construct a map $w\colon P \to \mathbb{R}^2$ defined on the 1-skeleton[34] of $\mathcal{T}$ which is

❹ *Globally distance non-expanding; i.e.*

$$|w(x) - w(y)|_{\mathbb{R}^2} \leqslant |x - y|_P$$

for any x and y in the 1-skeleton of $\mathcal{T}$.

❺ *Piecewise distance preserving on each edge for a yet finer triangulation*[35] $\mathcal{T}'$ *of the 1-skeleton.*

Once w is constructed, we can apply Brehm's theorem to each triangle in $\mathcal{T}$ in the same fashion as in the proof for the case where P is a convex polyhedron. (For a triangle Δ in $\mathcal{T}$ take as a's to be the set of all vertexes of $\mathcal{T}'$ on the boundary of Δ and set $b_i = w(a_i)$.) Since w is piecewise distance preserving on

[33] say with diameter of each triangle much smaller than $\frac{\varepsilon}{10}$.

[34] The k-skeleton of a simplicial complex $\mathcal{T}$ is the simplicial complex formed by all simplexes in $\mathcal{T}$ of dimension $\leqslant k$.

[35] In this case this is subdivision into segments.

each edge of $\mathcal{T}$, these maps will fit together to define a map f_ε that satisfies the conclusion of the theorem.[36]

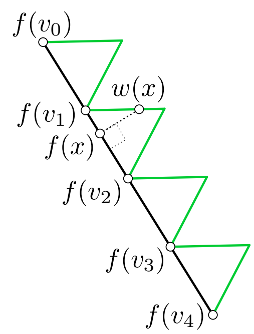
It remains to construct w. First we describe a zigzag construction for a piecewise distance preserving map of a segment. Then we describe a not quite working construction and afterwards, make a correction.

Zigzag construction. Fix a large integer n. Subdivide each edge E of the triangulation for f into n equal intervals. Denote by $v_0, v_1, \dots, v_n$ the endpoints of these interval in the order they appear on E.

Note that the image $f(E)$ is a line segment or a point, because f is linear when restricted to E. In the first case, take u to be a unit normal vector to $f(E)$; otherwise, take an arbitrary unit vector as u.

Set

$$k = \sqrt{\left(\frac{\operatorname{length} f(E)}{\operatorname{length} E}\right)^2 - 1}$$



and for $x \in E$, set $s(x) = \min_i\{|v_i - x|\}$. Define

$$w(x) = k \cdot s(x) \cdot u + f(x).$$

The choice of k is made to make w piecewise distance preserving on E. Let ℓ be the maximal length of all edges in $\mathcal{T}$. Note that

❻ $$|w(x) - f(x)| < \tfrac{\ell}{n}.$$

A not quite working construction of w. Without loss of generality, we may assume that there is $\delta > 0$ such that

❼ $$|f(x) - f(y)| \leqslant |x - y| - \delta \cdot |x - y|.$$

for any x and $y \in P$. If not, consider the sequence of maps $f_n = (1 - \frac{1}{n}) \cdot f$. Since P is compact, $f_n \to f$ converges uniformly[37]. Since f_n satisfy ❼ for $\delta < \frac{1}{n}$, the existence of an approximation for each f_n (as in the conclusion of the Theorem) would imply the existence of an approximation for f.

Repeating the zigzag construction for each edge, defines w on the 1-skeleton of $\mathcal{T}$. Since w is piecewise distance preserving on each edge, we have

❽ $$|w(x) - w(y)| \leqslant |x - y|_P \quad \text{if } x \text{ and } y \text{ lie on the same edge.}$$

For any x and y on the one skeleton of P, we have

$$|w(x) - w(y)|_{\mathbb{R}^2} \leqslant |x - y|_P - (\delta \cdot |x - y|_P - \vartheta),$$

if $\vartheta > 2 \cdot \frac{\ell}{n}$. Therefore for large enough n we have

❾ $$|w(x) - w(y)| \leqslant |x - y|_P \quad \text{or} \quad |x - y|_P < \tfrac{\vartheta}{\delta}$$

[36]This is the main nontechnical idea in the proof.

[37]i.e., $|f_n(x) - f(x)| < \varepsilon_n$ for any $x \in P$ and some fixed sequence $\varepsilon_n \to 0$.

for arbitrary fixed $\vartheta > 0$. From ❽, in the latter case, we can assume that the points x and y lie on the different edges of $\mathcal{T}$. It follows that in this case x and y have to lie near one vertex of $\mathcal{T}$.

The constructed map w almost satisfies the needed conditions; it only fails to meet ❹ for x and y near one of the vertex of $\mathcal{T}$. We now make a modification of the above construction to fix this issue.

Correction to the construction of w. Let us construct w as above so that in addition it satisfies the following property:

❿ *There is $\lambda > 0$ such that if*

$$|v - x|_P = |v - y|_P \leq \lambda$$

for two points on 1-skeleton x and y and a vertex v of $\mathcal{T}$ then

$$w(x) = w(y).$$

Once such a w is constructed, $|v - x|_P, |v - y|_P \leqslant \lambda$ would imply

$$\begin{aligned} |w(x) - w(y)|_{\mathbb{R}^2} &\leqslant \big||v - x|_P - |v - y|_P\big| \\ &\leqslant |x - y|_P \end{aligned}$$

for $\frac{\vartheta}{\delta} \ll \lambda$ the same way as in ❽.

To make such a modification, fix a small $\lambda > 0$. Denote by $\mathcal{T}^1$ the 1-skeleton of $\mathcal{T}$. Consider the map $q_\lambda\colon \mathcal{T}^1 \to \mathcal{T}^1$ which sends each edge to itself in such a way that any point of distance $\leqslant \lambda$ to a vertex maps to this vertex and the map is linear on the remaining part of edge. For each $\lambda > 0$ we can find minimal $\mu \geqslant 0$ such that $(1 - \mu)\cdot f \circ q_\lambda$ is distance nonexpanding.

It can be shown that $\mu \to 0$ as $\lambda \to 0$. Note that it is sufficienct to show this for three sides of a fixed triangle in $\mathcal{T}$. The latter can be done by straightforward calculations.

Choose sufficiently large n, so $\frac{\vartheta(n)}{\delta} \ll \lambda$ and apply the zigzag procedure to each of the three portions of each edge separately in such a way that all the λ-long intervals starting at one vertex are mapped in an identical way. The constructed w meets condition ❿. □

HWA 6; due Fri, Oct 7

Solve all the problems, write solutions for two problems of your choice.

6.A. Construct two triangles $\triangle a_1a_2a_3$ and $\triangle b_1b_2b_3$ in the plane such that

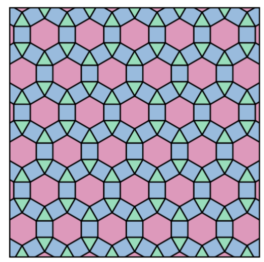
$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, but

$$\text{area}\,\triangle a_1a_2a_3 < \text{area}\,\triangle b_1b_2b_3.$$

Construct a map f as in Brehm's theorem for the collection of points a_1, a_2, a_3, b_1, b_2, b_3. What is the image of $\triangle a_1a_2a_3$ under f?

6.B. Consider the tessellation of the plane into regular hexagons, squares and triangles with side a as it shown on the picture.



Show that there is a piecewise distance preserving map $f_a\colon \mathbb{R}^2 \to \mathbb{R}^2$ which reverses the orientation of each square and preserves the orientation of each triangle and hexagon.[38]

Describe the map $f_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is obtained as the limit of f_a as $a \to 0$.

Try to think if it is possible to make a paper folding model for f_a.

The next problem generalizes the previous one.

6.C. Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^2$ with all triangles colored in black and white. Show that the following two conditions are equivalent.

a) There is a piecewise distance preserving map $\mathbb{R}^2 \to \mathbb{R}^2$ for this triangulation which preserves the orientation of each white triangle and reverses the orientation of each black triangle.

b) The sum of black angles around any vertex in $\mathcal{T}$ is either 0, π or $2{\cdot}\pi$.

6.D. Let Δ be a 3-dimensional simplex in $\mathbb{R}^3$ and $\partial\Delta$ be its boundary equipped with the induced intrinsic metric. Show that there is piecewise distance preserving map $f\colon \partial\Delta \to \mathbb{R}^2$ which is distance preserving on 2 out of 4 faces of Δ.

[38] Less formally you need to "fold" along each segment in this tessellation. More formally: any isometry of the plane can be presented as a coposition either of parallel tranlation and rotation, or a parallel translation and reflection. In the first case the isometry is called "orientation preserving", in the second case "orientation reversing".

7 Existence of a length-preserving map from the sphere to the plane

Proof of Corollary 6.3. Consider a nested sequence $K_0 \subset K_1 \subset \dots$ of convex polyhedra in $\mathbb{R}^3$ whose union is a unit ball. Clearly K_n converges to the unit ball in the Hausdorff sense (compare Lemma 2.3).

Denote by P_n the boundary of K_n equipped with the induced intrinsic metric. According to Exercise 5.8, P_n is a polyhedral space for each n. By Problem 4.D, P_n converges to $\mathbb{S}^2$ in the Gromov–Hausdorff sense.

The same construction as in Problem 2.A gives a distance nonexpanding map $\varphi_n\colon P_n \to P_{n-1}$ for each n. It is straightforward to see that φ_n are piecewise linear; i.e., there are triangulations of P_n and P_{n-1} such that for any simplex Δ in P_n, the image $\varphi_n(\Delta)$ lies in a simplex Δ' of P_{n-1} and the restriction $\varphi|_\Delta$ is linear (if written with respect to the barycentric coordinates of Δ and Δ').

Note that for any point $x \in \mathbb{S}^2$, there is unique sequence of points $x_n \in P_n$ such that $x_n \to x$ as $n \to \infty$ and $\varphi_n(x_n) = x_{n-1}$ for all $n \geqslant 1$. To show existence of this sequence, fix any sequence $z_n \in P_n$ such that $z_n \to x$. Consider the double sequence $y_{n,m} \in P_n$ defined for $n \leqslant m$ such that $y_{n,n} = z_n$ and $\varphi_n(y_{n,m}) = y_{n-1,m}$ for all $1 \leqslant n$. Then set

$$x_n = \lim_{m\to\infty} y_{n,m}.$$

7.1. Exercise. *Show that the limit above is defined.*

If $x_n \to x \in \mathbb{S}^2$ be a sequence as above, define $\psi_n\colon \mathbb{S}^2 \to P_n$ as $\psi_n(x) = x_n$. Clearly $\psi_n\colon \mathbb{S}^2 \to P_n$ is distance nonexpanding, $\psi_{n-1} = \varphi_n \circ \psi_n$ for each $n \geqslant 1$ and for any p and $q \in \mathbb{S}^2$ we have

❶ $$|p_n - q_n|_{P_n} \to |p - q|_{\mathbb{S}^2} \text{ as } n \to \infty$$

where $p_n = \psi_n(p)$ and $q_n = \psi_n(q)$.[39] In particular $\psi_n\colon \mathbb{S}^2 \to P_n$ is δ_n-isometry for some sequence $\delta_n \to 0$.

Recursive construction of the maps $f_n\colon P_n \to \mathbb{R}^2$. Assume we have a piecewise distance preserving map $f_{n-1}\colon P_{n-1} \to \mathbb{R}^2$. The composition $f_{n-1} \circ \varphi_n\colon P_n \to \to \mathbb{R}^2$ is a piecewise linear and distance nonexpanding. Thus given $\varepsilon_{n-1} > 0$ we can apply Theorem 6.10 to construct a piecewise distance preserving map $f_n\colon P_n \to \mathbb{R}^2$ which is ε_{n-1}-close to $f_{n-1} \circ \varphi_n$.

[39] Such sequences of spaces and maps

$$\dots \xrightarrow{\varphi_2} X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0$$

appear in all branches of mathematics. They are called *inverse systems.* A sequence $(x_0, x_1, x_2, \dots)$ such that $x_n \in X_n$ and $\varphi_n(x_n) = x_{n-1}$ could be considered as a point in a new space X which is called the *inverse limit* of the system and is denoted $X = \varprojlim X_n$. Given $x = (x_0, x_1, \dots) \in X$, the evaluation maps $\psi_n : X \to X_n$ given by $\psi_n(x) = x_n$ are called *projections.*

Denote by $M(n)$ the number of triangles in a triangulation for f_n; i.e. a triangulation of P_n such that f_n is distance preserving on each of its triangles. Set

$$\varepsilon_n = \frac{\varepsilon_{n-1}}{10{\cdot}M(n)}.$$

This way we recursively define the construction of f_n. It goes as follows:

1. Choose an arbitrary $\varepsilon_0 > 0$ and take a piecewise distance preserving map $f_0 \colon P_0 \to \mathbb{R}^2$ which is provided by Zalgaller's theorem (5.10).
2. Use φ_1, f_0 and ε_0 to construct f_1.
3. Use f_1 to construct ε_1.
4. Use φ_2, f_1 and ε_1 to construct f_2.
5. Use f_2 to construct ε_2.
6. ...

It remains to prove the following claim:

❷ *The sequence of maps $f_n \circ \psi_n \colon \mathbb{S}^2 \to \mathbb{R}^2$ converges to a length-preserving map $f \colon \mathbb{S}^2 \to \mathbb{R}^2$.*

Since ε_n decays faster than $\frac{\varepsilon_0}{10^n}$, the sequences $(f_n \circ \psi_n)(x)$ are Cauchy , for any fixed x; so the limit map $f(x)$ is defined. Since all $f_n \circ \psi_n$ are distance nonexpanding, we have that so is $f \colon \mathbb{S}^2 \to \mathbb{R}^2$. Note that this implies

$$\operatorname{length}(f \circ \alpha) \le \operatorname{length} \alpha$$

for any curve α in $\mathbb{S}^2$.

For the remaining part of proof we will need the following definition.

7.2. Definition of pre-length. *Let X be a metric space and $\alpha \colon [0,1] \to X$ be a curve. Set*

$$\ell_k(\alpha) \stackrel{def}{=\!=} \sup \left\{ \sum_{i=1}^{k} |\alpha(t_i) - \alpha(t_{i-1})|_X \;\middle|\; 0 = t_0 < t_1 < \dots < t_k = 1 \right\}.$$

Note that for any fixed α, the sequence $\ell_k(\alpha)$ is non-decreasing in k and

$$\operatorname{length} \alpha = \lim_{k \to \infty} \ell_k(\alpha).$$

Assume f is not length preserving. Since f is distance nonexpanding, it follows that for some curve γ in $\mathbb{S}^2$ we have

$$\operatorname{length}(f \circ \gamma) < \operatorname{length} \gamma.$$

Together with the definition of length (4.2), the latter implies that on γ one can choose two points, say p and q, such that

❸ $$\operatorname{length}(f \circ \alpha) < |p - q|_{\mathbb{S}^2},$$

where α denotes the part of γ from p to q.

Set $p_n = \psi_n(p)$ and $q_n = \psi_n(q)$ as above and let α_n be a curve from p_n to q_n. Note that one can exchange α_n for a shorter curve, say β_n which goes from p_n to q_n and whose image in any triangle of the triangulation of P_n is a line segment. It follows that $f_n \circ \beta_n$ is a broken line with at most $M(n)$ edges. Hence

$$\ell_{M(n)}(f_n \circ \beta_n) = \operatorname{length} \beta_n \leqslant \operatorname{length} \alpha_n.$$

Therefore

❹ $$|p_n - q_n|_{P_n} = \inf\{\ell_{M(n)}(f_n \circ \alpha_n)\},$$

where the infimum is taken for all curves α_n from p_n to q_n in P_n.

By the recursive construction of the maps, we have that

$$|(f_n \circ \psi_n)(x) - f(x)| < \varepsilon_n$$

for any $x \in \mathbb{S}^2$ and each n. It follows that

❺ $$|\ell_{M(n)}(f \circ \alpha) - \ell_{M(n)}(f_n \circ \psi_n \circ \alpha)| < \varepsilon_{n-1}.$$

Note that ❶ and ❸ implies that

$$\varepsilon_{n-1} < |p_n - q_n| - \operatorname{length} \alpha$$

for all large n, the latter contradicts ❺ and ❹. □

Comments on the proof of Gromov's theorem. The above argument can be also applied to proof our partial case of Gromov's theorem (6.2). One only needs to take ε_0 sufficiently small, choose the sequence P_n so that P_0 is sufficiently closet to $\mathbb{S}^2$ and construct a piecewise distance preserving map $f_0\colon P_0 \to \mathbb{R}^2$ so that the composition $f_0 \circ \varphi_0$ is close enough to f.

According to the theorem on approximation (6.10), in order to construct f_0 it is sufficient to construct a piecewise linear distance nonexpanding map $h_0\colon P_0 \to \mathbb{R}^2$ so that the composition $h_0 \circ \varphi_0$ is close enough to f.

A piecewise linear map $P_0 \to \mathbb{R}^2$ is uniquely determined by images of all the vertexes in a given triangulation. So given a vertex v, one can choose a value $h_0(v)$ in the set $(f \circ \psi^{-1})(v)$ and extend h_0 piecewise linearly to P.

In general the obtained map is not distance nonexpanding; see Problem 6.A. It turns out however that if f is smooth then given $\varepsilon > 0$ one can choose P_0 close enough to $\mathbb{S}^2$ so the map $(1-\varepsilon)\cdot h_0$ will be distance nonexpanding. If ε is small $(1-\varepsilon)\cdot h_0$ will be still close enough to f.

It remains to prove show the following lemma.

7.3. Lemma. *Any distance nonexpanding map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ can be approximated by a smooth distance nonexpanding map.*

To prove this lemma, extend f to $\mathbb{R}^3$, say the following way

$$\bar{f}(x) = f(\tfrac{x}{|x|}).$$

(The value $\bar f(0)$ is undefined, but we can still take the integral below.) Choose a mollifier, i.e., a sufficiently smooth nonnegative function $w\colon \mathbb{R}^3 \to \mathbb{R}$ such that $w(x) = 0$ if $|x| > \varepsilon$ and $\int_{\mathbb{R}^3} w = 1$. Consider convolution

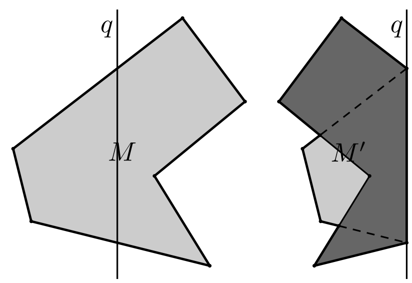
$$(\bar f * w)(x) \stackrel{def}{=\!=} \int_{\mathbb{R}^3} \bar f(y)\cdot w(x-y)\cdot dy$$

It is straightforward to check that the restriction of $\bar f * w$ to $\mathbb{S}^2$ is distance nonexpanding and that for small ε it has to be close enough to f.

Napkin folding problem

This section is meant to be relaxing. Here we will discuss the following problem posted by V. Arnold in 1956.

7.4. Problem. *Is it possible to fold a square on the plane so that the obtained figure will have a longer perimeter?*



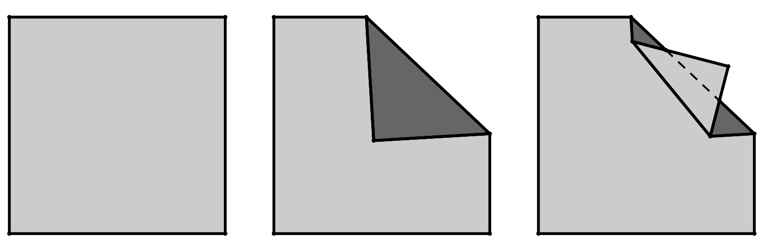
The answer to this problem depends on the meaning of word “fold”.

For example, one can consider sequence of *folding* such that all layers are folded simultaniousely along a line. From the following exercise, it follows that the perimeter is always non-increasing under ech folding of that type. Therefore, the resulting figure has to have smaller perimeter.

7.5. Exercise. *Show that each fold describes above indeed decreases perimeter.*

Note that in general, the intersection of of the line q with the polygon M on the picture might be a union of few line segments.

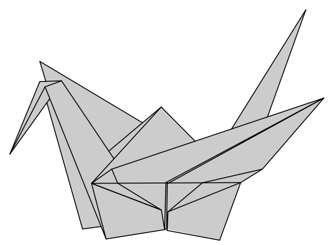
The *foldings* described above makes it impossible to unfold a layer which lies on top of an other layer, as it shown on the following picture.

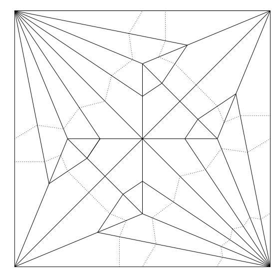


Note that the *unfold* increases the perimeter, although not beyond the perimeter of the original square. It is still unknown if it possible to increase perimeter by sequence of such “folds” and “unfolds”. (It is likely that no one wants to know.)

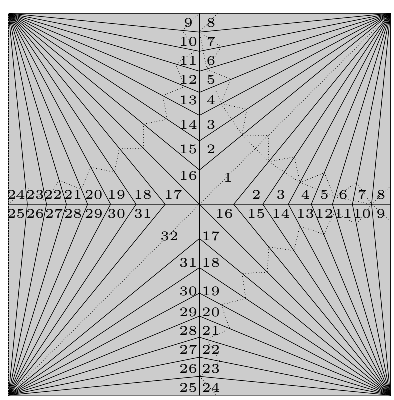
Japanese crane. Now let us consider a more general definition of folding. Imagine that we mark in advance the lines of folding and start to fold the napkin in such a way that each domain between folds remains flat all the time.

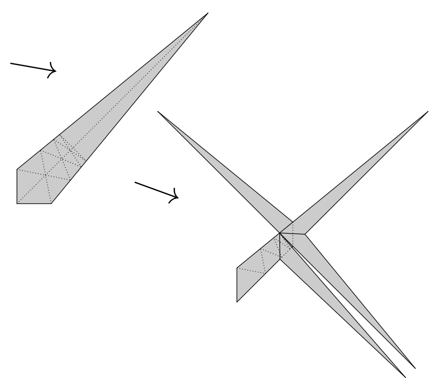
If you understand "folding" this way, then the answer to the problem is "yes; it is possible". In some sense, this problem was solved by origami practitioners well before it was even posed. The possibility to increase the perimeter slightly can be seen in the base for crane which was known by origami masters for centuries[40] but mathematicians learned this answer only in 1998[41].





The base for the crane, has 4 long ends and one short end. Two ends are used for wings, the other two have to be thinned, as one is used for the head and the other for the tail. Thinning twice each of the long ends makes it possible to produce a base which can then be folded into plane to obtain a figure with larger perimeter.





On the picture above, you can see the net of folds, the base, and the base with opened out ends. On the net of folds, you can see the number of the layer in the base. The dashed lines are the folds which appear at the opening out. The perimeter increases by about 0.5% and the number of layers in the end is equal to 80. I do not know of a way to increase the perimeter with a smaller number of layers.

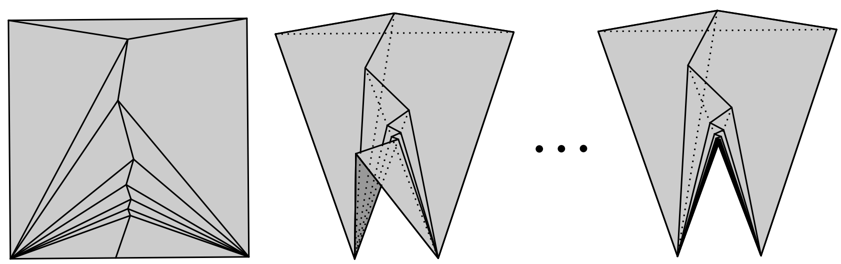
If a is the side of original square then it takes a bit less than a to go around each of 4 needles and it takes about $(\sqrt{2}-1)\cdot a$ to go around the short end. The

[40] It appears in the oldest known book on origami, "Senbazuru Orikata," dated 1797; but for sure it was known much earlier.

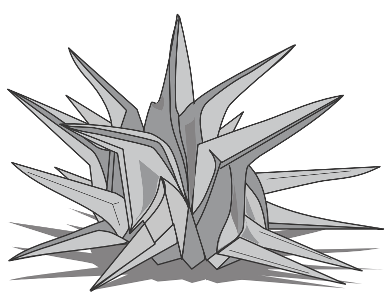
[41] Here is the html-file which tells how it happened.

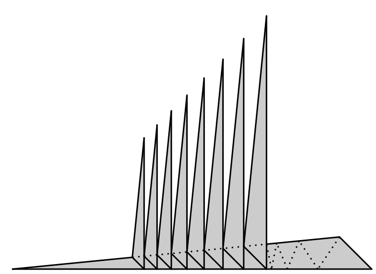
latter makes perimeter longer. Thinning the ends many times makes possible to increase the perimeter by a value arbitrarily close to $(\sqrt{2}-1)\cdot a$.

The following picture describes another way to increase the perimeter which is based on an idea of Yaschenko. It can be obtained by by recursive application of one simple move. If one repeats this move sufficiently many times, we obtain a figure with a longer perimeter; each iteration adds 2 layers near the concave corner. The total number of layers in this model is much bigger than in the crane base.



The sea urchin and the comb. It turns out the perimeter can be made arbitrarily large. This can be seen in the origami model for a sea urchin constructed by Robert Lang in 1987. In 2004 a complete solution was discovered independently by Alexei Tarasov; he constructs a folding of a "comb" which is shown on the picture on the right. He proved that the comb can be folded in a true way, in particular without starching and crooking the paper which is often used by in origami.

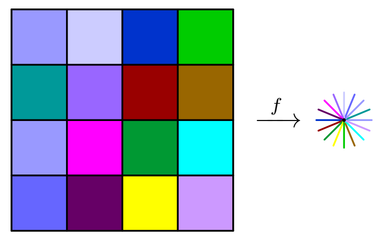




Foldings in 4-dimensional space. One can define a "folding" as a piecewise distance preserving map from the square to the plane. These foldings are yet more general than those which appear above. The piecewise distance preserving map discussed in Problem 6.B, shows that it is not always possible to construct a paper model for such maps. On the other hand, these obstructions disappear in the 4-dimensional case. Thus, one can regard piecewise distance preserving maps as paper folding in 4-dimensional space.

In this case, the existence of perimeter increasing folds follows from Brehm's theorem. It is sufficient to construct a distance nonexpanding map from the square to the plane so that the perimeter of its image is sufficiently long and then apply Brehm's theorem to approximate this map by a "folding".

The needed map can be constructed as follows: Fix a large n and divide the square $\square$ into n^2 squares with side length $\frac{a}{n}$. Let $d(x)$ denotes the distance from a point $x \in \square$ to the boundary of the small square which contains x. The function $d\colon \square \to \mathbb{R}$ takes values in $[0, \frac{a}{2\cdot n}]$. Further, let us enumerate the squares by integers from 1 to n^2. Given $x \in \square$ denote by $i(x)$ the (say minimal) number in this enumeration of a small square which contains x.



Now for each $i \in \{1, 2, \dots, n^2\}$ choose a unit vector $u_i \in \mathbb{R}^2$ so that $u_i \neq u_j$ if $i \neq j$. Consider the map $f\colon \square \to \mathbb{R}^2$ defined by

$$f(x) = d(x){\cdot}u_{i(x)}.$$

It is straightforward to check that the obtained map is distance nonexpanding. The image $f(\square)$ consists of n^2 segments of length $\frac{a}{2\cdot n}$ which start at the origin. So the perimeter of $f(\square)$ is equal to $2n^2{\cdot}\frac{a}{2\cdot n} = a{\cdot}n$. I.e., taking large enough n, one can make the perimeter of the image $f(\square)$ arbitrary large.

(The picture shows the case $n = 4$, in this case the perimeter of $f(\square)$ is $4{\cdot}a$; i.e. it is the same as perimeter of the original square, for $n > 4$ it gets larger. When you calculate the perimeter of the degenerate figure, imagine going up and down each segment as you "traverse the boundary" and count the length of each segment twice.)

Variations on a theme of Kirszbraun

This is yet another relaxing section.

7.6. Proposition. *Let $R > 0$ and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Then

$$\bigcap_{i=1}^{n} B_R(a_i) \neq \varnothing \quad \Longrightarrow \quad \bigcap_{i=1}^{n} B_R(b_i) \neq \varnothing.$$

Proof. Applying Kirszbraun's theorem (6.8), we get a distance noncontracting map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $f(a_i) = b_i$. Choose $x \in \bigcap_{i=1}^{n} B_R(a_i)$. Since f distance noncontracting,

$$|f(x) - b_i| \leqslant |x - a_i| \leqslant R$$

for each i. In particular, $f(x) \in B_R(b_i)$ for each i; hence the result. $\square$

Given a set $A \subset \mathbb{R}^m$, let $\mathrm{vol}_m(A)$ denote m-dimensional volume of A. For example vol_1 measures length in $\mathbb{R}^1$, vol_2 measures area in $\mathbb{R}^2$, and vol_3 measures the usual notion of volume in $\mathbb{R}^3$.

7.7. Conjecture. *Let $R > 0$ and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Then

❻
$$\mathrm{vol}_m\left(\bigcap_{i=1}^{n} B_R(a_i)\right) \leqslant \mathrm{vol}_m\left(\bigcap_{i=1}^{n} B_R(b_i)\right)$$

and

❼
$$\mathrm{vol}_m\left(\bigcup_{i=1}^{n} B_R(a_i)\right) \geqslant \mathrm{vol}_m\left(\bigcup_{i=1}^{n} B_R(b_i)\right).$$

7.8. Exercise. *Prove this conjecture in case $m = 1$.*

The inequality ❼ was conjectured by Poulsen in 1954 and Kneser in 1955 and Hadwiger in 1956. The inequality ❻ was conjectured much later, it appears in list of problems of Klee and Wagon published in 1991. Both inequalities are trivial in the case $m = 1$; both are open problems for $m > 2$. Both were proved in case $m = 2$ by Bezdek and Connelly in 2002. Here we will describe ideas in their proof without going into details.

Not quite working idea. Assume one can construct n smooth curves $\alpha_i\colon [0,1] \to \to \mathbb{R}^2$ such that

❽ $\alpha_i(0) = a_i$, $\alpha_i(1) = b_i$ and $\ell_{i,j}(t) = |\alpha_i(t) - \alpha_j(t)|$ is nonincreasing.

In this case one can consider functions

$$v(t) = \mathrm{area}\left(\bigcap_{i=1}^{n} B_R(\alpha(t))\right) \quad \text{and} \quad V(t) = \mathrm{area}\left(\bigcup_{i=1}^{n} B_R(\alpha(t))\right).$$

In order to prove ❻, it is sufficient to show that $v'(t) \geqslant 0$ for all t. Similarly, to prove ❼, it is sufficient to show that $V'(t) \leqslant 0$ for all t. The latter is a calculus problem; it is technical, but straightforward.

As one can see by the following exercise, this approach cannot lead to a solution directly.

7.9. Exercise. *Construct points $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, but there are no curves $\alpha_1, \alpha_2, \alpha_3, \alpha_4\colon [0,1] \to \mathbb{R}^2$ which satisfy ❽.

In their proof, Bezdek and Connelly found a work around which uses the following theorems.

7.10. Alexander's theorem. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$. Viewing $\mathbb{R}^{2\cdot m}$ as $\mathbb{R}^m \times \mathbb{R}^m$, we shall consider $\mathbb{R}^m = \mathbb{R}^m \times \{0\}$ as a coordinate subspace of $\mathbb{R}^{2\cdot m}$. Then there is a choice of curves $\alpha_i\colon [0,1] \to \mathbb{R}^{2\cdot m}$ such that $\alpha_i(0) = a_i = (a_i, 0)$, $\alpha_i(1) = b_i = (b_i, 0)$ and the function $\ell_{i,j}(t) = |\alpha_i(t) - \alpha_j(t)|$ is monotonic for each i and j.*

Proof. Straightforward calculations show that conclusion of the theorem hold for

$$\alpha_i(t) = \left(\frac{a_i + b_i}{2} + \cos(\pi\cdot t)\cdot\frac{a_i - b_i}{2},\ \sin(\pi\cdot t)\cdot\frac{a_i - b_i}{2}\right).$$

□

7.11. Archimedes's theorem. *Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Denote by $\Pi\colon \mathbb{S}^2 \to \mathbb{R}$ a coordinate projection. Then for any subinterval interval $[a,b] \subset \subset [-1,1]$, we have*

$$\operatorname{area}\left[\Pi^{-1}([a,b])\right] = 2\cdot\pi\cdot(b-a).$$

In other words, the area of the unit sphere which lies between two cutting parallel planes of distance h is equal to $2\cdot\pi\cdot h$.

Proof. The set $\Pi^{-1}([a,b])$ is a surface of revolution of function

$$f(x) = \sqrt{1 - x^2}$$

restricted to the interval $[a,b]$. Therefore

$$\begin{aligned}\operatorname{area}\left[\Pi^{-1}([a,b])\right] &= 2\cdot\pi\cdot\int_a^b f(x)\cdot\sqrt{1 + (f'(x))^2}\cdot dx\\ &= 2\cdot\pi\cdot(b-a).\end{aligned}$$

□

Let us denote by $\mathbb{B}^m$ the unit ball in $\mathbb{R}^m$ and vol_m the m-dimensional volume. Note that when we write $\operatorname{vol}_m \mathbb{S}^m$, we are considering the m-dimensional analogue of surface area of the unit sphere, not the $(m+1)$-dimensional volume of the region it bounds in $\mathbb{R}^{m+1}$. Archimedes's theorem admits the following generalization in higher dimensions. The proof goes along the same lines. The case $m = 1$ coincides with the original Archimedes's theorem; we will use only the case $m = 2$.

7.12. Generalized Archimedes's theorem. *Let m be a positive integer. Consider the unit sphere $\mathbb{S}^{m+1} \subset \mathbb{R}^{m+2}$. Denote by $\Pi\colon \mathbb{S}^{m+1} \to \mathbb{R}^m$ a coordinate projection; clearly $\Pi(\mathbb{S}^{m+1}) = \mathbb{B}^m$. Then for any domain $\Omega \subset \mathbb{B}^m$, we have*

$$\operatorname{vol}_{m+1}\left[\Pi^{-1}(\Omega)\right] = 2\cdot\pi\cdot\operatorname{vol}_m \Omega.$$

In particular

$$\mathrm{vol}_{m+1}\,\mathbb{S}^{m+1} = 2{\cdot}\pi{\cdot}\,\mathrm{vol}_m\,\mathbb{B}^m.$$

The working idea of Conjecture 7.7 for $m = 2$. According to Alexander's theorem we can construct curves $\alpha_i\colon [0,1] \to \mathbb{R}^4$ which satisfy ❽. Since we are considering $\mathbb{R}^2$ as a subspace of $\mathbb{R}^4$, given $x \in \mathbb{R}^2 \subset \mathbb{R}^4$ we need to distinguish the notions of balls centered at x in $\mathbb{R}^2$ and balls centered at x in $\mathbb{R}^4$. We will denote them $B_R(x;\mathbb{R}^2)$ and $B_R(x;\mathbb{R}^4)$ respectively. From the Generalized Archimedes's theorem, we get that

❾
$$\begin{aligned}
\mathrm{area}\left(\bigcap_{i=1}^{n} B_R(a_i;\mathbb{R}^2)\right) &= \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^{n} B_R(a_i;\mathbb{R}^4)\right)\\
\mathrm{area}\left(\bigcap_{i=1}^{n} B_R(b_i;\mathbb{R}^2)\right) &= \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^{n} B_R(b_i;\mathbb{R}^4)\right)
\end{aligned}$$

Now consider the function

$$w(t) = \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^{n} B_R(\alpha_i(t);\mathbb{R}^4)\right).$$

If $w(t)$ nonincreasing for all t, then together with ❾ it would imply ❻.

To show that $w(t)$ nonincreasing one needs to calculate its derivative and show that it is nonpositive. The latter follows since

$$w'(t) = -\sum_{i<j} \ell'_{i,j}(t){\cdot}\vartheta_{i,j}(t),$$

for some nonnegative values $\vartheta_{i,j}(t)$. In fact for any fixed t, the value $\vartheta_{i,j}(t)$ can be expressed using only the values $\ell_{i,j}(t)$ and R. (The inequality $\vartheta_{i,j} \geqslant 0$ follows from the geometric interpretation of this number as the area of certain sets in $\bigcap_{i=1}^{n} B_R(\alpha_i(t);\mathbb{R}^4)$, which we do not discuss here.)

The inequality ❼ is proved in a similar way. We have to define

$$W(t) = \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcup_{i=1}^{n} B_R(\alpha_i(t);\mathbb{R}^4)\right)$$

and prove that

$$W'(t) = \sum_{i<j} \ell'_{i,j}(t){\cdot}\Theta_{i,j}(t),$$

for some nonnegative values $\Theta_{i,j}(t)$. Then, apply the Generalized Archimedes theorem to get

$$\mathrm{area}\left(\bigcup_{i=1}^{n} B_R(a_i;\mathbb{R}^2)\right) = W(0),$$

$$\operatorname{area}\left(\bigcup_{i=1}^{n} B_R(b_i;\mathbb{R}^2)\right) = W(1).$$

Hence ❼.

7.13. Exercise. *Try to understand why the same idea does not work in the case $m = 3$.*

7.14. Exercise. *Construct points $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, but

$$\operatorname{length}\left(\partial \bigcup_{i=1}^{4} B_R(a_i)\right) < \operatorname{length}\left(\partial \bigcup_{i=1}^{4} B_R(b_i)\right).$$

7.15. Exercise. *Apply inequality ❼ for $R \to \infty$ to show that if a_1, $a_2, \dots, a_n$, b_1, $b_2, \dots, b_n \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j then

$$\operatorname{length}\left(\partial \operatorname{Conv}(a_1, a_2, \dots, a_n)\right) \geqslant \operatorname{length}\left(\partial \operatorname{Conv}(b_1, b_2, \dots, b_n)\right).$$

Mid-term exam

- ◇ On Wednesday, Oct 12, 10–12, we have an exam.
- ◇ Review is Tuesday, Oct 11, from 3 p.m. to the last student.

Problem 1+2. Two of the exercises from the list (the choice is mine):
1.2, 1.4 1.18, 1.23, 1.24, 1.30,
2.2, 2.4, 2.5, 2.10, 2.14, 2.15,
3.2, 3.3, 3.5, 3.6, 3.9, 3.10, 3.11, 3.13,
4.5, 4.8, 4.15, 4.16,
5.11, 5.12,
6.1, 6.11.

Problem 3. Two statements (or its part) from the list (the choice is mine):
1.25, 1.27,
2.1, 2.3, 2.6, 2.12,
3.4, 3.12, 3.A,
4.3, 4.6, 4.11, 4.12, 4.13,
5.5
6.6

Problem 4. One of HWA problems from the list (the choice is mine):
1.A,B,C,D;
2.A,C,D;
3.B,C;
4.A,B,C,D;
5.A,C,D;
6.A,B,C,D.

Problem 5. One new problem.

8 The surface of a convex polyhedron

Let K be a non-degenerate convex polyhedron in $\mathbb{R}^3$. The boundary of K equipped with the induced intrinsic metric will be called the *surface of K*. According to Exercise 5.8, the surface of a polyhedron is a polyhedral space.

8.1. Proposition. *Let P be the surface of a non degenerate convex polyhedron. Then P is homeomorphic to $\mathbb{S}^2$.*

Proof. Without loss of generality, we may assume that the origin $o \in \mathbb{R}^3$ lies in the interior of K. Consider the map $f\colon P \to \mathbb{S}^2$ defined by $f(x) = x/|x|$; in other words, a point $x \in P$ is mapped to the intersection of the ray $[ox)$ with the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

Since K is convex and the origin o is an interior point, for any $z \neq o$, the ray $[oz)$ intersects P at exactly one point. It follows that, $f\colon P \to \mathbb{S}^2$ is a bijection. Since $o \notin P$, the map $f\colon P \to \mathbb{S}^2$ is continuous. Hence $f\colon P \to \mathbb{S}^2$ is a homeomorphism; see Exercise 1.19. □

We shall be interested in when an abstract polyhedral space P can be realized as the surface of a convex polyhedron in $\mathbb{R}^3$. Proposition 8.1 says that a necessary condition for this is that P must be homeomorphic to $\mathbb{S}^2$. To formulate another necessary condition, we have to define the curvature at a point in P.

8.2. Definition. *Let P be a polyhedral space which is homeomorphic*[42] *to $\mathbb{S}^2$. Given $p \in P$, consider a triangulation of P for which p is a vertex. Denote by α_p the sum of the angles around p The value α_p will be called the* total angle around p *and the value*

$$\omega_p = 2{\cdot}\pi - \alpha_p$$

will be called the curvature of P at p.

8.3. Exercise. *Show that for any $p \in P$, there is a triangulation as described in the above definition, and that the curvature of P at p does not depend on the choice of such a triangulation.*

8.4. Exercise. *Assume P is a polyhedral space that is homeomorphic to a sphere and $\mathcal{T}$ is a triangulation of P. Show that if the curvature of P at p is nonzero then p is a vertex of $\mathcal{T}$.*

8.5. Exercise. *Assume P is a polyhedral space that is homeomorphic to a sphere and $\mathcal{T}$ is a triangulation of P. Show that the sum of the curvatures of P at all vertexes of $\mathcal{T}$ is equal to $4{\cdot}\pi$.*

Hint: Let k, l and m be the number of vertexes, edges, and triangles respectively in the triangulation $\mathcal{T}$. Use that sum of angles of in any triangle is π together with Euler's formula $k - l + m = 2$ and the identity $3{\cdot}m = 2{\cdot}l$.

[42] Instead of $\mathbb{S}^2$ one may take any 2-dimensional manifold, but I avoid to use the term "manifold" since I did not define it yet.

8.6. Exercise. *Let P be the surface of a convex polyhedron K. Show that the curvature of any point $p \in P$ is non-negative.*

Hint: Consider the space K_p, which is the intersection of K with a small sphere centered at p. Note that K_p is a convex spherical polygon. Then apply the idea used in the proof of Lemma 2.12.

Note that Proposition 8.1 holds in the degenerate case where K is a flat polygon, but in this case the "surface" of K should be defined differently. You have to imagine that you are living in $\mathbb{R}^3$ and can walk on a flat polygon made from rigid material; so to get from one of its sides to the other, you have to travel over a boundary edge.

More formally, we define the surface of a convex polygon K as its doubling; i.e., two copies of K glued along the boundary. Since K is homeomorphic to a disc, the doubling of K is homeomorphic to a sphere.

Proposition 8.1 and Exercise 8.6 together say that if a polyhedral space P is isometric to the surface of a convex polyhedron in $\mathbb{R}^3$, then P is homeomorphic to $\mathbb{S}^2$ and that the curvature of P is non-negative everywhere. Alexandrov's theorem provides a converse to this statement.

8.7. Alexandrov's theorem. *Let P be a polyhedral space which is homeomorphic to $\mathbb{S}^2$. Assume that the curvature of P is non-negative at each point. Then P is isometric to the surface of a convex polyhedron K in $\mathbb{R}^3$ (which may possibly degenerate to a polygon in a plane.)*

Moreover, the polyhedron K is unique up to isometry of $\mathbb{R}^3$.

This theorem is one of our big goals, the proof will take few lectures.

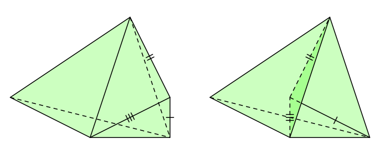
Uniqueness is an important step in the proof. Namely, we will first have to show the following.

8.8. Alexandrov's uniqueness theorem. *Any two convex polyhedra in $\mathbb{R}^3$ with isometric surfaces are congruent; i.e., they are the same up to isometry of $\mathbb{R}^3$.*

In particular, Alexandrov's uniqueness theorem implies Cauchy's theorem, which roughly says that *if we know each face of a convex polyhedron K and we know the rules that determine how these faces are attached to each other, then we know K up to isometry of $\mathbb{R}^3$.*

Flexible polyhedra

Without convexity, Alexandrov's uniqueness theorem (8.8) does not hold. In the picture you can see an example — two non-congruent polyhedra with isometric surfaces. One of the polyhedra is obtained from the other by reflecting 3 faces in a plane.



It is impossible to get from one these polyhedra to the other one by a continuous deformation which keeps each face rigid. In fact, both of these polyhedra are *rigid*; i.e., any continuous deformation of them which keeps each face rigid is given by isometries of $\mathbb{R}^3$.

8.9. Exercise. *Prove the last statement.*

In this section we construct examples of *flexible polyhedra*, i.e., a polyhedron which admits such a continuous deformation.

To make the statement precise we need the following technical definition. It is meant to capture a natural property of the following physical object.

Assume that we have a finite 2-dimensional simplicial complex $\mathcal{S}$ and a map $f\colon \mathcal{S} \to \mathbb{R}^3$ which is linear on each triangle. Equip $\mathcal{S}$ with the polyhedral metric so that f is distance preserving on each triangle. Let k be the number of vertexes in $\mathcal{S}$; label the images under f of the vertexes of $\mathcal{S}$ by $\{w_1, w_2, \dots, w_k\}$. If a pair (w_i, w_j) is connected by an edge in $\mathcal{S}$, let us connect w_i to w_j by a rigid bar and connect all the bars coming from one w_i with joint-hinge. If the obtained model admits a motion different from an isometry of Euclidean space, then f is said to be flexible; if not, then f is rigid.

Alternatively, one may also cut each triangle of $\mathcal{S}$ from a rigid material and join all these triangles by hinges along the edges. To make this new model, one has to assume in addition that f is an embedding; otherwise the triangles in $\mathcal{S}$ will intersect at their interior points (which would be hard to imagine in the physical world).

8.10. Definition. *Let $\mathcal{S}$ be a 2-dimensional simplicial complex and let $f\colon \mathcal{S} \to \mathbb{R}^3$ be a map that is linear on each triangle. The map f is called* flexible *if there is one parameter family of maps $f_t\colon \mathcal{S} \to \mathbb{R}^3$, $t \in [0,1]$ such that*

1. *$f_0 = f$;*
2. *the map $[0,1] \times \mathcal{S} \to \mathbb{R}^3$ defined as $(t,x) \mapsto f_t(x)$ is continuous*[43]*;*
3. *for each $t \in [0,1]$, the map f_t is linear on each triangle in $\mathcal{S}$, and for each edge e_j of $\mathcal{S}$ the length of $f_t(e_j)$ does not depend on t;*
4. *f_1 is different from f_0 in the sense that there is no isometry $\iota\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f_1 = \iota \circ f_0$.*

If the map $f\colon \mathcal{S} \to \mathbb{R}^3$ is not flexible, then f is called rigid.

8.11. Bricard–Connelly theorem. *There is a simplicial complex $\mathcal{S}$ that is homeomorphic to $\mathbb{S}^2$ which admits a flexible map $f\colon \mathcal{S} \to \mathbb{R}^3$.*

Moreover f can be chosen to be an embedding.

An example satisfying the first statement of this theorem was constructed by Raoul Bricard in 1897; his examples are called Bricard's octahedra, but they all fail to be embeddings because they have self-intersections. In 1977, Robert Connelly found a way to use Bricard's octahedra to produce a flexible embedding; this is the construction that additionally satisfies the second statement of

[43] A one parameter family of maps which satisfies these two conditions is called a *homotopy* of f.

theorem. The simplicial complex $\mathcal{S}$ in his construction has 24 vertexes which will be presented here.

Bricard's octahedra. There are two types Bricard's octahedra, which we shall refer to as the first and second type. In both cases, Bricard's octahedron, say P, is glued out of 8 triangles in the same way as a regular octahedron, and it has certain symmetry which depends on if it is the first or second type. The triangulation is the same as in an octahedron; it has 8 triangles, 12 edges and 6 vertexes. Bricard's octahedra are isometric to the surface of convex non-regular octahedra[44] , say K, but the flexible map f does not map P to the surface of K.

The 6 vertexes of the octahedra will be denoted as x, y, z, x', y', z', any pair of these vertexes is connected by an edge, with exception for the 3 pairs (x, x'), (y, y') and (z, z').

- ◇ In Bricard's octahedra of the first type, we assume that K has center of symmetry; i.e. the midpoints of the line segments $[xx']$, $[yy']$ and $[zz']$ coincide.
- ◇ In Bricard's octahedra of the second type, we assume there is a line of symmetry, say in the line (xx')[45].

Note that in both cases, the points x, x', y and y' lie on one plane, say Π. The flexible map is obtained by reflecting z' along with the 4 edges coming from z' through Π. The new map is distance preserving on each of the 8 triangles of $\mathcal{T}$.

To see that it is flexible, note first that the part glued out of 4 triangles with vertex z is flexible[46]; in other words we can fix vertexes x, y and z and move x' and y' along curves $x'(t)$, $y'(t)$ such that $x'(0) = x'$, $y'(0) = y'$ all of the distances

$$|z - x'(t)|, \quad |z - y'(t)|, \quad |y - x'(t)|, \quad |x'(t) - y'(t)|, \quad |y'(t) - x|$$

stay constant while t changes.

- ◇ For Bricard's octahedra of the first type, set $z'(t)$ to be the rotation of z by angle π around the line passing through the midpoints of line segments $[x, x'(t)]$ and $[y, y'(t)]$. In this case we have

$$\begin{aligned} |z'(t) - x| &= |z - x'(t)|, & |z'(t) - y| &= |z - y'(t)|, \\ |z'(t) - x'(t)| &= |z - x|, & |z'(t) - y'(t)| &= |z - y|. \end{aligned}$$

- ◇ For Bricard's octahedra of the second type set $z'(t)$ to be reflection of z in the plane passing through $x, x'(t)$ and the midpoint of line segment $[y, y'(t)]$. In this case we have

$$\begin{aligned} |z'(t) - x| &= |z - x|, & |z'(t) - y| &= |z - y'(t)|, \\ |z'(t) - x'(t)| &= |z - x'(t)|, & |z'(t) - y'(t)| &= |z - y|. \end{aligned}$$

[44] Well, we consider only these type of Bricard's octahedra.

[45] i.e. rotation by angle π arouind (xx') sends z to z' and y to y'.

[46] This is true if each angle $\angle xzy$, $\angle xzy'$, $\angle x'zy$, $\angle x'zy'$ is strictly larger than the sum of the remaining three angles. In particular, if the spherical quadrilateral Q with vertexes formed by unit vectors in the directions of rays $[zx)$, $[zy)$, $[zx')$ and $[zy')$ is not degenerate. Note that flexibility of Q in the sphere is equivalent to flexibility of our 4 triangles in $\mathbb{R}^3$.

In both cases, we have that

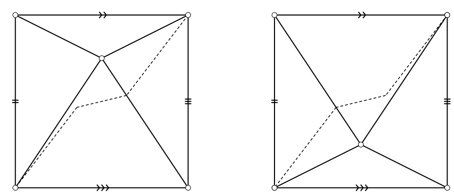
$$|z'(t) - x|, \qquad |z'(t) - y|, \qquad |z'(t) - x'(t)|, \qquad |z'(t) - y'(t)|$$

stay constant while t changes.

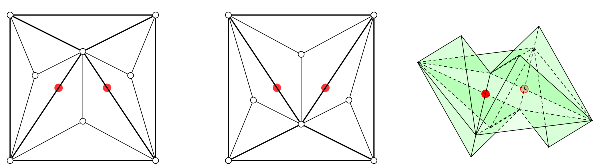
You may look at the animated images of Bricard's octahedron of the first type and the second type.

Connelly sphere. Now we will sketch Connelly's idea which uses 3 Bricard's octahedra to construct a flexible embedding of a polyhedral surface with 24 vertexes.

Start with a Bricard's octahedron of the first type, as on the picture. Here you see "upper" and "lower" sides[47] , both are squares and the triangulation is marked by solid lines. The dashed line is a line of self-intersection after a small deformation. The identified sides are decorated the same way.



Remove 3 big triangles from the upper side and exchange each by 3 triangles with common vertex quite a bit above of the corresponding triangle. This way you exchange 3 triangles to 9 and add 3 extra vertexes. Do the same for the lower side, but choose the vertex quite a bit below the corresponding triangle. The next picture shows how it will look from above, form below and a side view. The new model is still flexible and for small deformations its self intersections appear only at the spots marked by red.



Let us cover a red spot on the upper side by two small triangles joined along the edge. Construct a Bricard's octahedron of the second type with these two triangles as faces. This new model is still flexible and we can remove the two small triangles from it. For the right choice of the small triangles and the Bricard's octahedron, we can get rid of this self-intersection.

If the same is done to the other red spot, we get a flexible embedding.

Estimates of degree of freedom.

Here we explain roughly explain the reason why it is hard to construct a flexible map; i.e., why a randomly chosen map cannot be flexible (or better to say is not flexible with probability 1). We are not proving anything here, we only

[47]These sides on the same level, we call one upper and one lower to distiguish them.

count number of parameters and the number of constraints and show that these numbers are equal. So if the constraints were "independent," we would get rigidity.

Let k, l and m be the number of vertexes, edges, and faces respectively in a simplicial complex $\mathcal{S}$ which is homeomorphic to the sphere. Label the vertexes of $\mathcal{S}$ as $\{v_1, v_2, \dots, v_k\}$.

We are interested in maps $f\colon \mathcal{S} \to \mathbb{R}^3$ which are linear on each triangle. Each such map is completely determined by values

$$w_i = f(v_i) = (x_i, y_i, z_i) \in \mathbb{R}^3.$$

We want to distinguish maps up to an isometry of $\mathbb{R}^3$, so we can assume that w_1 is the origin of $\mathbb{R}^3$; w_2 lies on x-axis and w_3 lies in the xy-plane. In other words,

$$w_1 = (0,0,0), \qquad w_2 = (x_2,0,0), \qquad w_3 = (x_3,y_3,0)$$

Thus, up to isometry of $\mathbb{R}^3$, the map f is completely described by $3{\cdot}k - 6$ numbers

❶ $$(x_2), (x_3, y_3), (x_4, y_4, z_4), \dots, (x_k, y_k, z_k).$$

Now let us count number of constraints. We want to preserve the length of each edge in $\mathcal{S}$; i.e., we have if vertexes v_i and v_j are connected by an edge in $\mathcal{S}$ then

❷ $$|w_i - w_j|_{\mathbb{R}^3} = a_{ij},$$

where a_{ij} is the length of this edge. (The equation ❷ is quadratic if written in the variables x_i, y_i, z_i, x_j, y_j and z_j) All together f has to satisfy l equations of the form ❷.

According to Euler's formula, we get

$$k - l + m = \chi(\mathbb{S}^2) = 2.$$

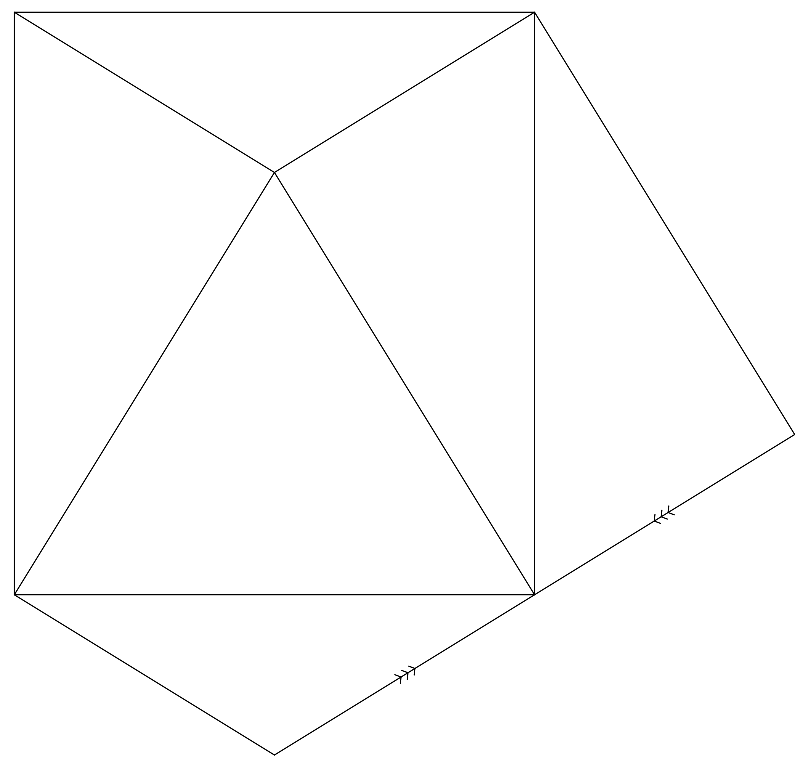
Further each edge appears as a side in exactly two triangles and each triangle has three sides; i.e., we have $3{\cdot}m = 2{\cdot}l$. Therefore

$$l = 3{\cdot}k - 6.$$

Thus, the number of equations (l) coincides with the number of parameters ($3{\cdot}k - 6$) in ❶.

HWA 7; due Fri, Oct 28

8.A. Print this page, cut the following figure and glue the marked sides.



You get 6 out of 8 triangles in the Bricard's octahedron. I hope you can imagine the remaining two triangles.

i) Is it first or second type of Bricard's octahedron?

ii) Move this model, folding only along marked lines to make another flat polygon (not the square which you started with).

Attach it to your homework.

Try to understand which Bricard's octahedra can be moved into two distinct flat positions.

8.B. Assume that the surface of a nonregular tetrahedra T has curvature π at each of it vertexes. Show that

i) all faces of T are congruent;

ii) the line passing through midpoints of opposite edges of T itersects these edges at right angles.

8.C. How many combinatorically different[48] triangulations of $\mathbb{S}^2$ with 5 vertexes are there? Show that any map for corresponding simplicial complex(es) such that no 4 vertexes lie in one plane is(are) rigid.

8.D. Do all exercises in the last lecture, write a solution of one of your choice.

---

[48] Two triangulations $\mathcal{T}$ and $\mathcal{T}'$ are the same combinatorially if there is a bijection $f : V \to V'$ between their vertex sets such that

i) there is an edge from v to w in $\mathcal{T}$ if and only if there is an edge from $f(v)$ to $f(w)$ in $\mathcal{T}'$;

ii) there is a triangle connecting u, v, w in $\mathcal{T}$ if and only if there is a triangle connecting $f(u), f(v), f(w)$ in $\mathcal{T}'$.

9 Cauchy's theorem

The first step in the proof of Alexandrov's theorem (8.7) is Alexandrov's uniqueness theorem (8.8) which in turn generalizes Cauchy's theorem formulated below. We start with the proof of Cauchy's theorem and then modify it to prove Alexandrov's uniqueness theorem. These proofs are written nicely in many books and the presentation below is just a short extract. We would recommend to read sections 2.1 and 3.1 in Alexandrov's book [1]. If you want to get an idea about other interesting proofs, look in Pak's lecture notes [3].

9.1. Cauchy's theorem. *Let K and K' be two non-degenerate convex polyhedra in $\mathbb{R}^3$, and denote their surfaces*[49] *by P and P'. If there is an isometry $P \to P'$ which sends each face of K to a face of K', then K is congruent to K'.*

Let us first break the proof into two parts, "local" and "global", which will be proved in the following sections.

Frame of the proof. Consider the graph Γ formed by the edges of K (equivalently, the edges of K').

For an edge e in Γ,

◇ denote by α_e the corresponding dihedral angle of K;

◇ denote by α'_e the corresponding dihedral angle of K'.

Mark an edge e of Γ with $(+)$ if $\alpha_e < \alpha'_e$ and with $(-)$ if $\alpha_e > \alpha'_e$.

Now remove from Γ everything which was not marked; i.e., leave only the edges marked by $(+)$ or $(-)$ and their endpoints. The statement of Cauchy's theorem is equivalent to the fact that Γ is an empty graph. Let us assume the contrary and try to arrive at a contradiction.

Note that Γ is embedded into P, which is homeomorphic to $\mathbb{S}^2$ (see Proposition 8.1). In particular, the edges coming from one vertex have a natural cyclical order. Given a vertex v of Γ, we can count the *number of sign changes* around v; i.e., the number of pairs of adjacent edges which are marked by different signs.

9.2. Local lemma. *For any vertex of Γ the number of sign changes is at least* 4.

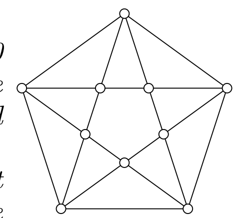
In other words, the local lemma states that at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$ and $(-)$ which are in the same cyclical order.

Once the Local lemma is proved, we get a contradiction by applying the following.

9.3. Global lemma. *Let Γ be a nonempty sub-graph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of Γ by $(+)$ or $(-)$ such that the number of sign changes around each vertex of Γ is at least* 4. □

[49] Their boundaries equipped with the induced intrinsic metric.

9.4. Exercise. *Assume that we glue one pentagon and 10 triangles in $\mathbb{R}^3$ along the rule shown in the picture. Assume that it forms a part of a surface of a convex polyhedron and each vertex is a vertex of the polyhedron.*



Show that this configuration is rigid; say one can not fix the position of the pentagon and continuously move the remaining 5 vertexes in a new position so that each triangle moves by a one parameter family of isometries of $\mathbb{R}^3$.

Local lemma

To prove the Local lemma, we will need the following.

9.5. Arm lemma. *Assume that $A = [a_0 a_1 \dots a_n]$ is a convex polygon in $\mathbb{R}^2$ and $A' = [a'_0 a'_1 \dots a'_n]$ be a closed broken line in $\mathbb{R}^3$ such that*

$$|a_i - a_{i+1}| = |a'_i - a'_{i+1}|$$

for any $i \in \{0, \dots, n-1\}$ and

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each $i \in \{1, \dots, n-1\}$. Then

$$|a_0 - a_n| \leqslant |a'_0 - a'_n|$$

and equality holds if and only if A is congruent to A'.

One may view the broken lines $[a_0 a_1 \dots a_n]$ and $[a'_0 a'_1 \dots a'_n]$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the "shoulder" and "tips of the fingers" increases. The following proof was given by Zaremba.

9.6. Exercise. *Show that the arm lemma does not hold if instead of the convexity condition, one only assumes that when you go along the broken line $a_0 a_1 \dots a_n$, then you only turn left.*

Proof. We will view $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$, so that both A and A' lie in $\mathbb{R}^3$. Let a_m be the vertex of A which has maximal distance to the line $(a_0 a_n)$.

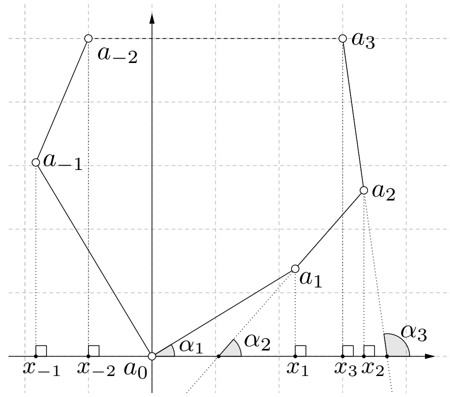
Cyclically shift indexes of a_i and a'_i down by m, so that

$$\begin{aligned} a_{-m} &:= a_0, & a'_{-m} &:= a'_0, \\ a_{-(m-1)} &:= a_1, & a'_{-(m-1)} &:= a'_1, \\ &\vdots & &\vdots \\ a_0 &:= a_m, & a'_0 &:= a'_m, \\ &\vdots & &\vdots \\ a_k &:= a_n & a_k &:= a_n, \end{aligned}$$

where $k = n - m$. The symbol ":=" means *an assignment statement* as in programming (the order of variables in an assignment statement is important: $a := b$ means that both a and b take the value b and $b := a$ means that both take the value a).

Without loss of generality, we may assume that

- ◇ $a_0 = a'_0$ and they both lie in the origin in $\mathbb{R}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m}a_k)$;
- ◇ the angle $\angle a'_0$ lies in xy-plane and contains the angle $\angle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a'_1 from a_0 appear in the same cyclic order.



Denote by x_i and x'_i the projections of a_i and a'_i to x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case

$$|a_k - a_{-m}| = x_k - x_{-m} \quad \text{and} \quad |a'_k - a'_{-m}| \geqslant x'_k - x'_{-m}.$$

The latter follows because projection is a distance nonincreasing map. Therefore it is sufficient to show that

$$x'_k - x'_{-m} \geqslant x_k - x_{-m}.$$

This holds if

❶ $$x'_i - x'_{i-1} \geqslant x_i - x_{i-1}.$$

for each i.

It remains to prove ❶; in the proof we assume that $i > 0$, the case $i \leqslant 0$ is done the same way. Let us

- ◇ denote by α_i the angle between the vector $w_i = a_i - a_{i-1}$ and the x-axis;
- ◇ denote by α'_i the angle between the vector $w'_i = a'_i - a'_{i-1}$ and the x-axis.

Note that

❷ $$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}|\cdot \cos\alpha_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}|\cdot \cos\alpha'_i \end{aligned}$$

for each $i > 0$. By construction $\alpha_1 \geqslant \alpha'_1$. Note that $\angle(w_{i-1}, w_i) = \pi - \angle a_i$. From convexity of $[a_1 a_1 \dots a_i]$ for $i > 0$, we have

$$\alpha_i = \alpha_1 + (\pi - \angle a_1) + \dots + (\pi - \angle a_i).$$

Since $\angle(w'_{i-1}, w'_i) = \pi - \angle a'_i$, from the triangle inequality for angles[50], we have

$$\alpha'_i \leqslant \alpha'_1 + (\pi - \angle a'_1) + \dots + (\pi - \angle a'_i).$$

Since $\angle a_j \leqslant \angle a'_j$ for each j, summing it we get

$$\alpha_i \geqslant \alpha'_i.$$

Applying ❷, we get ❶.

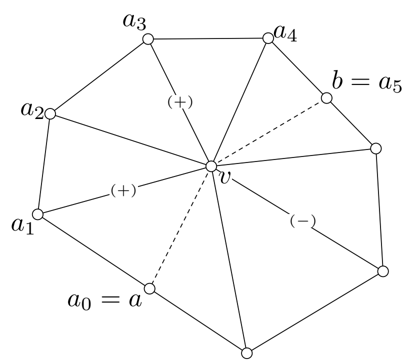
In the case of equality, we have $\alpha_i = \alpha'_i$ for each i, that implies $\angle a_i = \angle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the following exercise.

9.7. Exercise. *Consider a broken line $[a_0 a_1 a_2 a_3]$ in $\mathbb{R}^3$. Denote by $\vartheta_{i,j}$ the angle between the vectors $a_i - a_{i-1}$ and $a_j - a_{j-1}$. Then*

$$\vartheta_{1,3} \leqslant \vartheta_{1,2} + \vartheta_{2,3}$$

and in case of equality, the broken line lies in a plane. □

Proof of Local lemma (9.2). Assume that the Local lemma does not hold at the vertex v of Γ. Then one can choose two points a and b near v on the surface P so that on one side of the rays $[va)$ and $[vb)$ we have only $(+)$'s, and on the other side only $(-)$'s.



Cut a small pyramid with vertex v from K by a plane passing through a and b. The base polygon is formed by two broken lines with ends at a and b. Assume that

$$a = a_0,\ a_1,\ \dots,\ a_n = b$$

form the broken line along the side marked with $(+)$'s. Denote by

$$a' = a'_0,\ a'_1,\ \dots,\ a'_n = b'$$

the corresponding points in P'. Since each marked edge passing through a_i has a $(+)$ on it or nothing, we have

$$\angle a_{i-1} a_i a_{i+1} \leqslant \angle a'_{i-1} a'_i a'_{i+1}$$

for each i.

9.8. Exercise. *Prove the last statement.*

[50] Equivalently, this is the triangle inequality on the unit sphere $\mathbb{S}^2$ with the induced intrinsic metric. For any two nonzero vectors v_1 and v_2 in $\mathbb{R}^3$, one can consider their normalized (i.e. unit vector) versions and the corresponding points on $\mathbb{S}^2$. Note that the distance (in $\mathbb{S}^2$) between these points is the length of the arc of the great circle connecting these two points, which is equal to the angle between v_1 and v_2.

By construction we have $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ for all i. By the Arm lemma (9.5), we get $|a-b| \leqslant |a'-b'|$ and equality holds if no edge from v is marked with a $(+)$.

Repeating the same construction exchanging the places of K and K' gives $|a-b| \geqslant |a'-b'|$ and equality holds no edge from v is marked with a $(-)$.

Since we do have the equality $|a-b| = |a'-b'|$, it follows that no edge at v is marked, a contradiction. □

Global lemma

Before going into the proof, we suggest to do the following.

9.9. Exercise. *Show that all edges of an octahedron cannot be marked by $(+)$'s and $(-)$'s such that we have 4 sign changes at each vertex.*

The proof of the Global lemma is based on counting the sign changes in two ways; the first is as one moves around each vertex of Γ and the second is as one moves around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in moving around the region to whose boundary they belong. The converse is true as well. Therefore, both of the ways of counting give the same number.

Proof of 9.3. We can assume that Γ is connected[51] (if not, pass to any connected component of Γ.) Denote by k and l the number of vertexes and edges respectively in Γ. Denote by m the number of *regions* which Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

9.10. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❸ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertexes. By the Local lemma (9.2), we have

❹ $$4 \cdot k \leqslant s.$$

Now let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions which are bounded by n edges; if an edge appear twice when you go around the regions it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❺ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2 \cdot l = 3 \cdot m_3 + 4 \cdot m_4 + 5 \cdot m_5 + \dots .$$

[51] That is, one can get from any vertex to any other vertex by walking along edges.

Combining this with Euler's formula (❸), we get

❻ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even number which is at most n. Therefore

❼ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❹ and ❼ contradict ❻. □

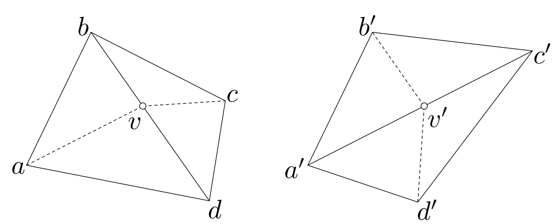
Alexandrov's uniqueness theorem

Alexandrov's uniqueness theorem states that the conclusion of Cauchy's theorem (9.1) still holds if one removes the phrase "which sends each face of K to a face of K'" from it. For your convenience we repeat the formulation here:

9.11. Alexandrov's uniqueness theorem. *Any two convex polyhedra in $\mathbb{R}^3$ with isometric surfaces are congruent.*

The proof is along the same lines as the proof of Cauchy's theorem. We will only describe the necessary modifications.

Let $\iota\colon P \to P'$ be an isometry. Mark in P all the edges of K and all the ι-preimages of edges of K', which will further be called fake edges. These lines divide P into convex polygons, say $\{Q_i\}$, and the restriction of ι to each Q_i is a rigid move. These polygons will play the role of faces in the proof of Cauchy's theorem.



A fake vertex $v \in K$ and the corresponding point $v' \in K'$.

A vertex of Q_i can be a vertex of K or it can be a fake vertex; i.e., lie on intersection of an edge and fake edge. For the first type of vertex, the Local lemma can be proved in exactly the same way. For a fake vertex v, it is easy to see that both parts of the edge coming through v are marked with $(+)$ while both of the fake edges at v are marked with $(-)$. Therefore, the Local lemma holds for the fake vertexes as well.

The remainder of the proof needs no further modifications.

HWA 8; due Fri, Nov 4

9.A. Show that the sum of the exterior angles of any closed broken line in $\mathbb{R}^3$ is at least $2{\cdot}\pi$.

Hint: Use induction on the number of edges.

Use the previous problem to solve the next one.

9.B. Consider a convex polygons $A = [a_1 a_2 \dots a_n]$ in $\mathbb{R}^2$ and a closed broken line $A' = [a'_1 a'_2 \dots a'_n]$ in $\mathbb{R}^3$. Let us enumerate the vertexes in an n-periodic way; i.e., set $a_{n+k} = a_k$ and $a'_{n+k} = a'_k$ for any k.

Assume $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ and $|a_i - a_{i-2}| \leqslant |a'_i - a'_{i-2}|$ for all i. Show that A is congruent to A'.

9.C. Give a complete proof of the Arm lemma (9.5) using the following plan.

Use induction on n. Prove the base case $n = 2$.

For a point b on the broken line $[a_0 a_1 \dots a_n]$, denote by b' the corresponding point on $[a'_0 a'_1 \dots a'_n]$; i.e., if $b \in [a_{i-1} a_i]$ then $b' \in [a'_{i-1} a'_i]$ and $|a_i - b| = |a'_i - b'|$.

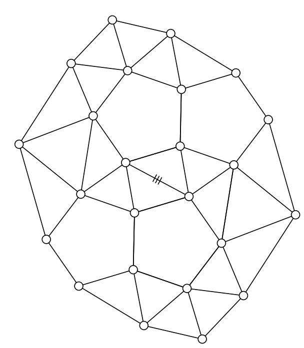
Assume $|a_0 - a_n| \geqslant |a'_0 - a'_n|$. Applying the induction hypothesis, we have the opposite inequality for all remaining pairs; i.e., $|a_i - a_j| \leqslant |a'_i - a'_j|$ if $|i - j| < n$. Moreover for any point b on the broken line $[a_0 a_1 \dots a_n]$, we have $|a_i - b| \leqslant |a'_i - b'|$ if $0 < i < n$.

Choose $b \in [a_{n-1} a_n]$ to be the closest point to a_n auch that $|a_0 - b| = |a'_0 - b'|$.

Show that the case $b = a_{n-1}$ can be reduced to the case $n = 2$.

In the remaining case, apply the previous problem to the broken lines $[a_0 a_1 \dots \dots a_{n-1} b]$ and $[a'_0 a'_1 \dots a'_{n-1} b']$.

9.D. Assume that we glue in $\mathbb{R}^3$ four regular pentagons and 22 equilateral triangles along the rule shown on the picture such that they form a part of surface of a convex polyhedron.



Use the Local lemma to show that this configuration has a rotational symmetry with axis passing through the midpoint of the marked edge.

9.E. Do all exercises in the last lecture, write a solution of one of your choice.

10 Existence of polyhedron with given surface

Now we turn to the proof of existence in Alexandrov's theorem (8.7); it can be formulated in the following way.

10.1. Alexandrov's existence theorem. *Let P be a polyhedral space which is homeomorphic to $\mathbb{S}^2$. If P has non-negative curvature at each point, then there is a convex polyhedron K whose surface is isometric to P.*

Recall that the convex polyhedron in the theorem can degenerate to a convex polygon. In this case its surface is defined as its doubling; i.e., two copies of the polygon glued along the boundary. Note that such a space is indeed homeomorphic to $\mathbb{S}^2$ and has non-negative curvature everywhere.

Proof. The proof is by induction on k, where k denotes the number of points in P with positive curvature. From now on, these points will be called *vertexes* of P.

10.2. Exercise. *Show that $k \geqslant 3$.*[52]

10.3. Exercise. *Show that k is finite.*[53]

Base case; $k = 3$. Assume that P has exactly three vertexes, say u, v, and w. It is sufficient to show that

❶ *P is isometric to a doubling of a planar triangle*[54].

Choose geodesics $[uv]$, $[vw]$ and $[wu]$ between each pair of points.

10.4. Exercise. *Show that these geodesics do not intersect each other at the interior points.*

Cut P along the geodesics $[uv]$, $[vw]$ and $[wu]$. As a result we get two triangles; since all the remaining points have curvature 0, these triangles are isometric to planar triangles. They are congruent since they have the same side lengths. Hence ❶ follows.

The proof of the induction step will occupy the next few following sections. Here we only break the proof into smaller pieces.

Let us denote by $\mathbf{M}_k$ the set of isometry classes of all polyhedral metrics on $\mathbb{S}^2$ with non-negative curvature and exactly k vertexes (i.e., points with curvature > 0). Denote by $\mathbf{P}_k$ the set of congruence classes of convex polyhedra in $\mathbb{R}^3$ with exactly k vertexes.

Given a convex polyhedron K with k vertexes, consider its surface P. According to Exercise 8.6, P also has k vertexes. In this way we construct a map $\Phi_k\colon \mathbf{P}_k \to \mathbf{M}_k$ which sends (the isometry class of) K to (the isometry class of) P. By Alexandrov's uniqueness theorem (8.8 and 9.11), Φ_k is injective. To prove the existence theorem, it is sufficient to show that

[52] Hint: show that curvature of each point is strictly less than $2\cdot\pi$ and use Exercise 8.5.
[53] Hint: Use Exercise 8.4.
[54] The doubling of a triangle is by definition the surface of this triangle in $\mathbb{R}^3$.

❷ $\Phi_k\colon \mathbf{P}_k \to \mathbf{M}_k$ *is a surjection.*

The case $k = 3$ is already proved. Further, we give a plan for the induction step; i.e., we prove ❷ assuming that all Φ_n are surjections if $3 \leqslant n < k$.

Part 1. We equip the polyhedral spaces from $\mathbf{M}_k$ and the polyhedra from $\mathbf{P}_k$ with extra structure — a chosen order of vertexes and an orientation. For brevity we will call the obtained objects *equipped polyhedral spaces* and *equipped polyhedra* correspondingly. The collections of equipped polyhedral spaces and equipped polyhedra form new sets $\mathbf{M}_k^\#$ and $\mathbf{P}_k^\#$ respectively. For these sets there is a natural map

$$\Phi_k^\#\colon \mathbf{P}_k^\# \to \mathbf{M}_k^\#$$

which is defined using $\Phi_k\colon \mathbf{P}_k \to \mathbf{M}_k$ and transferring the extra structure from the polyhedron to its surface.

Once we define $\mathbf{M}_k^\#$, $\mathbf{P}_k^\#$ and $\Phi_k^\#$, it will be clear that

- ⋄ $\Phi_k^\#$ is injective as well as Φ_k.
- ⋄ $\Phi_k^\#$ is surjective if and only if Φ_k is surjective.

For these sets $\mathbf{M}_k^\#$ and $\mathbf{P}_k^\#$ we will introduce metrics and show that

- ⋄ the map $\Phi_k^\#\colon \mathbf{P}_k^\# \to \mathbf{M}_k^\#$ is continuous with respect to these metrics.
- ⋄ both $\mathbf{M}_k^\#$ and $\mathbf{P}_k^\#$ are $(3{\cdot}k - 6)$-dimensional manifolds in the sense of the following definition.

10.5. Definition. *A metric space A is called an m-dimensional manifold if it is locally homeomorphic to $\mathbb{R}^m$; i.e., for any point $a \in A$ there is an open neighborhood*[55] *Ω of a which is homeomorphic to an open subset of $\mathbb{R}^m$.*

I.e., instead of proving claim ❷, it is sufficient to prove to the following

❸ $\Phi_k^\#\colon \mathbf{P}_k^\# \to \mathbf{M}_k^\#$ *is a surjection.*

The main advantage of proving ❸ instead of ❷ is that $\mathbf{M}_k^\#$ and $\mathbf{P}_k^\#$ are manifolds of the same dimension, whereas $\mathbf{M}_k$ and $\mathbf{P}_k$ are not manifolds.

Part 2. In order to prove ❸, we will show that the map $\Phi_k^\#$ satisfies the conditions in the following lemma.

10.6. Mapping Lemma. *Let A and B be two manifolds of the same dimension. Suppose $f\colon A \to B$ is a continuous injective map satisfying the following conditions:*

a) Each connected component[56] *of B contains the image of some point in A;*

b) For any sequence of points a_n in A, if the sequence $b_n = f(a_n)$ converges to $b_\infty \in B$, then a subsequence of a_n converges to a point $a_\infty \in A$. In particular $f(a_\infty) = b_\infty$.
In other words, the image $f(A)$ is closed in B.

[55] An open neighborhood of a means an open subset containing a.

[56] The connected component of the point $b \in B$ is the subset of all points in B which can be connected to b by a curve. This definition will be discussed in more detail later.

Then f is surjective.

The proof of the mapping lemma follows easily from the domain invariance theorem formulated below (see 10.11), the proof of the latter theorem will require one more excursion into topology.

The induction hypothesis will be used only to show that $\Phi_n^\#$ satisfies condition (a) in the mapping lemma. □

Manifolds: examples and connectedness

Here are a few examples of manifolds:

- Any open subset $\Omega \subseteq \mathbb{R}^n$ is an n-dimensional manifold. In particular $\mathbb{R}^n$ is an n-dimensional manifold.
- The unit n-sphere

$$\mathbb{S}^n = \left\{ (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \right\}$$

  is an n-dimensional manifold.
- The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$; i.e., the n-fold Cartesian product of $\mathbb{S}^1$, is an n-dimensional manifold.

The second and the third examples above are *connected*; i.e. any two points can be connected by a curve.[57]

The first example may be connected or not depending on the choice of Ω. In $\mathbb{R}$, the only connected open sets are open intervals (finite or infinite). As an example of a disconnected set, one can take the union of two intervals, say $\Omega = (0, 1) \cup (2, 3)$ in $\mathbb{R}$.

Part 1

Manifold of convex polyhedra

Consider the set $\mathbf{X}$ of all convex polyhedra with exactly k vertexes and with a chosen order of the vertexes; the elements of $\mathbf{X}$ will be called *equipped polyhedra.*

Let us introduce an equivalence relation $\sim$ on $\mathbf{X}$. Let $K, K' \in \mathbf{X}$; i.e., $K = \mathrm{Conv}(v_1, v_2, \dots, v_k)$ and $K' = \mathrm{Conv}(v_1', v_2', \dots, v_k')$ be two convex polyhedra with vertexes $v_1, v_2, \dots, v_k$ and $v_1', v_2', \dots, v_k'$ which are listed in the chosen order. We will write $K \sim K'$ if there is an orientation preserving isometry[58] $\iota\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\iota(v_i) = v_i'$ for each i.

Define $\mathbf{P}_k^\#$ to be the set of $\sim$-equivalence classes of $\mathbf{X}$.

The set $\mathbf{P}_k^\#$ can also be viewed in the following way. Given $K \in \mathbf{X}$ as above, note that triangle $\triangle v_1 v_2 v_3$ is non-degenerate. Therefore, there is a unique

[57] Formally a space where any two points can be connected by a curve is called *path connected* and the term *connected* is reserved for spaces where the only sets which are closed and open at the same time are the whole space and the empty set. However, for the manifolds these two definitions are equivalent.

[58] i.e., an isometry which can be presented as a composition of rotations around lines.

orientation preserving isometry of $\mathbb{R}^3$ which sends v_1 to the origin, v_2 to the open positive ray of the x-axis and v_3 to the open upper half of the xy-plane. I.e.,

$$\begin{aligned}
v_1 &= (0,0,0),\\
v_2 &= (x_2,0,0),\\
v_3 &= (x_3,y_3,0),\\
v_4 &= (x_4,y_4,z_4),\\
&\ \vdots\\
v_k &= (x_k,y_k,z_k).
\end{aligned}$$

Therefore the set $\mathbf{P}_k^\#$ can be identified with the subset Ω in $\mathbb{R}^{3\cdot k-6}$ formed by points with coordinates

$$(x_2),(x_3,y_3),(x_4,y_4,z_4),\dots,(x_k,y_k,z_k)$$

which satisfy the conditions $x_2>0$, $y_3>0$ and an additional k open conditions which make each v_i to be a vertex of $K=\mathrm{Conv}(v_1,v_2,\dots\dots,v_k)$.

This suggests to consider on $\mathbf{P}_k^\#$ the subspace metric induced from $\mathbb{R}^{3\cdot k-6}$ by identifying $\mathbf{P}_k^\#$ with $\Omega\subset\mathbb{R}^{3\cdot k-6}$. It is straightforward to check that Ω is open in $\mathbb{R}^{3\cdot k-6}$. Therefore, with this metric, $\mathbf{P}_k^\#$ is a $(3\cdot k-6)$-dimensional manifold.

Manifold of metrics

Denote by $\mathbf{Y}$ the class of all non-negatively curved polyhedral spaces homeomorphic to $\mathbb{S}^2$ with exactly k vertexes[59] and with a chosen order of vertexes and orientation. The elements of $\mathbf{Y}$ will be called *equipped polyhedral spaces*.

Let us introduce an equivalence relation $\sim$ on $\mathbf{Y}$. Let P and P' be two equipped polyhedral spaces with vertexes $v_1,v_2,\dots,v_k$ and $v_1',v_2',\dots,v_k'$ which are listed in the chosen order. We will write $P\sim P'$ if there is an orientation preserving isometry $\iota\colon P\to P'$ such that $\iota(v_i)=v_i'$ for each i.

An element in $\mathbf{M}_k^\#$ is a $\sim$-equivalence class of elements in $\mathbf{Y}$. If we take an element of $\mathbf{M}_k^\#$ and forget its orientation and the order of the vertexes, we obtain an element of $\mathbf{M}_k$. In other words, there is a natural map $\mathbf{M}_k^\#\to\mathbf{M}_k$. On the other hand, given polyhedral space P from $\mathbf{M}_k$, one can order its vertexes and choose an orientation to obtain an element of $\mathbf{M}_k^\#$. All together we can obtain at most $2\cdot k!$ such elements (it might be less if P has some symmetries). In particular the introduced map $\mathbf{M}_k^\#\to\mathbf{M}_k$ is surjective.

The map $\Phi^\#\colon\mathbf{P}_k^\#\to\mathbf{M}_k^\#$ is constructed in the following way. Let $K^\#$ be an equipped polyhedron and K be the corresponding (unequipped) polyhedron. Let P be the surface of K. Equip P with the order of the vertexes as in K and with the orientation induced from $\mathbb{R}^3$; i.e., if you look on an oriented basis

[59] Recall once more “vertex” = “a point with strictly positive curvature”

(w_1, w_2) on a face of K from outside of K, then w_2 obtained by counterclockwise rotation from w_1. Then $\Phi^\#(K^\#)$ is defined to be the obtained equipped polyhedral space.

To introduce a metric on $\mathbf{M}_k^\#$ one has to modify the Gromov–Hausdorff distance by taking into account orientation and the order of the vertexes. Namely, let P and P' be two oriented polyhedral spaces with vertexes $v_1, v_2, \dots, v_k$ and $v_1', v_2', \dots, v_k'$ which are listed in the chosen order. Define the distance $d(P, P')$ to be the infimum of all positive $\varepsilon > 0$ such that there is an orientation preserving piecewise linear homeomorphism $f\colon P \to P'$ which is an ε-isometry and such that $f(v_i) = v_i'$ for each i.

It is straightforward to check that d defines a metric on $\mathbf{M}_k^\#$. From now on, we will always consider $\mathbf{M}_k^\#$ equipped with the metric d.

10.7. Exercise. *The map $\Phi^\#\colon \mathbf{P}_k^\# \to \mathbf{M}_k^\#$ is continuous.*[60]

Our next aim is to show the following.

10.8. Proposition. *The space $\mathbf{M}_k^\#$ is a $(3{\cdot}k - 6)$-dimensional manifold.*

To prove this proposition, it is sufficient to find a homeomorphism of a neighborhood of (the equivalence class of) an equipped polyhedron P from $\mathbf{M}_k^\#$ to an open set in $\mathbb{R}^{3\cdot k-6}$. In other words, we need to find a coordinate system in this neighborhood.

In order to do this, we will prove the following lemma.

10.9. Definition. *Let X be a metric space a curve $\gamma\colon \mathbb{I} \to X$ is called* local geodesic *if for any $t_0 \in \mathbb{I}$ there is $\varepsilon > 0$ such that*

$$|\gamma(t_1) - \gamma(t_2)|_X = |t_1 - t_2|$$

for any $t_1, t_2 \in \mathbb{I}$ such that $|t_1 - t_0|, |t_2 - t_0| < \varepsilon$. def

10.10. Lemma on pseudo-triangulation. *There are $(3{\cdot}k-6)$ local geodesics between pairs of vertexes of P which intersect each other only at the common end points and which cut P into planar triangles.*

Note that this subdivision into triangles is not the same as a triangulation; two vertexes can be joined by two or more different edges and three vertexes can form the set of vertexes of different triangles.

The lemma will be proved next week; for now try to believe that it is true.

Note that the lengths of the local geodesics in the lemma can be taken as coordinates near P. Indeed, these lengths completely describe the triangles cut from P and therefore it describes P itself up to isometry. Moreover small changes of these length make small changes of each angles of each triangle. Therefore the polyhedral metric with coordinates close to the coordinates of P has positive curvature at each vertex. At the same time if P' is sufficiently close to P in $\mathbf{M}_k^\#$ then it can be cut into triangles along the same rule, i.e. these coordinates are defined for any element of $\mathbf{M}_k^\#$ near P.

[60] Compare with Problem 4.D.

Part 2

Mapping Lemma modulo Domain invariance

The mapping lemma follows easily from the following theorem.

10.11. Domain invariance theorem. *Let $U \subset \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}^n$ be an injective continuous map. Then $f(U)$ is an open subset of $\mathbb{R}^n$.*

Note that from Exercise 1.19 it follows that f is a homeomorphism from U to $f(U)$. So this theorem might seem trivial, since a homeomorphic mapping of one space onto another always takes open sets into open sets by the definition of homeomorphism. However, what was said above implies only that the image of an open set is open as a subset of $f(U)$, but not as a subset of $\mathbb{R}^n$. For example, under the identity mapping $[0, \infty) \to \mathbb{R}$ the image is not an open subset of $\mathbb{R}$, although it is open as a subset of $[0, \infty)$.

The domain invariance theorem will be proved next week. Now we give a proof of the mapping lemma modulo the domain invariance theorem. For your convenience we repeat the statement here.

10.12. Mapping Lemma. *Let A and B be two manifolds of the same dimension. Suppose $f: A \to B$ is a continuous injective map satisfying the following conditions:*

a) Each connected component *of B contains the image of some point in A;*

b) The image $f(A)$ is closed in B.

Then f is surjective.

Proof of the mapping lemma modulo Domain invariance theorem. We shall use the domain invariance theorem to show that $f(A)$ is open in B. Let $b \in f(A)$ and $a \in A$ be such that $f(a) = b$. Using the defining property of manifolds, there exists an open neighborhood U of a and an open neighborhood V of b that are both homeomorphic to open sets in $\mathbb{R}^n$, where n is the dimension of A and B. Moreover, by continuity of f, we can choose U so that $f(U) \subset V$.

So we have homeomorphisms

$$h : U \to \widetilde{U} \subset \mathbb{R}^n$$

$$k : V \to \widetilde{V} \subset \mathbb{R}^n$$

where $\widetilde{U}$ and $\widetilde{V}$ are open subsets of $\mathbb{R}^n$. Applying the domain invariance theorem to the map $k \circ f \circ h^{-1} : \widetilde{U} \to \mathbb{R}^n$ yields that $k(f(h^{-1}(\widetilde{U}))) = k(f(U))$ is open in $\mathbb{R}^n$, and hence is open in $\widetilde{V}$. Since k is a homeomorphism, $f(U)$ is an open subset of V.

I.e., $f(A)$ is open sinse together with any point $b \in f(A)$, the set $f(A)$ contains a neigborhood $f(U)$ of b.

On the other hand, from (b), the set $f(A)$ is closed in B. I.e., $f(A)$ is closed and open at the same time.

According to (a) any point $b \in B$ can be connected by a curve, say $\gamma: [0,1] \to$ $\to B$ to a point in $f(A)$. Let t_0 be infimum of values $t \in [0,1]$ so that $\gamma(t) \in f(A)$.

Since $f(A)$ is closed, $\gamma(t_0) \in f(A)$. Assume $t_0 > 0$. Since $f(A)$ is open, there is $\varepsilon > 0$ such that $\gamma(t_0 - \varepsilon) \in f(A)$, a contradiction. I.e. $t_0 = 0$ and $b = \gamma(0) \in f(A)$. □

Condition (b)

For the second part of the proof, we will need to check conditions (b) and (a) in the mapping lemma for the map $\Phi_k^\#$. We will prove (b) and leave (a) for next week.

The condition 10.6(b) holds for $\Phi_k^\#$. Let $K_1, K_2, \dots$ be a sequence of convex polyhedra with exactly k vertexes and $P_1, P_2, \dots$ be the corresponding surfaces. Assume P_n converges in the sense of Gromov–Hausdorff to $P_\infty \in \mathbf{M}_k$.

Note that since P_n is convergent and $\operatorname{diam} : \mathcal{M} \to [0, \infty)$ is continuous, their diameters are uniformly bounded; i.e., $\operatorname{diam} P_n \leqslant D$ for some fixed D. Note that $\operatorname{diam} K_n \leqslant \operatorname{diam} P_n$ for each n. In particular $\operatorname{diam} K_n \leqslant D$ for any n.

Without loss of generality we may assume that each K_n contains the origin of $\mathbb{R}^3$. In this case each K_n lie in a closed ball of radius D centered at the origin. Pass to a convergent subsequence of K_n in the Hausdorff metric of the ball of radius D; it exists according to Blaschke's theorem (2.1).

It is easy to see that the limit set K_∞ is a convex polyhedron. According to Exercise 10.7 (and/or Problem 4.D), its surface is isometric to P_∞.

It remains to equip K_∞ compatibly with P_∞. Simply order the vertexes of K_∞ the same way as in P_∞ and pass to a reflection of K_∞ in a plane if necessary. □

HWA 9; due Fri, Nov 11

Do 4 out of 6 problems.

10.A. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere.

1. Show that if two geodesics in P intersect at two points, then these are the end points for both geodesics.
2. Show that a geodesic in P cannot pass through a vertex of P

10.B. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere and let Δ be a triangle in P bounded by 3 geodesics. Denote by α, β and γ the angles of Δ. Show that $\alpha + \beta + \gamma - \pi$ is equal to the sum of curvatures of all points in the interior of Δ.

In particular, the sum of the angles of any triangle in P is at least π.

Hint: Pass to the doubling of Δ and apply 8.5 together with Problem 10.A.

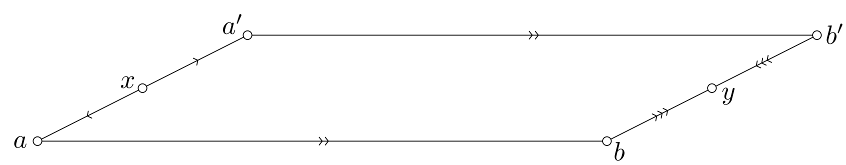
10.C. Let P be the surface of a regular tetrahedron. Find a periodic local geodesic[61] in P. Show that any two vertexes can be joined by arbitrary long local geodesic.

10.D. Let $aa'b'b$ be a parallelogram in the plane. Denote by x and y the midpoints of sides $[aa']$ and $[bb']$. Let us glue the following pairs of intervals:

$$([xa], [xa']),\ ([ab], [a'b']),\ ([yb], [yb']),$$

so after gluing $a = a'$ and $b = b'$. Note that we obtain a non-negatively curved polyhedral space homeomorphic to a sphere, so by Alexandrov's theorem it is isometric to a surface of a tetrahedron.

Try to understand how to draw the edges of this tetrahedron on the original parallelogram. Do it for the following example.



(You can make an experiment: cut it from the paper and glue along the rule.)

10.E. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere with exactly 4 vertexes a, b, c and d. Let us draw on P a geodesic between each pair of vertexes $[ab]$, $[bc]$, $[cd]$ and $[da]$. Show that either these geodesics intersect only at the common ends, or exactly two of them intersect at an interior point.

[61] see Definition 10.9

In the latter case, show that P is isometric to the doubling of a convex quadrilateral.

(You may use Alexandrov's theorem, doing this problem directly is harder.)

10.F. Show that $\mathbf{P}_k$ is connected.[62]

[62] Note that Alexandrov's theorem implies that $\mathbf{M}_k$ is also connected.

Last week we gave a proof of the Existence theorem (10.1) with a few gaps. Hopefully, the structure of the proof is clear by now.

It remains to prove the following:

◇ Prove the existence of a pseudo-triangulation (10.10),

◇ Prove the domain invariance theorem (10.11),

◇ Check the condition (a) in the mapping lemma (10.6) for $\Phi_k^{\#}$.

We will not give the proof of domain invariance theorem (you will see it in a first course in algebraic topology).

11 Existence of pseudo-triangulation

Let P be a polyhedral metric with non-negative curvature and vertexes $v_1, v_2, \dots$ $\dots, v_k$. Choose $k-1$ geodesics $[v_1v_2]$, $[v_1v_3], \dots, [v_1v_k]$. From Problem 10.A, these geodesics intersect only at the common vertex v_1. Let us cut[63] P along these geodesics.

We obtain a polyhedral space P' bounded by $2{\cdot}(k-1)$ edges; all of the interior points of P' have zero curvature. Note that P' is homeomorphic to a disc. Therefore there is a map $f\colon P' \to \mathbb{R}^2$ which is distance preserving in a neighborhood of any interior point.

11.1. Exercise. *Try to prove the last statement.*

The space P' can be then triangulated the same way as in the Problem 5.A. If f is injective, then P is isometric to a non-convex polygon equipped with the intrinsic metric, but even if not, the same proof works.

The sides of triangles in this triangulation are formed by geodesics in P', but in P they are only local geodesics. They divide P into triangles with vertexes $v_1, v_2, \dots, v_k$. In particular, k is the number of vertexes. Denote by l and m the number of edges and faces. Applying Euler formula together with the identity $2{\cdot}l = 3{\cdot}m$, we get $l = 3{\cdot}k - 6$.

Condition (a)

Given a polyhedral space $P \in \mathbf{M}_k$, we need to construct a continuous family of polyhedral spaces P_t in $\mathbf{M}_k$, such that $P_0 = P$ and P_1 is realizable as the surface of a convex polyhedron, that is, $P_1 \in \Phi_k(\mathbf{P}_k)$. Once this is done, one can easily cook up a corresponding family in $\mathbf{M}_k^{\#}$ by choosing an ordering of the vertexes and an orientation so that it changes continuously with t.

We first prove the existence of another family as described in the following lemma.

11.2. Lemma. *Assume $k \geqslant 4$. Then given $P \in \mathbf{M}_k$, there is a family of polyhedral spaces P_t with $t \in [0, t_{\max}]$ which is continuous with respect to the Gromov–Hausdorff metric and such that*

[63] Formally "cutting" an metric space X along a subset $A \subset X$ can be defined as following: equip $X \backslash A$ with the induced intrinsic metric and pass to the completion of the obtained space.

◇ $P_0 = P$;
◇ $P_t \in \mathbf{M}_k$ *for* $t < t_{\max}$;
◇ $P_{t_{\max}} \in \mathbf{M}_{k-1}$.

In other words, we can change P continuously in $\mathbf{M}_k$ so that exactly one of the vertexes disappear; i.e., its curvature will approach 0 as $t \to t_{\max}$ and so the limit metric will lie in $\mathbf{M}_{k-1}$.

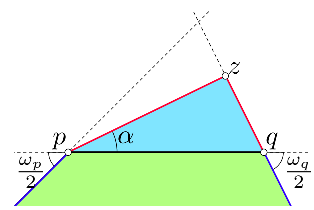
Note that the induction hypothesis is that $\Phi_{k-1}(\mathbf{P}_{k-1}) = \mathbf{M}_{k-1}$, and this lemma makes it possible to apply it.

Proof of Lemma 11.2. Choose two vertexes p and q in P with minimal positive curvatures ω_p and ω_q.

Cut P along an arbitrary geodesic $[pq]$. To obtain the family P_t we will glue the hole by a one-parameter family of patches which we now describe.

Take two congruent plane triangles, say $\triangle\hat p\hat q\hat z$ and $\triangle\check p\check q\check z$ such that $|\hat p - \hat q| = |\check p - \check q| = |p - q|_P$ and $\angle\hat q = \angle\check q = \omega_q/2$. All such triangles are uniquely determined by the angle $\alpha = \angle\hat p = \angle\check p$. Glue $\triangle\hat p\hat q\hat z$ to $\triangle\check p\check q\check z$ by gluing sides $[\hat p\hat z]$ to $[\check p\check z]$ and $[\hat q\hat z]$ to $[\check q\check z]$. Denote the obtained patch as Q_α. After gluing $\check z = \hat z$ and it will be also denoted as z.

Now let us glue Q_α along the cut $[pq]$ so that $\hat p, \check p \mapsto p$ and $\hat q, \check q \mapsto q$. Denote the obtained space as P_α, clearly $P_0 = P$.



Note that a neighborhood of $[pq]$ is isometric to two copies of the green quadrilateral on the picture, with identified sides marked by blue and black. After cutting P and gluing in the patch Q_α, it will look like two copies of the quadrilateral colored in green and blue with identified sides marked by blue and red; the patch is blue.

Straightforward calculations show that the obtained polyhedral metric has zero curvature at $\hat q = \check q = q$, it has curvature $2\cdot\pi - 2\cdot\angle\hat z = \omega_q + 2\cdot\alpha$ at $\hat z = \check z = z$ and it will have curvature $\omega_p - 2\cdot\alpha$ at $\hat p = \check p = p$.

Since the sum of two angles in a flat triangle has to be less than π, the value α can be chosen arbitrary in the interval $(0, \pi - \frac{\omega_q}{2})$.

If

❶ $$\omega_p + \omega_q < 2\cdot\pi,$$

the family P_α for $\alpha \in [0, \frac{\omega_p}{2}]$ satisfies the Lemma.

Recall that p and q are the vertexes in P with minimal possible curvature. Since the sum of all curvatures in P is $4\cdot\pi$, we always have $\omega_p + \omega_q \leqslant \frac{8}{k}\cdot\pi$. Therefore ❶ holds for $k \geqslant 5$.

In the remaining case $k = 4$, ❶ does not hold if and only if curvature at each vertex is π. In this case, we choose an arbitrary pair of vertexes and apply the same construction as above to move our metric continuously into a metric such that the curvature at one vertex is strictly less than π. For the obtained polyhedral space ❶ holds.

In the latter case our family P_t is joined from two families described above for two choices of pairs of vertexes. □

Apply the lemma and set $Q = P_{t_{\max}}$. Note that by the induction hypothesis, we can present Q as the surface of a convex polyhedron with $k-1$ vertexes, say K. Choose a pseudo-triangulation $\mathcal{T}$ of the surface of K.

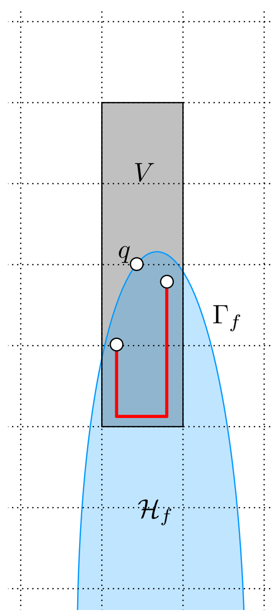
Mark a point p on K (the same notation as in the proof of the lemma). The point p lies on the side or face of some triangle of $\mathcal{T}$. Note that if we move p outside of K a bit then the convex hull p and K will form a convex polyhedron, say K', with k vertexes. Clearly we can choose K' to be arbitrarily close to K and therefore its surface, say Q' will be arbitrarily close to Q. To show that condition (a) holds for Φ_k, it is sufficient to connect Q' to some P_t by a continuous family in $\mathbf{M}_k$.

Subdivide $\mathcal{T}$ to make p one of its vertexes.

Let $\ell_1, \ell_2, \dots, \ell_{3\cdot k-6}$ be the lengths of the edges of $\mathcal{T}$. Let $\ell'_1, \ell'_2, \dots, \ell'_{3\cdot k-6}$ be a collection of real numbers, such that ℓ'_i is ε-close to ℓ_i for each i and some small enough $\varepsilon > 0$. We can construct triangles in $\mathcal{T}$ with the side-lengths ℓ'_i instead of ℓ_i; denote by $Q(\ell'_1, \ell'_2, \dots, \ell'_{3\cdot k-6})$ the constructed polyhedral metric. If $\varepsilon > 0$ is small enough, triangle inequalities still hold in each triangle and the curvature of each vertex except p is still positive.

Without loss of generality, we may assume that ℓ_1 is the length of the edge opposite from p in some triangle. Note that increasing ℓ_1 while leaving all the remaining edges fixed decreases the curvature at p.

Thus the set $(\ell'_1, \ell'_2, \dots, \ell'_{3\cdot k-6})$ for which the curvature at p is zero is formed by a graph $\ell_1 = f(\ell'_2, \dots, \ell'_{3\cdot k-6})$ for some continuous real-valued function f defined in a small neighborhood of $(\ell_2, \dots, \ell_{3\cdot k-6}) \in \mathbb{R}^{3\cdot k-7}$. Above this graph curvature at p is negative and below is positive. Thus the condition (a) boils down to the following lemma.



11.3. Lemma. *Let $U \subset \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}$ be a continuous function. Consider the graph and the open hypo-graph of f*

$$\Gamma_f = \{\, (x, y) \in U \times \mathbb{R} \mid y = f(x) \,\}$$

$$\mathcal{H}_f = \{\, (x, y) \in U \times \mathbb{R} \mid y < f(x) \,\}$$

Then any point $q \in \Gamma_f$ has an arbitrarily small neighborhood V such that $V \cap \mathcal{H}_f$ is connected.

11.4. Exercise. *Try to reconstruct the proof of this lemma from the picture.*

One application

Note that from Alexandrov's theorem and Problem 4.D we have the following:

11.5. Theorem. *Let P_∞ be a metric space. Then P_∞ is isometric to the surface of convex body in $\mathbb{R}^3$ if and only if P_∞ can be presented as Gromov–Hausdorff limit of a sequence of non-negatively curved polyhedral spaces P_n homeomorphic to $\mathbb{S}^2$.*

Assume you have a metric space P_∞ and you want to know if it is isometric to the surface of a convex body in $\mathbb{R}^3$. The above theorem can be applied easily if the answer is "yes"; it is seems to be harder to use this theorem to give a "no" answer to the question. It turns out that non-negative curvature can be defined for general metric spaces and this new definition will make it equally easy to answer either "yes" or "no" to the question.

This will be our topic for the next week.

HWA 10; due Fri, Nov 18

11.A. Consider a regular octahedra H, with vertexes a, a', b, b', c, c' and assume that the pairs (a, a'), (b, b') and (c, c') are opposite.

Cut from H a pyramid with vertex a by a plane Π parallel to the plane containing b, b', c and c'. Denote by P the remaining surface of H; it is bounded by a broken line $xyx'y'$ which bounds a square in Π.

Consider the patch R_α, which is a planar rhombus with the same side length as $xyx'y'$ and with angle α at one vertex. Glue R_α to P along $xyx'y'$ by a length-preserving map of its boundary vertex-to-vertex. Denote the obtained space as P_α.

1. For which α does the space P_α has non-negative curvature? For such α denote by K_α the convex polyhedron with surface isometric to P_α.
2. For which pairs of α, α' are the polyhedra K_α and $K_{\alpha'}$ congruent?
3. Construct another polygonal patch (not a rhombus) which gives a space with non-negative curvature.
4. Characterize all such patches.

11.B. Let P be a non-negatively curved polyhedral metric on $\mathbb{S}^2$. Cut a triangle $\triangle$ from P along geodesics and equip it with the intrinsic metric. Show that $\triangle$ is isometric to a planar triangle if and only if the sum of its angles is π.

Hint: Use Problem 10.B to prove the "only if" part. To prove the "if" part, construct a distance preserving map explicitly.

12 The ghosts of Euclid

Let X be a metric space.

Geodesics. Given a pair of points $x, y \in X$, we will denote by $[xy]$ the image of a geodesic from x to y.

In general, a geodesic between x and y need not exist and if it exists, it need not to be unique. However, once we write $[xy]$ we mean that we made a choice of a geodesic between x and y.

Also we will use the following notational short-cuts:

$$]xy[= [xy]\backslash\{x, y\}, \qquad]xy] = [xy]\backslash\{x\}, \qquad [xy[= [xy]\backslash\{y\}.$$

Triangles. For a triple of points $x, y, z \in X$, a choice of a triple of geodesics $([yz], [zx], [xy])$ will be called a *triangle* and we will use the short notation $[xyz] = ([yz], [zx], [xy])$.

Given a triple $x, y, z \in X$ there may be no triangle $[xyz]$ simply because one of the pairs of these points cannot be joined by a geodesic, and also there may be many different triangles with these vertexes, any of which can be denoted by $[xyz]$. Once we write $[xyz]$, it means that we made a choice of such a triangle, i.e. a choice of each $[yz]$, $[zx]$ and $[xy]$.

Model triangles. Given $x, y, z \in X$. Let us define its *model triangle* $[\tilde{x}\tilde{y}\tilde{z}]$ (briefly, $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}xyz$) to be a triangle in the Euclidean plane such that

$$|\tilde{x} - \tilde{y}|_{\mathbb{R}^2} = |x - y|_X, \quad |\tilde{y} - \tilde{z}|_{\mathbb{R}^2} = |y - z|_X, \quad |\tilde{z} - \tilde{x}|_{\mathbb{R}^2} = |z - x|_X.$$

Note that the model triangle is uniquely defined up to congruence.

In this case, a point $\tilde{p} \in [\tilde{x}\tilde{y}]$ is said to be *corresponding* to the point $p \in [xy]$ if

$$|\tilde{x} - \tilde{p}|_{\mathbb{R}^2} = |x - p|_X.$$

(Equivalently $|\tilde{y} - \tilde{p}|_{\mathbb{R}^2} = |y - p|_X$, or $\tilde{p}$ divides $[\tilde{x}\tilde{y}]$ in the same ratio as p divides $[xy]$.)

The definition

12.1. Definition. *A proper intrinsic space X has non-negative curvature in the sense of Alexandrov (briefly $X \in \mathrm{CBB}[0]$[64]) if the following inequality*

$$\text{❶} \qquad |z - p|_X \geqslant |\tilde{z} - \tilde{p}|_{\mathbb{R}^2}$$

holds for any triangle $[xyz]$ in X, its model triangle $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)$, any point $p \in]xy[$ and the corresponding point $\tilde{p} \in]\tilde{x}\tilde{y}[$.

[64] CBB[0] stays for “curvature bounded below by 0”. If in the definition of model triangle, one exchanges the Euclidean plane with a sphere or Lobachevsky plane of constant curvature k, then one gets the definition of *spaces with curvature $\geqslant k$ in the sense of Alexandrov*, which denoted as CBB$[k]$. We will only consider the case $k = 0$.

More ghosts of Euclid

Model angles. Let $[\tilde x\tilde y\tilde z]=\tilde\triangle(xyz)$ and

$$|x-y|,|x-z|>0,$$

the angle measure of $[\tilde x\tilde y\tilde z]$ at $\tilde x$ will be called the *model angle* of the triple x, y, z and it will be denoted by $\tilde\measuredangle(x\,{}^{y}_{z})$.

Since increasing one side of a planar triangle makes the opposite side bigger, we get the following.

12.2. Proposition. *The inequality ❶ can be rewritten in the following way*

$$\tilde\measuredangle(x\,{}^{p}_{z})\geqslant\tilde\measuredangle(x\,{}^{y}_{z}).$$

Hinges. Let $p,x,y\in X$ be a triple of points such that p is distinct from x and y. A pair geodesics $([px],[py])$ will be called a *hinge* and briefly, it will be denoted by $[p\,{}^{x}_{y}]=([px],[py])$.

From Proposition 12.2, we get the following.

12.3. Corollary. *Let $X\in\mathrm{CBB}[0]$. Given a hinge $[p\,{}^{x}_{y}]$ in X, consider the function of two arguments*

$$\alpha\colon(|p-\bar x|,|p-\bar y|)\mapsto\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})$$

where $\bar x\in\,]px]$ and $\bar y\in\,]py]$.

Then $\alpha(s,t)$ is nonincreasing in both arguments.

Angles. Given a hinge $[p\,{}^{x}_{y}]$, we define its *angle* as follows:

$$\measuredangle[p\,{}^{x}_{y}]\stackrel{def}{=\!=}\lim_{\bar x,\bar y\to p}\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x\in\,]px]$ and $\bar y\in\,]py]$.

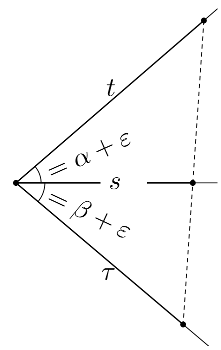
From Corollary 12.3, we get the following.

12.4. Proposition. *Let $X\in\mathrm{CBB}[0]$. Then for any hinge $[p\,{}^{x}_{y}]$ in X, the angle $\measuredangle[p\,{}^{x}_{y}]$ is defined.*

12.5. Triangle inequality for angles. *Let $[px]$, $[py]$ and $[pz]$ be three geodesics in a metric space. If all of the angles $\alpha=\measuredangle[p\,{}^{x}_{y}]$, $\beta=\measuredangle[p\,{}^{y}_{z}]$ and $\gamma=\measuredangle[p\,{}^{x}_{z}]$ are defined, then they satisfy the triangle inequality:*

$$\gamma\leqslant\alpha+\beta.$$

Proof. Since $\gamma \leqslant \pi$, we can assume that $\alpha + \beta < \pi$. Parametrize $[px]$. $[py]$ and $[pz]$ by arc-length starting from p and denote the obtained curves by σ_x, σ_y and σ_z. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_+$ we have



$$\begin{aligned}|\sigma_x(t) - \sigma_z(\tau)| &\leqslant |\sigma_x(t) - \sigma_y(s)| + |\sigma_y(s) - \sigma_z(\tau)| \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha+\varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\beta+\varepsilon)}\end{aligned}$$

Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain continues

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha+\beta+2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\gamma \leqslant \alpha + \beta + 2{\cdot}\varepsilon.$$

Hence the result.

To define $s(t, \tau)$, consider three rays $\tilde\sigma_x$, $\tilde\sigma_y$, $\tilde\sigma_z$ in the Euclidean plane starting at one point, such that $\measuredangle(\tilde\sigma_x, \tilde\sigma_y) = \alpha + \varepsilon$, $\measuredangle(\tilde\sigma_y, \tilde\sigma_z) = \beta + \varepsilon$ and $\measuredangle(\tilde\sigma_x, \tilde\sigma_z) = \alpha + \beta + 2{\cdot}\varepsilon$. We parametrize each ray by length from the common end. Given two positive numbers $t, \tau \in \mathbb{R}_+$, let $s = s(t, \tau)$ be the number such that $\tilde\sigma_y(s) \in [\tilde\sigma_x(t)\ \tilde\sigma_z(\tau)]$. Clearly $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. □

More definitions

The following theorem gives a number of equivalent ways to define CBB[0] spaces.

12.6. Theorem. *Let X be a proper intrinsic space. Then the following are equivalent.*

a) $X \in \mathrm{CBB}[0]$.

b) ((1+3)-point comparison) if p is distinct from x, y, and z then

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2{\cdot}\pi.$$

c) (adjacent angle comparison) for any geodesic $[xy]$ and $p \in\,]xy[$, $z \neq p$ we have

$$\tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant \pi.$$

d) (hinge comparison) for any hinge $[x\,{}^y_z]$, the angle $\measuredangle[x\,{}^y_z]$ is defined and

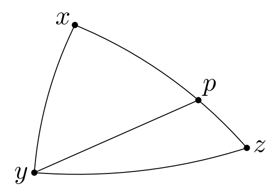
$$\measuredangle[x\,{}^y_z] \geqslant \tilde\measuredangle(x\,{}^y_z).$$

Moreover, if $p \in\,]xy[$, $z \neq p$ then

$$\measuredangle[p\,{}^y_z] + \measuredangle[p\,{}^z_x] \leqslant \pi$$

for any two hinges $[p\,{}^z_y]$ and $[p\,{}^z_x]$ with common side $[pz]$.[65]

[65] It is not known if the last inequality is necessary, even for spaces homeomorphic to $\mathbb{S}^2$.



In the proof we will need the following lemma.

12.7. Alexandrov's lemma. *Let x, y, z, p be distinct points in a metric space and $p \in\,]xz[$. Then the following expressions have the same sign:*

a) $\tilde\measuredangle(x\,{}^{y}_{p}) - \tilde\measuredangle(x\,{}^{y}_{z})$,

b) $\pi - \tilde\measuredangle(p\,{}^{y}_{x}) - \tilde\measuredangle(p\,{}^{y}_{z})$.

Proof. Consider the model triangle $[\tilde x\tilde y\tilde p] = \tilde\triangle xyp$. Take a point $\tilde z$ on the extension of $[\tilde x\tilde p]$ beyond $\tilde p$ so that $|\tilde x - \tilde z| = |x - z|$ (and therefore $|\tilde p - \tilde z| = |p - z|$).

Since increasing a side in a planar triangle increases the opposite angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde y}_{\tilde z}] - \tilde\measuredangle(x\,{}^{y}_{z})$;

(ii) $|\tilde y - \tilde z| - |y - z|$;

(iii) $\measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}] - \tilde\measuredangle(p\,{}^{y}_{z})$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde y}_{\tilde z}] = \measuredangle[\tilde x\,{}^{\tilde y}_{\tilde p}] = \tilde\measuredangle(x\,{}^{y}_{p})$$

and

$$\measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}] = \pi - \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \pi - \tilde\measuredangle(p\,{}^{x}_{y}),$$

the statement follows. □

Proof of Theorem 12.6. Note that X is a geodesic space[66], as it is proper and intrinsic (see Exercise 4.8).

(b) ⇒ (c). Since $p \in\,]xy[$, we have $\tilde\measuredangle(p\,{}^{x}_{y}) = \pi$. Thus, (1+3)-point comparison

$$\tilde\measuredangle(p\,{}^{x}_{y}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant 2\cdot\pi$$

implies

$$\tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant \pi.$$

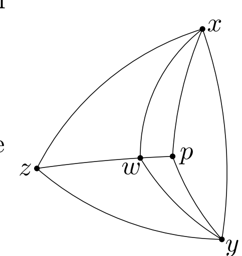
(c) ⇔ (a). Follows directly from Alexandrov's lemma (12.7).

(c) + (a) ⇒ (d). From Proposition 12.4, we get that for $\bar y \in\,]xy]$ and $\bar z \in\,]xz]$ the function $(|x - \bar y|, |x - \bar z|) \mapsto \tilde\measuredangle(x\,{}^{\bar y}_{\bar z})$ is nonincreasing in each argument. In particular, $\measuredangle[x\,{}^{y}_{z}] = \sup\{\tilde\measuredangle(x\,{}^{\bar y}_{\bar z})\}$. Thus, $\measuredangle[x\,{}^{y}_{z}]$ is defined and it is at least $\tilde\measuredangle(x\,{}^{y}_{z})$.

From above and (c), it follows that $\measuredangle[p\,{}^{y}_{z}] + \measuredangle[p\,{}^{z}_{x}] \leqslant \pi$.

(d) ⇒ (b). Consider a point $w \in\,]pz[$ close to p. From (d), it follows that

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{x}_{p}] \leqslant \pi \quad\text{and}\quad \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{y}_{p}] \leqslant \pi.$$



By the triangle inequality for angles (see 12.5), we have $\measuredangle[w\,{}^{x}_{y}] \leqslant \measuredangle[w\,{}^{x}_{p}] + \measuredangle[w\,{}^{y}_{p}]$. Therefore we get

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{x}_{y}] \leqslant 2\cdot\pi.$$

[66] I.e., any two points in X can be joined by a geodesic.

Applying the first inequality in (d), we obtain

$$\tilde\measuredangle(w\,{}^{x}_{z}) + \tilde\measuredangle(w\,{}^{y}_{z}) + \tilde\measuredangle(w\,{}^{x}_{y}) \leqslant 2\cdot\pi.$$

Passing to the limits $w \to p$, we obtain

$$\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{x}_{y}) \leqslant 2\cdot\pi.$$

□

12.8. Proposition. *Let (X_n) be a sequence of compact* CBB[0] *spaces which converge to a compact space X_∞ in the sense of Gromov–Hausdorff.*

Then $X_\infty \in$ CBB[0].

Proof. According to Exercise 4.5, X_∞ is intrinsic.

For each natural number n, choose $f_n\colon X_n \to X_\infty$ to be an ε_n-isometry for some sequence $\varepsilon_n \to 0$ as $n \to \infty$. Take arbitrary points $p_\infty, x_\infty, y_\infty, z_\infty \in X_\infty$ and choose points $p_n, x_n, y_n, z_n \in X_n$ such that $f_n(p_n) \to p_\infty$, $f_n(x_n) \to x_\infty$, $f_n(y_n) \to y_\infty$, $f_n(z_n) \to z_\infty$ as $n \to \infty$.

Since $X_n \in \mathrm{CBB}[0]$, from 12.6b we have

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) + \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) + \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \leqslant 2\cdot\pi.$$

Clearly,

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) \to \tilde\measuredangle(p_\infty\,{}^{x_\infty}_{y_\infty}),\ \ \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) \to \tilde\measuredangle(p_\infty\,{}^{y_\infty}_{z_\infty}),\ \ \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \to \tilde\measuredangle(p_\infty\,{}^{z_\infty}_{x_\infty})$$

as $n \to \infty$. Therefore

$$\tilde\measuredangle(p_\infty\,{}^{x_\infty}_{y_\infty}) + \tilde\measuredangle(p_\infty\,{}^{y_\infty}_{z_\infty}) + \tilde\measuredangle(p_\infty\,{}^{z_\infty}_{x_\infty}) \leqslant 2\cdot\pi.$$

The proposition follows from 12.6b. □

Surface of convex body

12.9. Theorem. *A metric space X is isometric to the surface of a convex body*[67] *if and only if X is homeomorphic to $\mathbb{S}^2$ and $X \in$* CBB[0].

The proof is using Theorem 11.5, Proposition 12.8 and the following two propositions:

12.10. Proposition. *Any non-negatively curved polyhedral space homeomorphic to*[68] *$\mathbb{S}^2$ is a* CBB[0] *space.*

12.11. Proposition. *Given a* CBB[0] *space X homeomorphic to a sphere, there is a non-negatively curved polyhedral space $\tilde X$ homeomorphic to $\mathbb{S}^2$ for which the Gromov–Hausdorff distance $d_{GH}(X, \tilde X)$ arbitrarily small.*

[67] We allow a convex body to degenerate to a planar figure but not to a segment. As in the case of convex polyhedra, the surface of a planar figure is defined as its doubling.

[68] Instead of $\mathbb{S}^2$, we could say homeomorphic to a two-dimensional manifold.

Proof with cheating. We will use two claims without proof, here is the first one:

❷ *Given $\varepsilon > 0$, X admits a triangulation into triangles with geodesic sides such that the diameter of each triangle is less than ε.*

Fix small $\varepsilon > 0$ and choose a triangulation $\mathcal{T}$ of X provided by the claim.

12.12. Exercise. *The sum of the angles at one of the vertexes of $\mathcal{T}$ is $\leqslant 2{\cdot}\pi$.*

For each triangle in $\mathcal{T}$, construct a model triangle and glue them together the same way as the corresponding triangles in X. Denote the obtained polyhedral space by $\tilde{X}$, and given a vertex v_i of $\mathcal{T}$, we will denote by $\tilde{v}_i$ the corresponding point in $\tilde{X}$. Applying the hinge comparison (12.6d) we have that the sum of the angles around a vertex $\tilde{v}_i$ of $\tilde{X}$ is at most as big as the sum of the angles around the corresponding vertex v_i in X. Hence $\tilde{X}$ is non-negatively curved.

Here is the second claim which we use without proof:

❸ *The triangulation of X can be chosen on such a way that in the constructed polyhedral space $\tilde{X}$ each triangle is convex; i.e., any two points in one triangle can be joined by a geodesic inside this triangle.*

Now to prove the proposition, it is sufficient to show that if $\varepsilon > 0$ is small enough then $\tilde{X}$ is sufficiently close to X.

First note that the set of vertexes $\{v_i\}$ forms an ε-net in X and the set of vertexes $\{\tilde{v}_i\}$ forms an ε-net in $\tilde{X}$. Therefore it is sufficient to show that the inequalities

❹ $$|\tilde{v}_i - \tilde{v}_j|_{\tilde{X}} \leqslant |v_i - v_j|_X$$

❺ $$|v_i - v_j|_X - 4{\cdot}\pi{\cdot}\varepsilon \leqslant |\tilde{v}_i - \tilde{v}_j|_{\tilde{X}}$$

hold for any i and j.

To prove ❹, consider a geodesic $[v_i v_j]$ in X. Let $v_i = x_0, x_1, \dots, x_n = v_j$ be the points of intersection of $[v_i v_j]$ with the edges of triangulation listed in the order from v_i to v_j.[69]

Fix $k \in \{1, \dots, n\}$. Let $[pqr]$ be the triangle in $\mathcal{T}$ which contains $[x_{k-1}x_k]$ inside. Without loss of generality, we can assume that $x_{k-1} \in [pq]$ and $x_k \in [pr]$. Applying definition of CBB[0] spaces twice, first for the triangle $[pqr]$ and $x_k \in [pr]$ and then for the triangle $[pqx_k]$ and $x_{k-1} \in [pq]$ we get that

$$|\tilde{x}_k - \tilde{x}_{k-1}|_{\tilde{X}} \leqslant |x_k - x_{k-1}|_X$$

holds for each k. Summing up, we get ❹.

The proof of ❺ is similar. Consider a geodesic $[\tilde{v}_i\tilde{v}_j]$ in $\tilde{X}$. Let $\tilde{v}_i = \tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_m = v_j$ be the points of intersection of $[\tilde{v}_i\tilde{v}_j]$ with the edges of

[69]Note that according problem 12.A and 12.C, if a geodesic $[v_iv_j]$ intersect an edge at two points then it contains this edge. In this case one can take as x any point on this edge. Taking this into account, we have that n is finite.

triangulation listed in the order from $\tilde v_i$ to $\tilde v_j$. According to Claim ❸, $[\tilde v_i\tilde v_j]$ intersects each triangle at most once.

Fix $k\in\{1,\dots,n\}$. Let $[\tilde p\tilde q\tilde r]$ be the triangle in $\tilde X$ which contains $[\tilde y_{k-1}\tilde y_k]$ inside. Without loss of generality, we can assume that $\tilde y_{k-1}\in[\tilde p\tilde q]$ and $\tilde y_k\in[\tilde p\tilde r]$. Set

$$\begin{aligned}
&\alpha=\measuredangle[p\,{}^{q}_{r}], &&\beta=\measuredangle[q\,{}^{p}_{r}], &&\gamma=\measuredangle[r\,{}^{p}_{q}],\\
&\tilde\alpha=\tilde\measuredangle(p\,{}^{q}_{r}) &&\tilde\beta=\tilde\measuredangle(q\,{}^{p}_{r}) &&\tilde\gamma=\tilde\measuredangle(r\,{}^{p}_{q})
\end{aligned}$$

By hinge comparison, (12.6d) we have

$$\alpha\geqslant\tilde\alpha,\qquad\beta\geqslant\tilde\beta,\qquad\gamma\geqslant\tilde\gamma$$

For the triangle $\triangle=[pqr]$ we define its *curvature* as

$$\kappa(\triangle)=\alpha+\beta+\gamma-\pi=\alpha+\beta+\gamma-(\tilde\alpha+\tilde\beta+\tilde\gamma).$$

From the above $\kappa(\triangle)\geqslant 0$. Together with the rule of cosines, straightforward estimates give the following:

$$\begin{aligned}
|y_k-y_{k-1}|_X&\leqslant\sqrt{|p-y_{k-1}|^2+|p-y_k|^2-2|p-y_{k-1}|\cdot|p-y_k|\cdot\cos\alpha}\\
&\leqslant|\tilde y_k-\tilde y_{k-1}|_{\tilde X}+\varepsilon\cdot(\alpha-\tilde\alpha)\\
&\leqslant|\tilde y_k-\tilde y_{k-1}|_{\tilde X}+\varepsilon\cdot\kappa(\triangle).
\end{aligned}$$

Summing it up, we get

$$\begin{aligned}
|v_i-v_j|_X&\leqslant\sum_{k=1}^{m}|y_k-y_{k-1}|_X\\
&\leqslant\sum_{k=1}^{m}|\tilde y_k-\tilde y_{k-1}|_{\tilde X}+\varepsilon\cdot\sum_{\triangle}\kappa(\triangle).
\end{aligned}$$

where the last sum is taken over all triangles $\triangle$ in $\mathcal{T}$. Hence ❺ boils down to the inequality

$$\sum_{\triangle\text{ in }\mathcal{T}}\kappa(\triangle)\leqslant 4\cdot\pi.$$

12.13. Exercise. *Prove the last inequality.*[70] □

Theorem 12.9 and Proposition 12.10 will be proved next week.

[70]Hint: Use Exercise 12.12 and Euler's formula the same way as Exercise 8.5.

HWA 11; due Mon, Nov 28

12.A. Let $X \in \mathrm{CBB}[0]$ and $[x\,{}^y_z]$ be a hinge in X. Assume $\measuredangle[x\,{}^y_z] = 0$. Show that either $[xy] \subset [xz]$ or $[xz] \subset [xy]$.

12.B. Let $X \in \mathrm{CBB}[0]$. Show that given three distinct points x, y and p in X, there is at most one geodesic from x to y which passes through p.

12.C. Let $X \in \mathrm{CBB}[0]$ and $[z\,{}^x_p]$ and $[z\,{}^y_p]$ be two hinges with common side $[zp]$ in X. Assume that points p, x, y and z are distinct and $z \in [xy]$. Show that

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] = \pi.$$

12.D.[71] Let X a metric space with hinge $[x\,{}^y_z]$. Assume the angle $\alpha = \measuredangle[x\,{}^y_z]$ is defined. Show that

$$|z - \bar{y}| \leqslant |z - x| - |x - \bar{y}| \cdot \cos\alpha + o(|x - \bar{y}|)$$

for $\bar{y} \in\,]xy]$.

[71] Hint: Apply the definition of angle and triangle inequality

$$|z - \bar{y}| \geqslant |\bar{z} - \bar{y}| + |\bar{z} - z|$$

for $\bar{z} \in\,]xz]$.

13 Proof of Proposition 12.10

We will now show that if P is a non-negatively curved polyhedral space that is homeomorphic to $\mathbb{S}^2$, then P is a CBB[0] space. Let $[pxy]$ be a triangle in P and let $[\tilde{p}\tilde{x}\tilde{y}]$ be the model triangle of $[pxy]$. Set $\ell = |x-y|_P = |\tilde{x}-\tilde{y}|_{\mathbb{R}^2}$.

Denote by $\gamma(t)$ the geodesic $[xy]$ parametrized by length starting from x and let $\tilde{\gamma}(t)$ be the geodesic $[\tilde{x}\tilde{y}]$ parametrized by length starting from $\tilde{x}$. It is sufficient to show that

❶ $$|\tilde{p}-\tilde{\gamma}(t)| \leqslant |p-\gamma(t)|$$

for any t in $[0,\ell]$. Without loss of generality, we may assume that p is a *regular point*[72]; in other words p is not a vertex of P. (A vertex can be approximated by regular points, so if inequality ❶ holds for all regular points it also holds for the vertexes.)

From the cosine law, we get that the function

$$\tilde{f}(t) = |\tilde{p}-\tilde{\gamma}(t)|^2 - t^2$$

is linear. Consider the function

$$f(t) = |p-\gamma(t)|^2 - t^2.$$

Note that

$$f(0) = \tilde{f}(0),$$
$$f(\ell) = \tilde{f}(\ell).$$

Further the condition ❶ is equivalent to

❷ $$f(t) \geqslant \tilde{f}(t).$$

To prove ❷ it is sufficient to show that f is a concave function. The latter follows once we prove the following:

❸ *For any $t_0 \in\,]0,\ell[$ there is a* supporting *linear function $h(t)$; i.e.,*

$$h(t_0) = f(t_0) \quad \textit{and} \quad h(t) \geqslant f(t)$$

for any $t \in (t_0-\varepsilon, t_0+\varepsilon)$ and some fixed $\varepsilon > 0$.

Note that according to Problem 10.A, $\gamma(t_0)$ is regular. Since p is regular, a geodesic $[p\gamma(t)]$ contains only regular points. Therefore there is a neighborhood $\Omega \supset [p\gamma(t)]$ with only regular points. We may assume that Ω is homeomorphic to a disc; in this case it is easy to construct a locally distance preserving embedding $\iota\colon \Omega \to \mathbb{R}^2$. The image $\iota([p\gamma(t)])$ is a line segment and so is the image $\iota(\Omega\cap[xy])$. Thus $\iota(\Omega)$ contains a triangle with base $\iota([\gamma(t_0-\varepsilon)\ \gamma(t_0+\varepsilon)])$ for some small $\varepsilon > 0$ and vertex $\iota(p)$.

[72] i.e., the sum of angles around p is equal to $2{\cdot}\pi$

Clearly, for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we have

$$|\iota(p) - \iota(\gamma(t))| \geqslant |p - \gamma(t)|.$$

Note that the function

$$h(t) = |\iota(p) - \iota(\gamma(t))|^2 - t^2$$

is linear and it satisfies the condition ❸. □

Proof of Theorem 12.9

"Only if" part. Assume X is a surface of a convex body B (B might degenerate to a flat figure, but not a line segment). The same argument as in Proposition 8.1, shows that X is homeomorphic to $\mathbb{S}^2$. The convex body B can be approximated by a sequence of convex polyhedra K_n in the sense of Hausdorff. The proof of the last statement is the same as in Lemma 2.13.

Denote by P_n the surface of K_n. According to Problem 4.D, P_n converge to X in the sense of Gromov–Hausdorff. Applying Propositions 12.8 and 12.10, we get that $X \in \mathrm{CBB}[0]$.

"If" part. Assume X is a $\mathrm{CBB}[0]$ space which is homeomorphic to a sphere. According to Proposition 12.11, there is a sequence of non-negatively curved polyhedral spaces P_n which converge to X in Gromov–Hausdorff sense and such that each P_n is homeomorphic to $\mathbb{S}^2$.

Applying Alexandrov's existence theorem (10.1), we obtain a sequence of convex polyhedra K_n such that the surface of K_n is isometric to P_n for each n. Note that for all n the diameter of K_n is bounded by the diameter of P_n. Since $P_n \to X$ in the sense of Gromov–Hausdorff we have that $\operatorname{diam} P_n \to \operatorname{diam} X$ as $n \to \infty$ because diam is a continuous function (see Exercise 3.2.) In particular, $\operatorname{diam} P_n \leqslant C$ for some fixed constant C and all n.

Without loss of generality we may assume that each K_n contains the origin of $\mathbb{R}^3$. Therefore K_n lies in a fixed bounded region for all large n. Applying Blaschke's theorem (2.1), we can pass to a Hausdorff-converging subsequence of K_n. Denote by B its limit.

According to Problem 4.D, X is isometric to the surface of B. □

Isometric actions

Let X be a metric space. Denote by $\operatorname{Isom} X$ the set of all isometries of X. Let G be a nonempty subset of $\operatorname{Isom} X$ such that the following condition[73] holds:

- ◇ *Given two isometries $f, g \in G$ the maps $f \circ g\colon X \to X$ as well as the inverse $f^{-1}\colon X \to X$ are in G.*

[73] In the language of group theory, $\operatorname{Isom} X$ is a group and G is a subgroup of $\operatorname{Isom} X$.

In this case we say that *group G acts on X* by isometries. For example, one can take $G = \operatorname{Isom} X$ or $G = \{\mathrm{id}_X\}$.

In this case, given $g \in G$ and $x \in X$ the g-image of x will denoted as $g{\cdot}x$, that is, $g{\cdot}x = g(x)$. Consider the relation "$\sim$" on X such that $x \sim y$ if and only if there if there is $g \in G$ such that $g{\cdot}x = y$. Given $x \in X$, the G-orbit of x is defined as

$$G{\cdot}x = \{\, g{\cdot}x \mid g \in G. \}$$

Note that $G{\cdot}x = G{\cdot}y$ if and only if $x \sim y$; i.e., $\sim$ is an equivalence relation and G-orbits form the $\sim$-equivalence classes.

The set of G-orbits of X is denoted as X/G. Denote by $\pi\colon X \to X/G$ the natural surjective map, $\pi\colon x \mapsto G{\cdot}x$.

In the case that every G-orbit is a closed subset of X, then the set of orbits X/G can be equipped with the following metric

$$|\pi(x) - \pi(y)|_{X/G} \stackrel{def}{=\!=} \inf_{g \in G}\{|x - g{\cdot}y|_X\}.$$

13.1. Exercise. *Check that $|{*} - {*}|_{X/G}$ is a metric on X/G.*

The set X/G equipped with this metric is called the quotient space; for the quotient space, we keep the same notation X/G.

Note that

❹ $$|\pi(x) - \pi(y)| \leqslant |x - y|$$

for any $x, y \in X$; i.e. $\pi\colon X \to X/G$ is a distance non-expanding map.

Examples:

- ⋄ $\mathbb{Z}$ acts on $\mathbb{R}$ by shifts $n{\cdot}x \stackrel{def}{=\!=} 2{\cdot}\pi{\cdot}n + x$; in this case $\mathbb{R}/\mathbb{Z}$ is isometric to $\mathbb{S}^1 \subset \mathbb{R}^2$ with the induced intrinsic metric.
- ⋄ The set of all isometries of $\mathbb{R}$ which can be presented as a composition of a finite number of shifts $x \mapsto x+1$ and reflections $x \mapsto -x$ forms a group, say G; in this case $\mathbb{R}/G$ is isometric to the interval $[0, \frac{1}{2}]$.
- ⋄ $\mathbb{S}^1$ acts on $\mathbb{R}^2$ by rotations which fix the origin; in this case $\mathbb{R}^2/\mathbb{S}^1$ is isometric to the ray $[0, \infty)$.

13.2. Proposition. *Let X be a proper intrinsic space such that a group G acts on X by isometries and has closed orbits. Then X/G a proper intrinsic space.*

Proof. Given $\bar{x}$ and $\bar{y}$ in X/G, choose an arbitrary $x \in X$ such that $\pi(x) = \bar{x}$. Further, since X is proper and the orbits are closed, we can choose y such that $\pi(y) = \bar{y}$ and

$$|x - y| = |\bar{x} - \bar{y}|.$$

I.e. for any $\bar{y} \in \bar{B}_r(\bar{x})$ there is $y \in \bar{B}_r(x)$ such that $\pi(y) = \bar{y}$, or equivalently $\pi(\bar{B}_r(x)) = \bar{B}_r(\bar{x})$ for any $r > 0$. Since X is proper, $\bar{B}_r(x)$ is compact; hence $\bar{B}_r(\bar{x}) \subset X/G$ is compact for any $r > 0$ because it is the continuous image of a compact set, and therefore X/G is proper.

It remains to show that X/G is intrinsic. Let z be the midpoint for x and y; i.e.,

$$|z-x| = |z-y| = \tfrac{1}{2}\cdot|x-y|$$

(The existence of z follows from Exercise 4.8.) Consider $\bar z = \pi(z)$. From ❹, we get

$$|\bar x - \bar z| \leqslant |x-z| \qquad\qquad |\bar y - \bar z| \leqslant |y-z|$$

All together, this implies that $\bar z$ is the midpoint of $\bar x$ and $\bar y$. □

13.3. Proposition. *Let $X \in \mathrm{CBB}[0]$ such that a group G acts on X by isometries and has closed orbits. Then $X/G \in \mathrm{CBB}[0]$*

Proof. Applying Proposition 13.2, we get that X/G is proper and intrinsic. By Theorem 12.6b, it remains to show that

❺
$$\tilde\measuredangle(\bar p\,{}^{\bar x}_{\bar y}) + \tilde\measuredangle(\bar p\,{}^{\bar y}_{\bar z}) + \tilde\measuredangle(\bar p\,{}^{\bar z}_{\bar x}) \leqslant 2\cdot\pi.$$

holds for any $\bar p, \bar x, \bar y, \bar z \in X/G$.

Choose arbitrary $p \in X$ such that $\pi(p) = \bar p$. Since X is proper and orbits are closed, we can choose x, y and $z \in X$ such that

$$\begin{aligned} \pi(x) &= \bar x, & |p-x|_X &= |\bar p - \bar x|_{X/G},\\ \pi(y) &= \bar y, & |p-y|_X &= |\bar p - \bar y|_{X/G},\\ \pi(z) &= \bar z, & |p-z|_X &= |\bar p - \bar z|_{X/G}. \end{aligned}$$

By the definition of metric in X/G, we have

$$|x-y|_X \geqslant |\bar x - \bar y|_{X/G}, \quad |y-z|_X \geqslant |\bar y - \bar z|_{X/G}, \quad |z-x|_X \geqslant |\bar z - \bar x|_{X/G}.$$

Taking all this into account we get

❻
$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar x}_{\bar y}), \quad \tilde\measuredangle(p\,{}^{y}_{z}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar y}_{\bar z}), \quad \tilde\measuredangle(p\,{}^{z}_{x}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar z}_{\bar x}),$$

Since $X \in \mathrm{CBB}[0]$, the we have

$$\tilde\measuredangle(p\,{}^{x}_{y}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant 2\cdot\pi.$$

This inequality plus ❻ implies ❺. □

An application

Now we describe a simple application of Proposition 13.3, which relies on the following problem in discrete geometry.

Erdős problem

13.4. Problem. *Assume $x_1, x_2, \dots, x_m$ is a collection of points in n-dimensional Euclidean space such that $\angle[x_i\,{}^{x_j}_{x_k}] \leqslant \frac{\pi}{2}$ for any distinct i, j and k. Show that $m \leqslant 2^n$ and moreover, if $m = 2^n$ then the x_i form the set of vertexes of a right parallelepiped.*

This problem was posed by Erdős and solved by Danzer and Grünbaum.

Proof. Let K be the convex hull of $x_1, x_2, \dots, x_m$. Without loss of generality we may assume that K is non-degenerate convex polyhedron. (Otherwise, instead of $\mathbb{R}^n$ take the minimal subspace which contain K; its dimension has to be $< n$.)

First let us prove the following

❼ $\measuredangle[x_i\,{}^{v}_{w}] \leqslant \frac{\pi}{2}$ *for each i and any $v, w \in K$.*

Indeed, assume contrary. For fixed x_i and v, the set of points H_v containing x_i and all $w \in \mathbb{R}^n$ such that $\measuredangle[x_i\,{}^{v}_{w}] \leqslant \frac{\pi}{2}$ is a half-space. Thus, if $\measuredangle[x_i\,{}^{v}_{w}] > \frac{\pi}{2}$ for some $w \in K$ then $K \not\subset H_v$ and so $x_j \notin H_v$ for some j; i.e., $\measuredangle[x_i\,{}^{v}_{x_j}] > \frac{\pi}{2}$ for some j. Repeating the same argument for x_j instead of v, we get that $\measuredangle[x_i\,{}^{x_k}_{x_j}] > \frac{\pi}{2}$ for some j and k, a contradiction.

For each x_i denote by K_i the dilation of K with center x_i and coefficient $\frac{1}{2}$.

❽ *For any $i \neq j$, the polyhedra K_i and K_j have no common interior points. In particular,* $\operatorname{vol}(K_i \cap K_j) = 0$.

Assume there is an interior point v of $K_i \cap K_j$. Without loss of generality, we can assume that $|v - x_i| \neq |v - x_j|$. Then there are points $y_i, y_j \in K$ such that v is the midpoint of $[x_i y_i]$ and $[x_j y_j]$. Hence $x_i y_j y_i x_j$ is a parallelogram, therefore $\measuredangle[x_i\,{}^{x_j}_{y_j}] + \measuredangle[x_j\,{}^{x_i}_{y_i}] = \pi$. From ❼ we get that $\measuredangle[x_i\,{}^{x_j}_{y_j}] = \measuredangle[x_j\,{}^{x_i}_{y_i}] = \frac{\pi}{2}$. I.e., $x_i y_j y_i x_j$ is a rectangle and therefore $|v - x_i| = |v - x_j|$, a contradiction.

Clearly $\operatorname{vol} K_i = \frac{1}{2^n} \cdot \operatorname{vol} K$ and $K_i \subset K$ for each i. From ❽, we get

$$\sum_{i=1}^{m} \operatorname{vol} K_i \leqslant \operatorname{vol} K.$$

Hence the result. □

Counting isolated fixed points

Let $X \in \mathrm{CBB}[0]$. A point $p \in X$ is called *extremal* if $\measuredangle[p\,{}^{x}_{y}] \leqslant \frac{\pi}{2}$ for any hinge $[p\,{}^{x}_{y}]$ in X.

For example if $X \stackrel{iso}{=\!=} [0,1]$ then both ends 0 and 1 are extremal points and the remaining points are not extremal. Further in a n-dimensional cube, each of 2^n vertexes is an extremal point and the remaining points are not extremal. In a regular triangle, the vertexes are the only extremal points, In a regular pentagon there are no extremal points.

Let G act by isometries on $\mathbb{R}^n$. According to Proposition 13.3, $\mathbb{R}^n/G \in \mathrm{CBB}[0]$. Assume that the action of G on $\mathbb{R}^n$ is *properly discontinuous*; i.e., given a compact set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, there are only finitely many elements $g \in G$ such that $g \cdot x \in K$.

In this case $\mathbb{R}^n/G$ is a polyhedral space; it is not hard to prove, but we will not give a proof here. In particular, we can talk about volume and dimension of $\mathbb{R}^n/G$.

It tuns out that a point $\bar{x} \in \mathbb{R}^n/G$ is extremal if and only if one (and therefore any) point $x \in \mathbb{R}^n$ such that $\pi(x) = \bar{x}$ is an isolated fixed point for some subgroup of G. More precisely, if G_x is the subset of all isometries g in G such that $g \cdot x = x$ [74] then for any $y \neq x$ there is $g \in G_x$ such that $g \cdot y \neq x$. Therefore counting the number of such G-orbits is equivalent to the counting extremal points in $\mathbb{R}^n/G$. All this means that the following theorem implies some nontrivial information about the G-action.

13.5. Theorem. *Suppose G acts by isometries on $\mathbb{R}^n$ and this action is properly discontinuous. Then $\mathbb{R}^n/G$ has at most 2^n extremal points.*

Before going into proofs, let us consider a couple of examples which show that $\mathbb{R}^n/G$ can have exactly 2^n extremal points.

Consider the set of all isometries which can be presented as a composition of parallel translations $x \mapsto x + v$ with a vector v with all integer coordinates, and all the reflections in the coordinate hyperplanes. This defines a group action, say G on $\mathbb{R}^n$ for which the quotient $\mathbb{R}^n/G$ is isometric to the cube $[0, \frac{1}{2}]^n$. Note that each vertex of the cube is an extremal point.

[74] The subset G_x is in fact a subgroup of G and it is called the *stabilizer* of x.

14 Final exam

In the ticket you will see one theoretical question and three simple[75] problems from the exercises and the homework problems from the lecture notes. You will need to solve only 2 problems out of 3 (your choice).

Theoretical questions for the final exam

1. Compactness, nets and maximal packings. Theorem 1.25.
2. Lebesgue's number lemma 4.12; Theorem 4.13.
3. Hausdorff convergence and Blaschke's theorem (2.1).
4. Isoperimetric inequality in the plane. Theorem 1.27 = Theorem 1.26 + Theorem 2.7; structure of the proof + details in you favorite part.
5. Gromov–Hausdorff metric; definitions of Gromov–Hausdorff convergence via ε-isometries, Theorem 2.17.
6. Gromov's compactness theorem (Problem 3.A); structure of the proof + details in you favorite part.
7. Intrinsic spaces; Semicontinuity of length 4.3; Exercise 4.5; Proposition 4.6
8. Hopf–Rinow theorem 4.11; Exercise 4.8
9. Nerves and partition of unity (5.5)
10. Polyhedral spaces, Zalgaller's theorem 5.10 (compact 2-dimensional case).
11. Brehm's theorem 6.6; structure of the proof + details in you favorite part.
12. Akopyan's Theorem 6.10 (compact 2-dimensional case); structure of the proof + details in you favorite part.
13. Corollary 6.3; structure of the proof + details in you favorite part.
14. Bezdek–Connelly theorem (Conjecture 7.7 in case dim = 2); structure of the proof + details in you favorite part.
15. Bricard–Connelly theorem 8.11; structure of the proof + details in you favorite part.
16. Arm Lemma 9.5.
17. Cauchy's theorem 9.1 modulo Arm Lemma; structure of the proof + details in you favorite part.
18. Alexandrov existance theorem 10.1; structure of the proof + details in you favorite part.
19. Different definitions of Alexandrov space (Theorem 12.6); statement + your favorite non-trivial part.
20. Isometric group actions, propositions 13.2 and 13.3.

[75] A simple problem is one that is NOT considered hard. Here is the list of problems that ARE considered hard:
hard problems: 2.B, 3.A, 5.D, 9.C, 12.D
hard exercise is any exercise with word "try" inside plus the followng: 3.3, 3.6, 6.4, 7.14, 7.15, 10.7,

Procedure

Here is bit more info about the procedure; it is extracted from e-mail I got from Sergei Tabachnikov:

The exams are oral, and a student has 1 hour for presentation. After that, s/he is asked to wait outside, the grade is discussed by the committee, the student is called back and informed about the grade. We use all shades of letters, starting with A+ (not an official Penn State grade, but we use it for our purposes). We do hope that the great majority of the grades are flavors of A and B, but we can use lower grades if need be.

The student has 1 hour for preparation. S/he arrives at a reserved room, draws a random "ticket" with theoretical question(s) and problem(s), and prepares, in a proctored, closed book environment.

The examination committee consists of three people: the instructor, the TA, and a guest.

Each exam consists of three parts: the answer to the ticket question, an open-ended discussion with the committee (which may want to probe deeper or ask other questions on the course material), and the presentation of the research project (this part is prepared in advance and may use computer or slides).

The examination days are 12/08, 12/10, and 12/12. We shall need to meet after the last exam on the 12th to discuss special awards and fellowships. On 12/13, at 10 am, we shall have an official graduation ceremony.

Sample ticket

Theoretical question. Compactness, nets and maximal packings. Give a proof the following theorem:

Theorem. *Let X be a complete metric space. Then the following conditions are equivalent:*

a) X is compact;
b) $\operatorname{pack}_\varepsilon X$ is finite for any $\varepsilon > 0$;
c) X is totally bounded*; i.e., for any $\varepsilon > 0$ there is a finite ε-net in X.*

Problems. Do two of the following three problems:

A. Show that the set of (isometry classes of) intrinsic spaces in $\mathcal{M}$ is closed.

B. Let P be a (possibly nonconvex) polygon eqipped with the induced intrinsic metric. Show that P admits a triangulation[76] such that the set of vertexes of the triangulation is the set of vertexes of P.

C. Let $X \in \mathrm{CBB}[0]$ and $[x\,{}^y_z]$ be a hinge in X. Assume $\measuredangle[x\,{}^y_z] = 0$. Show that either $[xy] \subset [xz]$ or $[xz] \subset [xy]$.

[76] Recall that triangulations of polyhedral space are always assumed to be linear; in particular the sides of the triangles are formed by linesegments in the plane.

References

[1] Alexandrov, A. D. *Convex Polyhedra.* Springer Monographs in Mathematics 2005

[2] Burago, D.; Burago, Yu.; Ivanov, S. *A course in metric geometry.*

[3] Pak, I. *Lectures on Discrete and Polyhedral Geometry*