Exercises in orthodox geometry

Edited by A. Petrunin
orthodox.geometry@gmail.com

Abstract

Here is a collection of interesting problems in geometry. Solution of each problem requires only one non-trivial idea. The project is oriented to graduate students who want to learn fast simple tricks in geometry.

For problem solvers. Easy problems are marked by "$\circ$" and hard problems by "$*$". If the solution requires knowledge of some not quite standard theorem it is marked by "+", in this case the needed theorem are mentioned in the beginning of the section. The problems which are known to have solutions based on different ideas are marked by "$\sharp$".

To get a hint, send an e-mail to the above address with the number and the name of the problem.

For problem makers. This collection is under permanent development. If you have suitable problems or corrections please e-mail it to me.

Many thanks. I want to thank everyone sharing the problems. Also I want to thank R. Matveyev, P. Petersen, S. Tabachnikov and number of students in my classes for their interest in this list and for correcting number of mistakes.

1 Curves and surfaces

1.1. *Kneser's spiral.* Let γ be a plane curve with strictly monotonic curvature function. Prove that γ has no self-intersections. *[Ovsienko–Tabachnikov]*

1.2. *Closed curve.* A smooth closed *simple curve* with curvature at most 1 bounds a rigion Ω in a plane. Prove that Ω contains a disc or radius 1. *???*

1.3$^\sharp$ *A curve in a sphere.* Let γ be a closed curve in a unit sphere which intersects each equator, prove that its length is at least 2π. *N. Nadirashvili*

1.4$^\sharp$ *A spring in a tin.* Let α be a closed smooth immersed curve inside a unit disc. Prove that the average absolute curvature of α is at least 1, with equality if and only if α is the unit circle possibly traversed more than once. *???*

If instead of a disc we have a region bounded by closed convex curve γ then it is still true that the average absolute curvature of α is at least as big as average absolute curvature of γ. The proof is not that simple, see ???.

1.5. *A minimal surface.* Let Σ be a minimal surface in $\mathbb{R}^3$ which has boundary on a unit sphere. Assume Σ passes through the center of the sphere. Show that area of Σ is at least π. *???*

The problem is simpler if you assume that Σ is a topological disc. If Σ does not pass through the center and we only know the distance from center to Σ then optimal bound is not known.

1.6. *Asymptotic line.* Consider a smooth surface Σ in $\mathbb{R}^3$ given as a graph $z = f(x, y)$. Let γ be a closed smooth asymptotic line on Σ. Assume Σ is strictly saddle in a neighborhood of γ. Prove that projection of γ on xy-plane is not star shaped. *[D. Panov$_\alpha$]*

1.7. *Closed twisted geodesics.* Give an example of a closed Riemannian 2-manifold which has no closed smooth curve with constant geodesic curvature = 1.
[V. Ginzburg]

1.8. *Non contractable geodesics.* Give an example of a non-flat metric on 2-torus such that it has no contractible geodesics.
Y. Colin de Verdière, [M. Gromov$_\delta$, 7.8(1/2)+]

1.9. *Fat curve.* Construct a *simple plane curve* with non-zero Lebesgue's measure. *???*

2 Comparison geometry

For doing most of problems in this section it is enough to know second variation formula. Knowledge of some basic results such as O'Neil formula, Gauss formula, Gauss–Bonnet theorem, Toponogov's comparison theorem, Soul theorem, Toponogov splitting theorems and Synge's lemma also might help. Problem 2.12 requires Liouville's theorem for geodesic flow.

2.1. *Totally geodesic hypersurface.* Prove that if a compact positively curved m-manifold M admits a totally geodesic embedded hypersurface then M or its double cover is homeomorphic to the m-sphere. *P. Petersen*

2.2. *Immersed convex hypersurface I.* Let M be a complete simply connected Riemannian manifold with nonpositive curvature and $\dim M \geqslant 3$. Prove that any immersed locally convex hypersurface in M is globally convex, i.e. it is an embedded hypersurface which bounds a convex set. *[S. Alexander]*

2.3.* *Immersed convex hypersurface II.* Prove that any immersed locally convex hypersurface in a complete positively curved manifold M of dimension $m \geqslant 3$, is the boundary of an immersed ball. I.e. there is an immersion of a closed ball $\bar{B}^m \to M$ such that the induced immersion of its boundary $\partial\bar{B}^m \to M$ gives our hypersurface. *[M. Gromov$_\gamma$], [J. Eschenburg], [B. Andrews]*

2.4. *Almgren's inequalities.* Let Σ be a closed k-dimensional *minimal surface* in the unit $\mathbb{S}^n$. Prove that $\operatorname{vol}\Sigma \geqslant \operatorname{vol}\mathbb{S}^k$. *[F. Almgen], [M. Gromov$_\alpha$]*

2.5. *Hypercurve.* Let $M^m \hookrightarrow \mathbb{R}^{m+2}$ be a closed smooth m-dimensional submanifold and let g be the induced Riemannian metric on M^m. Assume that sectional curvature of g is positive. Prove that curvature operator of g is positively defined[1]. *[A. Weinstein$_\beta$]*

> *In particular, it follows from [Micallef–Moore]/[Böhm–Wilking] that the universal cover of M is homeomorphic/diffeomorphic to a standard sphere.*

2.6. *Horosphere.* Let M be a complete simply connected manifold with negatively pinched sectional curvature (i.e. $-a^2 \leqslant K \leqslant -b^2 < 0$). And let $\Sigma \subset M$ be an horosphere in M (i.e. Σ is a level set of a *Busemann function* in M). Prove that Σ with the induced intrinsic metric has polynomial volume growth[2]. *V. Kapovitch*

2.7. *Minimal spheres.* Show that a positively curved 4-manifold can not contain two distinct *equidistant minimal* 2-spheres. *D. Burago*

2.8. *Fixed point of conformal mappings.* Let (M, g) be an even-dimensional positively curved closed oriented Riemannian manifold and $f\colon M \to M$ be a conformal orientation preserving map. Prove that f has a fixed point.

[A. Weinstein$_\alpha$]

2.9. *Totally geodesic immersion.* Let (M, g) be a simply connected positively curved m-manifold and $N \hookrightarrow M$ be a totally geodesic immersion. Prove that if $\dim N > \frac{m}{2}$ then N is embedded. *B. Wilking*

2.10. *Minimal hypersurfaces.* Show that any two compact minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature must intersect.

[T. Frankel]

2.11. *Ricci curvature vs. symmetry.* Let (M, g) be a closed Riemannian manifold with negative Ricci curvature. Prove that isometry group of (M, g) is finite.

???

2.12.$^+$ *Scalar curvature vs. injectivity radius.* Let (M, g) be a closed Riemannian m-manifold with scalar curvature $\mathrm{Sc}_g \geqslant m(m-1)/2$ (i.e. bigger then scalar curvature of $\mathbb{S}^m$). Prove that the injectivity radius of (M, g) is at most π. *???*

2.13. *Almost flat manifold.* Show that for any $\epsilon > 0$ there is $n = n(\epsilon)$ such that there is a compact n-dimensional manifold M which is not a finite factor of a *nil-manifold* and which admits a Riemannian metric with diameter $\leqslant 1$ and sectional curvature $|K| < \epsilon$. *[G. Guzhvina]*

---

[1] The Riemannian curvature tensor R can be viewed as an operator $\mathbf{R}$ on bi-vectors defined by

$$\langle \mathbf{R}(X \wedge Y), Z \wedge T\rangle = \langle R(X, Y)Z, T\rangle,$$

The operator $\mathbf{R}\colon \bigwedge^2 T \to \bigwedge^2 T$ is called curvature operator and it is positively defined if $\langle \mathbf{R}(\varphi), \varphi\rangle > 0$ for all non zero bi-vector $\varphi \in \bigwedge^2 T$.

[2] i.e. $\operatorname{vol} B_r(p) \leqslant C(r^k + 1)$, where $B_r(p)$ is the ball in Σ of radius r centered at $p \in \Sigma$ and C, k are real constants.

2.14. *Lie group.* Show that the space of non-negatively curved left invariant metrics on a compact Lie group G is contractable. *B. Wilking*

2.15. *Simple geodesic.* Let g be a complete Riemannian metric with positive curvature on $\mathbb{R}^2$. Show that there is a two-sided infinite geodesic in $(\mathbb{R}^2, g)$ with no self-intersections. *[V. Bangert]*

2.16.$^\sharp$ *Polar points.* Let (M, g) be a Riemannian manifold with sectional curvature $\geqslant 1$. A point $p^* \in M$ is called polar to $p \in M$ if $|px| + |xp^*| \leqslant \pi$ for any point $x \in M$. Prove that for any point in (M, g) there is a polar. *[A. Milka]*

2.17. *Deformation to a product.* Let (M, g) be a compact Riemannian manifold with non-negative sectional curvature. Show that there is a continuous one parameter family of non-negatively curved metrics g_t on M, $t \in [0, 1]$, such that a finite Riemannian cover of (M, g_1) is isometric to a product of a flat torus and a simply connected manifold. *[B. Wilking]*

2.18.* *Isometric section.* Let $s\colon (M, g) \to (N, h)$ be a Riemannian submersion. Assume that g is positively curved. Show that s does not admit an isometric section; i.e. there is no isometry $\imath\colon (N, h) \hookrightarrow (M, g)$ such that $s \circ \imath = \mathrm{id}_N$. *[G. Perelman]*

2.19.$^\sharp$ *Minkowski space.* Let us denote by $\mathbb{M}^m$ the set $\mathbb{R}^m$ equipped with the metric induced by the ℓ^p-norm. Prove that if $p \neq 2$ then $\mathbb{M}^m$ can not be a Gromov–Hausdorff limit of Riemannian m-manifolds (M_n, g_n) such that $\mathrm{Ric}_{g_n} \geqslant C$ for some contant $C \in \mathbb{R}$.

2.20. *An island of scalar curvature.* Construct a Riemannian metric g on $\mathbb{R}^3$ which is Euclidean outside of an open bounded set Ω and scalar curvature of g is negative in Ω. *[J. Lohkamp]*

3 Curvature free

Most of the problems in this section require no special knowledge. Solution of 3.1 relies on Gromov's pseudo-holomorphic curves; problem 3.4 uses Liouville's theorem for geodesic flow.

3.1.$^+$ *Minimal foliation.* Consider $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with a Riemannian metric g which is C^∞-close to the product metric. Prove that there is a conformally equivalent metric λg and re-parametrization of $\mathbb{S}^2 \times \mathbb{S}^2$ such that each sphere $\mathbb{S}^2 \times x$ and $y \times \mathbb{S}^2$ forms a *minimal surface* in $(\mathbb{S}^2 \times \mathbb{S}^2, \lambda g)$.

3.2. *Smooth doubling.* Let N be a Riemannian manifold with boundary which is isometric to $(M, g)/\mathbb{S}^1$, where g is an $\mathbb{S}^1$-invariant complete smooth Riemannian metric on M. Prove that the *doubling* of N is a smooth Riemannian manifold. *[A. Lytchak]*

3.3. *Loewner's theorem.* Given $\mathbb{R}\mathrm{P}^n$ equipped with a Riemannian metric g conformally equivalent to the canonical metric g_{can} let ℓ denote the minimal length of curves in $(\mathbb{R}\mathrm{P}^n, g)$ not homotopic to zero. Prove that

$$vol(\mathbb{R}\mathrm{P}^n, g) \geqslant vol(\mathbb{R}\mathrm{P}^n, g_{\mathrm{can}})(\ell/\pi)^n$$

and in case of equality $g = c g_{\mathrm{can}}$ for some positive constant c. *???*

3.4.$^+$ *Convex function — infinite volume.* Let M be a complete Riemannian manifold which admits a non-constant convex function. Prove that M has infinite volume. *[S. Yau]*

3.5. *Besikovitch inequality.* Let g be a Riemannian metric on a n-dimensional cube $Q = (0,1)^n$ such that any curve connecting opposite faces has length $\geqslant 1$. Prove that $\mathrm{vol}(Q, g) \geqslant 1$ and equality holds if and only if (Q, g) is isometric to the interior of the unit cube. *???*

3.6. *Mercedes-Benz sphere.* Construct on $\mathbb{S}^3$ a Riemannian metric g and involution $\imath\colon \mathbb{S}^3 \to \mathbb{S}^3$ such that $|x\,\imath(x)|_g > 1$ and $\mathrm{vol}\, g$ is arbitrary small. *[C. Croke]*

Note that for $\mathbb{S}^2$ such thing is not possible.

4 Metric geometry.

The necessary definitions can be found in [Burago–Burago–Ivanov]. It is very hard to do 4.2 without using Kuratowski embedding. To do problem 4.5 first do problem 4.4; to do this problem you have to know a construction of compact manifolds of constant negative curvature of given dimension m. To do problem 4.11 you should be familiar with the proof of Nash–Kuiper theorem. Problems 4.12 and 4.13 are similar, in both you have to know Rademacher's theorem on differentiability of Lipschitz maps.

4.1.$^\circ$ *Noncontracting map.* Let X be a compact metric space and $f\colon X \to X$ be a noncontracting map. Prove that f is an isometry.

4.2.$^+$ *Embedding of a compact.* Prove that any compact metric space is isometric to a subset of a compact *length space.*

4.3. *Bounded orbit.* Let X be a *proper metric space* and $\imath\colon X \to X$ is an isometry. Assume that for some $x \in X$, the the orbit $\imath^n(x)$, $n \in \mathbb{Z}$ has a partial limit in X. Prove that for one (and hence for any) $y \in X$, the orbit $\imath^n(y)$ is bounded. *[A. Całka]*

4.4. *Covers of figure eight.* Let (Φ, d) be a "figure eight"; i.e. a metric space which is obtained by gluing together all four ends of two unit segments.

Prove that any compact *length space* X is a Gromov–Hausdorff limit of a sequence of metric covers $(\widetilde{\Phi}_n, \tilde{d}/n) \to (\Phi, d/n)$.

4.5.$^+$ *Constant curvature is everything.* Given $m \in \mathbb{Z}_+$, prove that any compact *length space* X is a Gromov–Hausdorff limit of a sequence of m-dimensional manifolds M_n with curvature $-n^2$.

4.6. *2-sphere is far from a ball.* Show that there is no sequence of Riemannian metrics on $\mathbb{S}^2$ which converge in Gromov–Hausdorff topology to the standard ball $\bar{B}^2 \subset \mathbb{R}^2$.

4.7. *3-sphere is close to a ball.* Construct a sequence of Riemannian metrics on $\mathbb{S}^3$ which converge in Gromov–Hausdorff topology to the standard ball $\bar{B}^3 \subset \mathbb{R}^3$. *???*

4.8.$^\circ$ *Macrodimension.* Let M be a simply connected Riemannian manifold with the following property: any closed curve can be shrunk to a point in an ε-neighborhood of itself. Prove that M is 1-dimensional on scale $10^{10}\varepsilon$, i.e. there is a graph Γ and a continuous map $f\colon M \to \Gamma$, such that for any $x \in \Gamma$ we have $\operatorname{diam}(f^{-1}(x)) \leqslant 10^{10}\varepsilon$. *N. Zinoviev*

4.9. *Anti-collapse.* Construct a sequence of Riemannian metric g_i on a 2-sphere such that $vol\, g_i < 1$ such that induced distance-metrics $d_i\colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}_+$ converge to a metric $d\colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}_+$ with arbitrary large Hausdorff dimension. *[Burago–Ivanov–Shoenthal]*

4.10.* *No short embedding.* Construct a length-metric on $\mathbb{R}^3$ which admits no local *short* embeddings into $\mathbb{R}^3$. *[Burago–Ivanov–Shoenthal]*

4.11.$^+$ *Sub-Riemannian sphere.* Prove that any *sub-Riemannian metric* on the n-sphere is isometric to the intrinsic metric of a hypersurface in $\mathbb{R}^{n+1}$.

4.12.$^+$ *Path isometry.* Show that there is no *path isometry* $\mathbb{R}^2 \to \mathbb{R}$.

4.13.$^+$ *Minkowski plane.* Let $\mathbb{M}^2$ be a *Minkowski plane* which is not isometric to the Euclidean plane. Show that $\mathbb{M}^2$ does not admit a *path isometry* to $\mathbb{R}^3$.

4.14. *Hyperbolic space.* Show that the hyperbolic 3-space is *quasi-isometric* to a subset of product of two hyperbolic planes. *???*

4.15. *Kirszbraun's theorem.* Let $X \subset \mathbb{R}^2$ be an arbitrary subset and $f\colon X \to \mathbb{R}^2$ be a *short map.* Show that f can be extended as a short map to whole $\mathbb{R}^2$; i.e. there is a short map $\bar{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that its restriction to X coinsides with f. *???*

4.16. *Hilbert problem.* Let F be a convex plane figure. Construct a complete Finsler metric d on the interior of F such that any line segment in F forms a geodesic of d. *???*

5 Topology.

5.1. *Milnor's disks.* Construct two "topologically different" smooth immersions of the disk into the plane which coincide near the boundary. (Two immersions $f_1, f_2\colon D \to \mathbb{R}^2$ are topologically different if there is no diffeomorphism $h\colon D \to D$ such that $f_1 = f_2 \circ h$) *[M. Gromov$_\beta$, ???]*

5.2. *Conic neighborhood.* Let $p \in X$ be a point in a topological space X. We say that an open neighborhood U_p of $p \in X$ is conic if there is a homeomorphism from a cone to U_p which sends its vertex to p. Show that any two conic neighborhoods of p are homeomorphic to each other. *[K. Kwun]*

Note that for two cones $\mathrm{Cone}(\Sigma_1)$ *and* $\mathrm{Cone}(\Sigma_2)$ *might be homeomorphic while* Σ_1 *and* Σ_2 *are not.*

5.3.$^\circ$ *Positive Dehn twist.* Let Σ be an oriented surface with non empty boundary. Prove that any composition of *positive Dehn twists* of Σ is not homotopic to identity *rel* boundary. *R. Matveyev*

5.4.$^\circ$ *Simmetric square.* Let X be a connected topological space. Note that $X \times X$ admits natural $\mathbb{Z}_2$-action by $(x, y) \mapsto (y, x)$. Show that fundamental group of $X \times X/\mathbb{Z}_2$ is commutative. *R. Matveyev*

5.5.$^\circ$ *Sierpinski triangle.* Find the group of homeomorphisms of Sierpinski triangle. *B. Kliener*

5.6. *Knaster's circle.* Construct a bounded open set in $\mathbb{R}^2$ whose boundary does not contain a *simple curve.* *[L. Wayne]*

6 Descrete geometry

It is suggested to do Kirszbraun's theorem (4.15) before doing problem 6.3. One of the solutions of 6.8 uses mixed volumes.

6.1.$^\circ$ *4-polyhedron.* Give an example of a convex 4-dimensional polyhedron with 100 vertices, such that any two vertices are connected by an edge.

6.2. *PL-isometry I.* Let P be a compact m-dimensional *polyhedral space.* Construct a *PL-isometry* $f\colon P \to \mathbb{R}^m$. *[V. Zalgaller]*

6.3.$^+$ *PL-isometry II.* Prove that any *short map* to $\mathbb{R}^2$ which is defined on a finite subset of $\mathbb{R}^2$ can be extended to a *PL-isometry* $\mathbb{R}^2 \to \mathbb{R}^2$. *[Akopyan–Tarasov]*

6.4. *Minimal polyhedron.* Consider the class of polyhedral *surfaces* in $\mathbb{R}^7$ with fixed boundary curve such that each (1) is homeomorphic to a 2-disc and (2) is glued out n triangles. Let Σ_n be a surface of minimal area in this class. Show that Σ_n is a *saddle surface.*

Note that it is not longer true if Σ *minimizes area only in the class of polyhedral surfaces with fixed triangulation.*

6.5. *Convex triangulation.* A triangulation of a convex polygon is called convex if there is a convex function which is linear on each triangle and changes the gradient if you come trough any edge of the triangulation.

Find a non-convex triangulation. *[Gelfand–Kapranov–Zelevinsky]*

6.6.* *A sphere with one edge.* Let P be a finite 3-dimensional simplicial complex with *spherical polyhedral metric.* Let us denote by P_s the subset of singular[3] points of P.

Construct P which is homeomorphic to $\mathbb{S}^3$ and such that P_s is formed by a knotted circle. Show that in such an example the total length of P_s can be arbitrary large and the angle around P_s can be made strictly less than 2π. *[D. Panov$_\beta$]*

6.7. *Monotonic homotopy.* Let F be a finite set and $h_0, h_1 \colon F \to \mathbb{R}^m$ be two maps. Consider $\mathbb{R}^m$ as a subspace of $\mathbb{R}^{2m}$. Show that there is a homotopy $h_t : F \to \mathbb{R}^{2m}$ from H_1 to h_2 such that for any $x, y \in F$ the function $t \mapsto |h_t(x) - h_t(y)|$ is monotonic. *[R. Alexander]*

6.8.$^\sharp$ *Box in a box.* Assume that one rectangular box with sizes a, b, c is inside another with sizes A, B, C. Show that $A + B + C \geqslant a + b + c$. *[A. Shen]*

6.9.* *Vanishing sectors.* Show that one can cut a unit plane disc by radii into sectors and then move each sector by a parallel translation on such a way that its union will have arbitrary small area. *???*

6.10. *Boys and girls in a Lie group.* Let L_1 and L_2 be two discrete subgroups of a Lie group G, h be a left invariant metric on G and ρ_i be the induced left invariant metric on L_i. Assume $L_i\backslash G$ are compact and moreover

$$\mathrm{vol}(L_1\backslash(G,h)) = \mathrm{vol}(L_2\backslash(G,h)).$$

Prove that there is bi-Lipschitz one-to-one mapping (not necessarily a homomorphism) $f \colon (L_1, \rho_1) \to (L_2, \rho_2)$. *D. Burago*

7 Dictionary

Busemann function. Let X be a metric spaces and $\gamma \colon [0, \infty) \to X$ is a geodesic ray; i.e. it is a one side infinite geodesic which is minimizing on each interval. The Busemann function of γ is defined by

$$b_\gamma(p) = \lim_{t\to\infty} (|\gamma(t)\, p| - t).$$

From the triangle inequality, it is clear that the limit above is well defined.

[3]A point is called *regular* if it has a neighborhood isometric to an open set of standard sphere; it is called singular point *otherwise*.

Dehn twist. Let Σ be a surface and $\gamma\colon \mathbb{R}/\mathbb{Z} \to \Sigma$ be *simple noncontractible closed curve*. Let U_γ be a neighborhood of γ which admits a homeomorphism $h\colon U_\gamma \to \mathbb{R}/\mathbb{Z} \times (0,1)$. Dehn twist along γ is a homeomorphism $f\colon \Sigma \to \Sigma$ which is identity outside of U_γ and $h \circ f \circ h^{-1}\colon (x,y) \mapsto (x+y,y)$.

If Σ is orientable, then the Dehn twist described above is called *positive* if h is orientation preserving.

Doubling of a manifold M with boundary ∂M is is two copies of M_1, M_2 identified along corresponding points on the boundary $\partial M_1, \partial M_2$.

Equidistant subsets. Two subsets A and B in a metric space are called equidistant if dist_A is constant on B and dist_B is constant on A.

Length space. A complete metric space X is called *length space* if the distance between any pair of points in X is equal to the infimum of lengths of curves connecting these points.

Minimal surface. Let Σ be a k-dimensional smooth surface in a Riemannian manifold M and $T(\Sigma)$ and $N(\Sigma)$ correspondingly tangent and normal bundle. Let $s\colon T \otimes T \to N$ denotes the second fundamental form of Σ. Let e_i is an orthonormal basis for T_x, set $H_x = \sum_i s(e_i, e_i) \in N_x$; it is the mean curvature vector at $x \in \Sigma$.

We say that Σ is *minimal* if $H \equiv 0$.

Minkowski space — $\mathbb{R}^m$ with a metric induced by a norm.

Nil-manifolds form the minimal class of manifolds which includes a point, and has the following property: the total space of any oriented $\mathbb{S}^1$-bundle over a nil-manifold is a nil-manifold.

It also can be defined as a factor of a connected nilpotent Lie group by a lattice.

Path isometry A map $f : X \to Y$ of length spaces X and Y is a path isometry if for any path $\alpha : [0,1] \to X$, we have

$$\mathrm{length}(\alpha) = \mathrm{length}(f \circ \alpha).$$

Polyhedral space — a simplicial complex with a metric such that each simplex is isometric to a simplex in a Euclidean space.

It admits the following generalizations:

spherical (hyperbolic) polyhedral space — a simplicial complex with a metric such that each simplex is isometric to a simplex in a unit sphere (corresp. hyperbolic space of constant curvature -1).

Proper metric space. A metric space X is called *proper* if any closed bounded set in X is compact.

PL-isometry — a piecewise linear map from a polyhedral space which is isometric on each simplex. More precisely: Let P and Q be polyhedral spaces, a map $f\colon P \to Q$ is called PL-isometry if there is a triangulation $\mathcal{T}$ of P such that at any simplex $\Delta \in \mathcal{T}$ the restriction $f|_\Delta$ is globally isometric.

Quasiisometry. A map $f\colon X \to Y$ is called a quasiisometry if there is a constant $C < \infty$ such that $f(X)$is a C-net in Y and

$$\frac{|xy|}{C} - C \leqslant |f(x)f(y)| \leqslant C|xy| + C$$

Note that a quasiisometry is not assumed to be continuous, for example any map between compact metric spaces is a quasiisometry.

Saddle surface. A smooth surface Σ in $\mathbb{R}^3$ is saddle (correspondingly strictly saddle) if the product of the principle curvatures at each point is $\leqslant 0$ (correspondingly < 0).

It admits the following generalization to non-smooth case and arbitrary dimension of the ambient space: A surface Σ in $\mathbb{R}^n$ is saddle if the restriction $\ell|_\Sigma$ of any linear function $\ell\colon \mathbb{R}^3 \to \mathbb{R}$ has no strict local minima at interior points of Σ.

One can generalize it further to an arbitrary ambient space, using convex functions instead of linear functions in the above definition.

Short map — a distance non increasing map.

Simple curve — an image of a continuous injective map of a real segment or a circle.

References

[Akopyan–Tarasov] Akopyan, A.; Tarasov, A. ???

[R. Alexander] Alexander, R., *Lipschitzian mappings and total mean curvature of polyhedral surfaces,* no. I, Trans. Amer. Math. Soc. 288 (1985), no. 2, 661–678.

[S. Alexander] Alexander, S., Proc. Amer. Math. Soc. 64 (1977), no. 2, 321–325;

[F. Almgen] Almgren, F. *Optimal isoperimetric inequalities.* Indiana Univ. Math. J. 35 (1986), no. 3, 451–547.

[B. Andrews] Andrews, B., *Contraction of convex hypersurfaces in Riemannian spaces.* J. Differential Geom. 39 (1994), no. 2, 407–431.

[V. Bangert] Bangert, V, *On the existence of escaping geodesics.* Comment. Math. Helv. 56 (1981), no. 1, 59–65.

[Burago–Burago–Ivanov] Burago, D.; Burago, Yu.; Ivanov, S., *A course in metric geometry.* Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. xiv+415 pp. ISBN: 0-8218-2129-6

[Burago–Ivanov–Shoenthal] Burago, D.; Ivanov, S.; Shoenthal, D. Two counterexamples in low dimensional length geometry, St.Petersburg Math. J. Vol. 19 (2008), No. 1, Pages 33–43

[Böhm–Wilking] Böhm, C.; Wilking, B., *Manifolds with positive curvature operators are space forms.* International Congress of Mathematicians. Vol. II, 683–690, Eur. Math. Soc., Zürich, 2006.

[A. Całka] Całka, A., *On conditions under which isometries have bounded orbits.* Colloq. Math. 48 (1984), 219–227

[C. Croke] Croke, C. Small volume on big n-spheres. Proc. Amer. Math. Soc. 136 (2008), no. 2, 715–717

[T. Frankel] Frankel, T. O*n the fundamental group of a compact minimal submanifold.* Ann. of Math. (2) 83 1966 68–73.

[V. Ginzburg] Ginzburg, V. L., *On the existence and non-existence of closed trajectories for some Hamiltonian flows.* Math. Z. 223 (1996), no. 3, 397–409.

[Gelfand–Kapranov–Zelevinsky] Gelfand, I. M.; Kapranov, M. M.; Zelevinsky, A. V., *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994, x+523, 0-8176-3660-9.

[M. Gromov$_\alpha$] Gromov's Appendix in Milman, Vitali D.; Schechtman, Gideon Asymptotic theory of finite-dimensional normed spaces. Lecture Notes in Mathematics, 1200. Springer-Verlag, Berlin-New York, 1986. viii+156 pp.

[M. Gromov$_\beta$] Gromov, M., Partial Differential relations

[M. Gromov$_\gamma$] Gromov, M., Sign and geometric meaning of curvature. Rend. Sem. Mat. Fis. Milano 61 (1991), 9–123 (1994). 53C20 (53C23)

[M. Gromov$_\delta$] Gromov, M., Metric structures for Riemannian and non-Riemannian spaces. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. xx+585 pp. ISBN: 978-0-8176-4582-3; 0-8176-4582-9

[G. Guzhvina] Guzhvina, G., Gromov's pinching constant, arXiv:0804.0201v1

[J. Eschenburg] Eschenburg, J.-H. *Local convexity and non-negative curvature — Gromov's proof of the sphere theorem.* Invent. Math. 84 (1986), no. 3, 507–522.

[K. Kwun] Kwun, K. W., *Uniqueness of the open cone neighborhood,* Proc. Araer. Math. Soc. 15 (1964) 476–473

[J. Lohkamp] Lohkamp, J. ???

[A. Lytchak] ???

[A. Milka] Milka, A. D. *Multidimensional spaces with polyhedral metric of non-negative curvature.* I. (Russian) Ukrain. Geometr. Sb. Vyp. 5–6 1968 103–114.

[Micallef–Moore] Micallef, M. J.; Moore, J. D., *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes.* Ann. of Math. 127 pp.199–227 1988

[Ovsienko–Tabachnikov] Ovsienko, V., Tabachnikov, S., Projective differential geometry, old and new: from Schwarzian derivative to cohomology of diffeomorphism groups, Cambridge Univ. Press, 2005.

[D. Panov$_\alpha$] Panov, D. A., *Parabolic curves and gradient mappings.* Local and global problems of singularity theory (Russian). Tr. Mat. Inst. Steklova 221 (1998), 271–288.

[D. Panov$_\beta$] Panov, D., *Polyhedral Käler Manifolds.* Geometry and Topology. ???

[G. Perelman] Perelman, G. *Proof of the soul conjecture of Cheeger and Gromoll.* J. Differential Geom. 40 (1994), no. 1, 209–212.

[A. Shen] Shen A., *???* Math. Intelligencer 21 (1999), no. 3.

[L. Wayne] Wayne, L., *The Pseudo-Arc,* Bol. Soc. Mat. Mexicana Volume 5. (1999), 25–77.

[A. Weinstein$_\alpha$] Weinstein, A., *A fixed point theorem for positively curved manifolds.* J. Math. Mech. 18 1968/1969 149–153.

[A. Weinstein$_\beta$] Weinstein, A. *Positively curved n-manifolds in $\mathbb{R}^{n+2}$.* J. Differential Geometry 4 1970 1–4.

[B. Wilking] Wilking, B., *On fundamental groups of manifolds of nonnegative curvature.* Differential Geom. Appl. 13 (2000), no. 2, 129–165.

[S. Yau] Yau, S. T., *Non-existence of continuous convex functions on certain Riemannian manifolds.* Math. Ann. 207 (1974), 269–270.

[V. Zalgaller] Zalgaller, V. A. *Isometric embedding of polyhedra.* (Russian) Dokl. Akad. Nauk SSSR 123 1958 599–601.