Alexandrov meets Lott–Villani–Sturm

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Abstract

Here I show compatibility of two definition of generalized curvature bounds — the lower bound for sectional curvature in the sense of Alexandrov and lower bound for Ricci curvature in the sense of Lott–Villani–Sturm.

Introduction

Let me denote by $\mathrm{CD}[m,\kappa]$ the class metric-measure spaces which satisfy a weak curvature-dimension condition for dimension m and curvature κ (see preliminaries). By $\mathrm{Alex}^m[\kappa]$, I will denote the class of all m-dimensional Alexandrov spaces with curvature $\geqslant \kappa$ equipped with the volume-measure (so $\mathrm{Alex}^m[\kappa]$ is a class of metric-measure spaces).

Main theorem. $\mathrm{Alex}^m[0] \subset \mathrm{CD}[m,0]$.

The question appears first in [Lott–Villani, 7.48]. In [Villani], it is formulated more generally: $\mathrm{Alex}^m[\kappa] \subset \mathrm{CD}[m,(m-1)\kappa]$. The later statement can be proved, along the same lines, but I do not write it down.

About the proof. The idea of the proof is the same as in Riemannian case (see [CMS, 6.2] or [Lott–Villani, 7.3]). One only needs to extend certain calculus to Alexandrov spaces. To do this, I used the same technique as in [Petrunin 03]. I will illustrate the idea on a very simple problem.

Let M be a 2-dimensional non-negatively curved Riemannian manifold and $\gamma_\tau\colon [0,1] \to M$ be a continuous family of unit-speed geodesics such that

$$|\gamma_{\tau_0}(t_0)\,\gamma_{\tau_1}(t_1)| \geqslant |t_1 - t_0|. \qquad ❶$$

Set $\ell(t)$ to be the total length of curve $\sigma_t\colon \tau \mapsto \gamma_\tau(t)$. Then $\ell(t)$ is a concave function — that is easy to prove.

Now, assume you have $A \in \mathrm{Alex}^2[0]$ instead of M and a non-continuous family of unit-speed geodesics $\gamma_\tau(t)$ which satisfies ❶. Define $\ell(t)$ as the 1-dimensional Hausdorff measure of image of σ_t. In this case ℓ is also concave.

Here is an idea how one can proceed; it is not the simplest one but the one which admits proper generalization. Consider two functions $\psi = \mathrm{dist}_{\mathrm{Im}\,\sigma_0}$ and $\varphi = \mathrm{dist}_{\mathrm{Im}\,\sigma_1}$. Note that geodesics $\gamma_\tau(t)$ are also gradient curves of ψ and φ. It implies that $\Delta\varphi + \Delta\psi$ vanish almost everywhere on the image of the map $(\tau,t) \to \gamma_\tau(t)$ (the Laplasians $\Delta\varphi$ and $\Delta\psi$ are Radon sign-measure). Then result follows from the second variation formula from [Petrunin 98] and calculus on Alexandrov spaces developed in [Perelman].

Remark. My original motivation to do this problem was a misunderstanding — I thought that CD-spaces are analogous to Alexandrov spaces. But although $\mathrm{CD}[m,\kappa]$ is a very natural class of metric-measure spaces, some basic tools in Ricci comparison can not work there in principle. For instance, *there are* $\mathrm{CD}[m,0]$*-spaces which do not satisfy the Abresch–Gromoll inequality*, (see [AG]). Thus, in order to be suitable for substantial applications in Riemannian geometry, one has to modify the definition of the class $\mathrm{CD}[m,\kappa]$.

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1 Preliminaries

Prerequisite. The reader is supposed to be familiar with basic definition and notion in optimal transport as in [Villani], measure theory in Alexandrov spaces from [BGP], DC-structure on Alexandrov's spaces from [Perelman] and technique and notation of gradient flow as in [Petrunin 07].

What needs to be proved. Let me recall the definition of class $\mathrm{CD}[m,0]$ only — it is sufficient for understanding this paper. The definition of $\mathrm{CD}[m,\kappa]$ can be found in [Villani, 29.8].

Similar definitions were given in [Lott–Villani] and [Sturm]. The idea behind these definitions — convexity of certain functionals in the Wasserstein space over a Riemannian manifold, appears in [Otto–Villani], [CMS], [Sturm–v.Renesse]. In the Euclidean context, this notion of convexity goes back to [McCann]. More on the history of the subject can be found in [Villani].

For a metric-measure space X, I will denote by $|xy|$ the distance between points $x,y\in X$ and by $\operatorname{vol} E$ the distinguished measure of Borel subset $E\subset X$ (I will call it *volume*). Let us denote by P_2X the set of all probability measures with compact support in X equipped with Wasserstein distance of order 2, see [Villani, 6.1].

Further, we assume X is proper geodesic space; in this case P_2X is geodesic.

Let μ be a probability measure on X. Denote by μ^r the absolutely continuous part of μ with respect to volume. I.e. μ^r coincides with μ outside a Borel subset of volume zero and there is a Borel function $\rho\colon X\to\mathbb{R}$ such that $\mu^r=\rho\cdot\mathrm{vol}$. Define

$$U_m\mu \stackrel{def}{=\!=} \int\limits_X \rho^{1-\frac{1}{m}}\,d\,\mathrm{vol} = \int\limits_X \frac{1}{\sqrt[m]{\rho}}\,d\mu^r.$$

Then $X\in\mathrm{CD}[m,0]$ if the functional U_m is concave in P_2X; i.e for any two measures $\mu_0,\mu_1\in\mathrm{P}_2X$, there is a geodesic path μ_t, in P_2X, $t\in[0,1]$ such that the real function $t\mapsto U_m\mu_t$ is concave.

Calculus in Alexandrov spaces. Let $A\in\mathrm{Alex}^m[\kappa]$ and $S\subset A$ be the subset of singular points; i.e. $x\in S$ if its tangent space T_x is not isometric to Euclidean m-space $\mathbb{E}^m$. The set S has zero volume ([BGP, 10.6]). The set of regular points $A\backslash S$ is convex ([Petrunin 98]); i.e. any geodesic connecting two regular points consits only of regular points.

According to [Perelman], if $f\colon A\to\mathbb{R}$ is a semiconcave function and $\Omega\subset A$ is an image of a DC_0-chart, then $\partial_k f$ and components of metric tensor g^{ij} are functions of locally bounded variation which are continuous in $\Omega\backslash S$.

Further, for almost all $x\in A$ the the Hession of f is well defined. I.e. there is a subset of full measure $\operatorname{Reg} f\subset A\backslash S$ such that for any $p\in\operatorname{Reg} f$ there is a bi-linear form[1] $\operatorname{Hess}_p f$ on T_p such that

$$f(q)=f(p)+d_pf(v)+\operatorname{Hess}_p f(v,v)+o(|v|^2),$$

where $v=\log_p q$. Moreover, the Hessian can be found using standard calculus in the DC_0-chart. In particular,

$$\operatorname{Trace}\operatorname{Hess} f\overset{a.e.}{=\!=}\frac{\partial_i(\det g\cdot g^{ij}\cdot\partial_j f)}{\det g}$$

The following is an extract from second variation formula [Petrunin 98, 1.1B] reformulated with formalism of ultrafilters. Let ω be a nonprinciple ultrafiler, $A\in\operatorname{Alex}^m[0]$ and $[pq]$ be a minimizing geodesic in A which is extendable beyond p and q. Assume further that one of (and therefore each) of points p and q is regular. Then there is a model configuration $\tilde p,\tilde q\in\mathbb{E}^m$ and isometries $\imath_p\colon \mathrm{T}_pA\to\mathrm{T}_{\tilde p}\mathbb{E}^m$, $\imath_q\colon\mathrm{T}_qA\to\mathrm{T}_{\tilde q}\mathbb{E}^m$ such that

$$\left|\exp_p(\tfrac1n\cdot v)\ \exp_q(\tfrac1n\cdot w)\right|\leqslant\left|\exp_{\tilde p}\circ\imath_p(\tfrac1n\cdot v)\ \exp_{\tilde q}\circ\imath_q(\tfrac1n\cdot w)\right|+o(n^2)$$

for ω-almost all n (once the left-hand side is well defined).

If $\tilde\tau\colon\mathrm{T}_{\tilde p}\to\mathrm{T}_{\tilde q}$ is the parallel translation in $\mathbb{E}^m$, then the isometry $\tau\colon\mathrm{T}_p\to\mathrm{T}_q$ which satisfy identity $\imath_q\circ\tau=\tilde\tau\circ\imath_p$ will be called "parallel transportation" from p to q.

Laplacian of semiconcave function. Here are some facts from [Petrunin 03].

Given a function $f\colon A\to\mathbb{R}$, define its *Laplacian* Δf to be a Radon sign-measure which satisfies the following identity

$$\int\limits_A u\,d\Delta f=-\int\limits_A\langle\nabla u,\nabla f\rangle\,d\operatorname{vol}$$

for any Lipschitz function $u\colon A\to\mathbb{R}$.

1.1. Claim. *Let $A\in\operatorname{Alex}^m[\kappa]$ and $f\colon A\to\mathbb{R}$ be λ-concave Lipschitz function. Then Laplacian Δf is well defined and*

$$\Delta f\leqslant m\lambda\cdot\operatorname{vol}.$$

In particular, $\Delta^s f$ — the singular part Δf is negative.

Moreover,

$$\Delta f=\operatorname{Trace}\operatorname{Hess} f\cdot\operatorname{vol}+\Delta^s f.$$

Proof. Let us denote by $F_t\colon A\to A$ the f-gradient flow for time t.

Given a Lipschitz function $u\colon A\to\mathbb{R}$, consider family $u_t(x)=u\circ F_t(x)$. Clearly, $u_0\equiv u$ and u_t is Lipschitz for any $t\geqslant 0$. Further, for any $x\in A$ we have $\left|\tfrac{d^+}{dt}u_t(x)|_{t=0}\right|\leqslant\mathrm{Const}$. Moreover

$$\tfrac{d^+}{dt}u_t(x)|_{t=0}\overset{a.e.}{=\!=}d_xu(\nabla_xf)\overset{a.e.}{=\!=}\langle\nabla_xu,\nabla_xf\rangle.$$

[1]Note that $p\in A\backslash S$, thus T_p is isometric to Euclidean m-space.

Further,

$$\int\limits_A u_t\, d\,\mathrm{vol} = \int\limits_A u\, d(F_t\#\,\mathrm{vol}),$$

where # stands for push-forward. Since $|F_t(x)F_t(y)| \leqslant e^{\lambda t}|xy|$, for any $x, y \in A$ we have

$$F_t\#\,\mathrm{vol} \geqslant \exp(-m\lambda t)\cdot \mathrm{vol}\,.$$

Therefore, for any non-negative Lipschitz function $u\colon A \to \mathbb{R}$,

$$\int\limits_A u_t\, d\,\mathrm{vol} = \int\limits_A u\, d(F_t\#\,\mathrm{vol}) \geqslant \exp(-m\lambda t)\int\limits_A u\, d\,\mathrm{vol}\,.$$

Therefore

$$\int\limits_A \langle\nabla u, \nabla f\rangle\, d\,\mathrm{vol} = \tfrac{d^+}{dt}\int\limits_A u_t\, d\,\mathrm{vol}\Big|_{t=0} \geqslant -m\lambda \int\limits_A u\, d\,\mathrm{vol}\,.$$

I.e. there is a Radon measure χ on A, such that

$$\int\limits_A u\, d\chi = \int\limits_A \left[\langle\nabla u, \nabla f\rangle + m\lambda u\right]\, d\,\mathrm{vol}$$

Set $\Delta f = -\chi + m\lambda$, it is a Radon sign-measure and $\chi = -\Delta f + m\lambda \geqslant 0$.

To prove the second part of theorem, assume u is a non-negative Lipschitz function with support in a DC_0-chart $U \to A$, where $U \subset \mathbb{R}^m$ is an open subset. Then

$$\int\limits_A \langle\nabla u, \nabla f\rangle = \int\limits_U \det g\cdot g^{ij}\cdot \partial_i u\cdot \partial_j f\cdot dx^1dx^2\dots dx^m =$$

$$= -\int\limits_U u\cdot \partial_i(\det g\cdot g^{ij}\cdot \partial_j f)dx^1dx^2\dots dx^m,$$

Thus

$$\Delta f = \partial_i(\det g\cdot g^{ij}\cdot\partial_j f)dx^1dx^2\dots dx^m \overset{a.e.}{=\!=} \mathrm{Trace}\,\mathrm{Hess}\, f. \qquad \square$$

Gradient curves. Here I extend notion of gradient curves to a family of functions, see [Petrunin 07] for all necessary definitions.

Let $\mathbb{I}$ be an open real interval and $\lambda\colon \mathbb{I} \to \mathbb{R}$ be a continuous function. A one parameter family of functions $f_t\colon A \to \mathbb{R}$, $t \in \mathbb{I}$ will be called $\lambda(t)$-*concave* if the function $(t, x) \mapsto f_t(x)$ is locally Lipschitz and f_t is $\lambda(t)$-concave for each $t \in \mathbb{I}$.

We will write $\alpha^\pm(t) = \nabla f_t$ if for any $t \in \mathbb{I}$, the right/left tangent vector $\alpha^\pm(t)$ is well defined and $\alpha^\pm(t) = \nabla_{\alpha(t)} f_t$. The solutions of $\alpha^+(t) = \nabla f_t$ will be also called f_t-gradient curves.

The following is a slight generalization of [Petrunin 07, 2.1.2&2.2(2)]; it can be proved along the same lines.

1.2. Proposition-definition. *Let $A \in \mathrm{Alex}^m[\kappa]$, $\mathbb{I}$ be an open real interval, $\lambda\colon \mathbb{I} \to \mathbb{R}$ be a continuous function and $f_t\colon A \to \mathbb{R}$, $t \in \mathbb{I}$ be $\lambda(t)$-concave family.*

Then for any $x \in A$ and $t_0 \in \mathbb{I}$ there is an f_t-gradient curve α which is defined in a neighborhood of t_0 and such that $\alpha(t_0) = x$.

More over, if $\alpha, \beta\colon \mathbb{I} \to A$ *f_t-gradient then for any* $t_0, t_1 \in \mathbb{I}$*,* $t_0 \leqslant t_1$*,*

$$|\alpha(t_1)\beta(t_1)| \leqslant L|\alpha(t_0)\beta(t_0)|,$$

where $L = \exp\left(\int_{t_0}^{t_1} \lambda(t)\,dt\right)$.

Note that above proposition implies that the value $\alpha(t_0)$ of f_t-gradient curve $\alpha(t)$ uniquely determines it for all $t \geqslant t_0$ in $\mathbb{I}$. Thus we can define *f_t-gradient flow* — a family of maps $F_{t_0,t_1}\colon A \to A$ such that

$$F_{t_0,t_1}(\alpha(t_0)) = \alpha(t_1) \quad \text{if} \quad \alpha^+(t) = \nabla f_t.$$

1.3. Claim. *Let* $f_t\colon A \to \mathbb{R}$ *be a* $\lambda(t)$*-concave family and* F_{t_0,t_1} *be* f_t*-gradient flow. Let* $E \subset A$ *be a bounded Borel set, fix* t_1 *and consider function* $v(t) = \operatorname{vol} F^{-1}_{t,t_1}(E)$*. Then*

$$v\Big|_t^{t_1} = \int\limits_t^{t_1} \Delta f_\xi\big[F^{-1}_{\xi,t_1}(E)\big]\;d\xi$$

Proof. Let $u\colon A \to \mathbb{R}$ be a Lipschitz function with compact support. Set $u_t = u \circ F_{t,t_1}$. Clearly all $(x,t) \mapsto u_t(x)$ is locally Lipschitz. Thus, the function

$$w_u\colon t \mapsto \int\limits_A u_t\,d\operatorname{vol}$$

is locally Lipschitz. Further

$$w'_u(t) \overset{a.e.}{=\!=} -\int\limits_A \langle\nabla u_t, \nabla f_t\rangle\,d\operatorname{vol} = \int\limits_A u_t\,d\Delta f_t.$$

Therefore

$$w_u\Big|_t^{t_1} = \int\limits_t^{t_1} d\xi \int\limits_A u_t\,d\Delta f_t.$$

The last formula extendeds to arbitrary Borel function $u\colon A \to \mathbb{R}$ with bounded support. Applying it to characteristic function of E we get the result. □

2 Games with Hamilton–Jacobi shifts.

Let $A \in \mathrm{Alex}^m[0]$. For a function $f\colon A \to \mathbb{R}\cup\{+\infty\}$, let us define its Hamilton–Jacobi shift[2] $\mathcal{H}_t\, f\colon A \to \mathbb{R}$ for time $t > 0$,

$$\mathcal{H}_t\, f(x) \overset{def}{=\!=} \inf_{y\in A}\left\{f(y) + \tfrac{1}{2t}|xy|^2\right\}.$$

We say that $\mathcal{H}_t\, f$ is well defined if the above infimum is $> -\infty$ everywhere in A. Clearly,

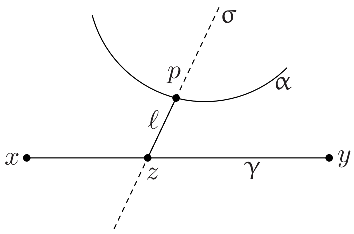
$$\mathcal{H}_{t_0+t_1}\, f = \mathcal{H}_{t_1}\, \mathcal{H}_{t_0}\, f, \qquad ❷$$

[2]There is a lot of similarity between Hamilton–Jacobi shift of function and equidistant for hypersurface.

for any $t_0, t_1 > 0$.

Note that for $t > 0$, $f_t = \mathcal{H}_t f$ forms a $\frac{1}{t}$-concave family, thus, we can apply 1.2 and 1.3. The next theorem gives a more delicate property of the gradient flow, for such families; it is an analog of [Petrunin 07, 3.3.6].

2.1. Claim. *Let $A \in \mathrm{Alex}^m[0]$, $f_0\colon A \to \mathbb{R}$ be function and $f_t = \mathcal{H}_t f_0$ is well defined for $t \in (0,1)$. Assume $\gamma\colon [0,1] \to A$ is a geodesic path which is an f_t-gradient in $(0,1)$ and $\alpha\colon (0,1) \to A$ is an other f_t-gradient curve. Then if for some $t_0 \in (0,1)$, $\alpha(t_0) = \gamma(t_0)$ then $\alpha(t) = \gamma(t)$ for all $t \in (0,1)$.*



Proof. Note that function $\ell = \ell(t) = |\alpha(t)\gamma(t)|$ is locally Lipschitz in $(0,1)$. According to 1.2, it is sufficient to show that

$$\ell' \geqslant -[\tfrac{1}{t} + \tfrac{2}{1-t}] \cdot \ell$$

for almost all t.

Since α is locally Lipschitz, for almost all t, $\alpha^+(t)$ and $\alpha^-(t)$ are well defined and *opposite*[3] to each other.

Fix such t and set $x = \gamma(0)$, $z = \gamma(t)$, $y = \gamma(1)$, $p = \alpha(t)$, so $\ell(t) = |pz|$. Note that function

$$f_t + \tfrac{1}{2(1-t)} \operatorname{dist}_y^2 \qquad ❸$$

has a minimum at z. Extend a geodesic $[zp]$ by a both-sides infinite unit-speed quasigeodesic[4] $\sigma\colon \mathbb{R} \to A$, so $\sigma(0) = z$ and $\sigma^+(0) = \uparrow_{[zp]}$. The function $f_t \circ \sigma\colon \mathbb{R} \to \mathbb{R}$ is $\frac{1}{t}$-concave and from ❸,

$$f_t \circ \sigma(s) \geqslant f_t(z) + \langle \gamma^+(t), \uparrow_{[zp]} \rangle \cdot s - \tfrac{1}{2(1-t)} \cdot s^2.$$

It follows that

$$\begin{aligned} \langle \nabla_p f_t, \sigma^+(\ell) \rangle &\geqslant d_p f_t(\sigma^+(\ell)) = \\ &= (f_t \circ \sigma)^+(\ell) \geqslant \\ &\geqslant \langle \gamma^+(t), \uparrow_{[zp]} \rangle - [\tfrac{1}{t} + \tfrac{2}{1-t}] \cdot \ell. \end{aligned}$$

Now,

1. Vectors $\sigma^\pm(\ell)$ are polar, thus $\langle \alpha^\pm(t), \sigma^+(\ell) \rangle + \langle \alpha^\pm(t), \sigma^-(\ell) \rangle \geqslant 0$.
2. Vectors $\alpha^\pm(t)$ are opposite, thus $\langle \alpha^+(t), \sigma^\pm(\ell) \rangle + \langle \alpha^-(t), \sigma^\pm(\ell) \rangle = 0$.
3. $\alpha^+(t) = \nabla_p f_t$ and $\sigma^-(\ell) = \uparrow_{[pz]}$

Thus, $\langle \nabla_p f_t, \sigma^+(\ell) \rangle + \langle \alpha^+(t), \uparrow_{[pz]} \rangle = 0$. Therefore

$$\ell' = -\langle \alpha^+(t), \uparrow_{[pz]} \rangle - \langle \gamma^+(t), \uparrow_{[zp]} \rangle \geqslant -[\tfrac{1}{t} + \tfrac{2}{1-t}] \cdot \ell.$$

□

2.2. Proposition. *Let $A \in \mathrm{Alex}^m[0]$, $f\colon A \to \mathbb{R}$ be bounded and continuous function and $f_t = \mathcal{H}_t f$. Assume $\gamma\colon (0,a) \to A$ is a f_t-gradient curve which is also a constant-speed geodesic. Assume that function*

$$h(t) \overset{def}{=\!=} \operatorname{Trace} \operatorname{Hess}_{\gamma(t)} f_t$$

[3] I.e. $|\alpha^+(t)| = |\alpha^-(t)|$ and $\measuredangle(\alpha^+(t), \alpha^-(t)) = \pi$

[4] A careful proof of existence of quasigeodesics can be found in [Petrunin 07].

is defined for almost all $t \in (0,a)$*. Then*

$$h' \leqslant -\tfrac{1}{m}h^2$$

in the sense of distributions; i.e. for any non-negative Lipschitz function $u\colon (0,a)\to\mathbb{R}$ *with compact support*

$$\int_0^a \left(\tfrac{1}{m}h^2u - hu'\right)\,dt \geqslant 0.$$

Proof. Since h are defined a.e., all $\mathrm{T}_{\gamma(t)}$ for $t\in(0,a)$ are isometric to Euclidean m-space. From ❷,

$$f_{t_1}(x) = \inf_{y\in A}\left\{ f_{t_0}(y) + \frac{|xy|^2}{2(t_1-t_0)}\right\}.$$

Thus, for a parallel transportation $\tau\colon \mathrm{T}_{\gamma(t_0)}\to\mathrm{T}_{\gamma(t_1)}$ along γ, we have

$$\mathrm{Hess}_{\gamma(t_1)} f_{t_1}(y) \leqslant \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(x) + \frac{|\tau(x)\,y|^2}{2(t_1-t_0)}$$

for any $x\in\mathrm{T}_{\gamma(t_0)}$ and $y\in\mathrm{T}_{\gamma(t_1)}$. Taking trace leads to the result. □

3 Proof of the main theorem

Let $A\in\mathrm{Alex}^m[0]$; in particular A is proper geodesic space. Let μ_t be a family of probability measures on A for $t\in[0,1]$ which forms a *geodesic path*[5] in P_2A and both μ_0 and μ_1 are absolutely continuous with respect to volume on A.

It is sufficient[6] to show that function

$$\Theta\colon t\mapsto U_m\mu_t$$

is concave.

According to [Villani, 7.22], there is a probability measure Π on the space of all geodesic paths in A which satisfy the following: If $\Gamma = \operatorname{supp}\Pi$ and $e_t\colon\Gamma\to A$ is evaluation map $e_t\colon\gamma\mapsto\gamma(t)$ then $\mu_t = e_t\#\Pi$.

The measure Π is called *dynamical optimal coupling* for μ_t and the measure $\pi = (e_0,e_1)\#\Pi$ is the corresponding *optimal transference plan.* The space Γ will be considered further equipped with the metric $|\gamma\gamma'| = \max_{t\in[0,1]}|\gamma(t)\gamma'(t)|$.

First we present μ_t *as push-forward of each other for gradients flow of a family of functions.* According to [Villani, 5.10], there are optimal price functions $\varphi,\psi\colon A\to\mathbb{R}$ such that

$$\varphi(y)-\psi(x)\leqslant\tfrac12|xy|^2$$

for any $x,y\in A$ and equality holds for any $(x,y)\in\operatorname{supp}\pi$. We can assume that $\psi(x)=+\infty$ for $x\notin\operatorname{supp}\mu_0$ and $\varphi(y)=-\infty$ for $y\notin\operatorname{supp}\mu_1$.

[5] i.e. constant-speed minimizing geodesic defined on $[0,1]$

[6] It follows from [Villani, 30.32] since Alexandrov's spaces are nonbranching.

Consider two families of functions

$$\psi_t = \mathcal{H}_t\,\psi \quad \text{and} \quad \varphi_t = \mathcal{H}_{1-t}(-\varphi).$$

Clearly, ψ_t forms a $\frac{1}{t}$-concave family for $t \in (0,1]$ and φ_t forms[7] a $\frac{1}{1-t}$-concave family for $t \in [0,1)$.

It is straightforward to check that for any $\gamma \in \Gamma$ and $t \in (0,1)$

$$\pm\langle\gamma^{\pm}(t), v\rangle = d_{\gamma(t)}\psi_t(v) = -d_{\gamma(t)}\varphi_t(v);$$

in particular,

$$\gamma^{+}(t) = \nabla\psi_t \quad \text{and} \quad \gamma^{-}(t) = \nabla\varphi_t. \qquad ❹$$

For $0 < t_0 \leqslant t_1 \leqslant 1$, let us consider the maps $\Psi_{t_0,t_1} \colon A \to A$ — the gradient flow of ψ_t, defined by

$$\Psi_{t_0,t_1}\alpha(t_0) = \alpha(t_1) \quad \text{if} \quad \alpha^{+}(t) = \nabla\psi_t.$$

Similarly, $0 \leqslant t_0 \leqslant t_1 < 1$, define map $\Phi_{t_1,t_0} \colon A \to A$

$$\Phi_{t_1,t_0}\beta(t_1) = \beta(t_0) \quad \text{if} \quad \beta^{-}(t) = \nabla\varphi_t.$$

According to 1.2,

$$\Psi_{t_0,t_1} \text{ is } \tfrac{t_1}{t_0}\text{-Lipschitz} \quad \text{and} \quad \Phi_{t_1,t_0} \text{ is } \tfrac{1-t_0}{1-t_1}\text{-Lipschitz}. \qquad ❺$$

From ❹, $e_{t_1} = \Psi_{t_0,t_1} \circ e_{t_0}$ and $e_{t_0} = \Phi_{t_1,t_0} \circ e_{t_1}$. Thus, for any $t \in (0,1)$, the map $e_t \colon \Gamma \to A$ is bi-Lipschits. In particular, for any measure χ on A, there is uniquely determined one-parameter family of "pull-back" measures χ_t^* on Γ, i.e. such that $\chi_t^* E = \chi(e_t E)$ for any Borel subset $E \subset \Gamma$.

Fix some $z_0 \in (0,1)$ (one can take $z_0 = \frac{1}{2}$) and equip Γ with the measure $\nu = \mathrm{vol}_{z_0}^*$. Thus, from now on "almost everywhere" has sense in Γ, $\Gamma \times (0,1)$ and so on.

Now we will represent Θ in terms of families of functions on Γ. Note that $\mu_1 = \Psi_{t,1}\#\mu_t$ and $\Psi_{t,1}$ is $\frac{1}{t}$-Lipschitz. Since μ_1 is absolutely continuous, so is μ_t for all t. Set $\mu_t = \rho_t \cdot \mathrm{vol}$. Note that from ❺, we get that

$$\left(\frac{1-t_1}{1-t_0}\right)^m \leqslant \frac{\rho_{t_1}(\gamma(t_1))}{\rho_{t_0}(\gamma(t_0))} \leqslant \left(\frac{t_1}{t_0}\right)^m$$

for almost all $\gamma \in \Gamma$ and $0 < t_0 < t_1 < 1$. For $\gamma \in \Gamma$ set $r_t(\gamma) = \rho_t(\gamma(t))$. Then

$$\Theta(t) = \int\limits_A \rho_t^{-\frac{1}{m}}\,d\mu_t = \int\limits_\Gamma r_t^{-\frac{1}{m}}\,d\Pi. \qquad ❻$$

In particular, Θ locally Lipschitz in $(0,1)$.

Next we show that measure $\Delta\varphi_t$ is absolutely continuous on $e_t\Gamma$ and that $r_t(\gamma(t)) = \rho_t(\gamma(t))\Delta\varphi_t$ in some weak sense. From ❺, $\mathrm{vol}_t^* = e^{w_t}\nu$ for some Borel function $w_t \colon \Gamma \to \mathbb{R}$. Thus

$$\mathrm{vol}\, e_t E = \int\limits_E e^{w_t}\,d\nu$$

[7] Note that usually φ_t is defined with opposite sign, but I wanted to work with semiconcave functions only.

for any Borel subset $E \subset \Gamma$. Moreover, for almost all $\gamma \in \Gamma$, we have that function $t \mapsto w_t(\gamma)$ is locally Lipschitz in $(0,1)$ (more precicely, $t \mapsto w_t(\gamma)$ coinsides with a Lipschitz function outside of a set of zero measure). In particular $\frac{\partial w_t}{\partial t}$ is well defined a.e. in $\Gamma \times (0,1)$ and moreover

$$w_t \overset{a.e.}{=\!=} \int\limits_{z_0}^{t} \frac{\partial w_\xi}{\partial \xi}\, d\xi.$$

Further, from 2.1, if $0 < t_0 \leqslant t_1 < 1$ then for any $\gamma \in \Gamma$,

$$\Psi_{t_0,t_1}(x) = \gamma(t_1) \iff x = \gamma(t_0),$$

$$\Phi_{t_1,t_0}(x) = \gamma(t_0) \iff x = \gamma(t_1).$$

Thus, for any Borel subset $E \subset \Gamma$,

$$e_{t_1}E = \Psi_{t_0,t_1} \circ e_{t_0}E = \Phi^{-1}_{t_1,t_0}(e_{t_0}E)\,,$$

$$e_{t_0}E = \Phi_{t_1,t_0} \circ e_{t_1}E = \Psi^{-1}_{t_0,t_1}(e_{t_1}E)$$

Set

$$v(t) \overset{def}{=\!=} \operatorname{vol} e_t E = \int\limits_E e^{w_t}\, d\nu.$$

From 1.3,

$$v'(t) \overset{a.e.}{=\!=} \Delta\psi_t\, e_t E \overset{a.e.}{=\!=} -\Delta\varphi_t\, e_t E.$$

Thus, $\Delta\psi_t + \Delta\varphi_t = 0$ everywhere on $e_t\Gamma$. From 1.1,

$$\Delta\psi_t \leqslant \tfrac{m}{t} \operatorname{vol}, \qquad \Delta\varphi_t \leqslant \tfrac{m}{1-t} \operatorname{vol}.$$

Thus, both restrictions $\Delta\psi_t|_{e_t\Gamma}$ and $\Delta\varphi_t|_{e_t\Gamma}$ are absolutely continuous with respect to volume. Therefore

$$v'(t) \overset{a.e.}{=\!=} \int\limits_{e_t E} \operatorname{Trace}\operatorname{Hess}\varphi_t\, d\operatorname{vol}.$$

For one parameter family of functions $h_t(\gamma) = \operatorname{Trace}\operatorname{Hess}_{\gamma(t)}\varphi_t$, we have

$$v\big|_{z_0}^{t} = \int\limits_E (e^{w_t} - 1)\, d\nu = \int\limits_{z_0}^{t} d\xi \int\limits_E h_\xi e^{w_\xi}\, d\nu$$

or any Borel set $E \subset \Gamma$. Equivalently,

$$\frac{\partial w_t}{\partial t} \overset{a.e.}{=\!=} h_t$$

From 2.2,

$$\frac{\partial h_t}{\partial t} \leqslant -\tfrac{1}{m} h_t^2$$

Thus, for almost all $\gamma \in \Gamma$, the following inequality holds in the sense of distributions:

$$\frac{\partial^2}{\partial t^2} \exp\left(\frac{w_t(\gamma)}{m}\right) = \left(\tfrac{1}{m^2} h_t^2 + \tfrac{1}{m}\frac{\partial h_t}{\partial t}\right) \exp\left(\frac{w_t(\gamma)}{m}\right) \leqslant 0;$$

i.e. $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ is concave — more precisely, $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ coinsides with a concave function almost everywhere.

Clearly, for any t we have $\mu = r_t e^{w_t} \cdot \nu$. Thus, for almost all γ there is a non-negative Borel function $a\colon \Gamma \to \mathbb{R}_{\geqslant}$ such that $r_t \overset{a.e.}{=\!=} a e^{-w_t}$. Continue ❻,

$$\Theta(t) = \int\limits_{\Gamma} r_t^{-\frac{1}{m}}\, d\Pi = \int\limits_{\Gamma} e^{\frac{w_t}{m}} \cdot \sqrt[m]{a}\, d\Pi$$

I.e. Θ is concave as an average of concave functions. Again, more precisely, Θ coinsides with a concave function a.e., but since Θ is locally Lipschitz in $(0, 1)$ we get that Θ is concave. □

References

[AG] Abresch, U., Gromoll, D., *On complete manifolds with nonnegative Ricci curvature,* J. Amer. Math. Soc. 3 (1990), no 2 355–374.

[Bertrand] Bertrand, J., *Existence and uniqueness of optimal maps on Alexandrov spaces.* Adv. Math. 219, 3 (2008), 838–851.

[BGP] Burago, Yu.; Gromov, M.; Perelman, G., *A. D. Aleksandrov spaces with curvatures bounded below.* (Russian) Uspekhi Mat. Nauk 47 (1992), no. 2(284), 3–51, 222; translation in Russian Math. Surveys 47 (1992), no. 2, 1–58

[CMS] Cordero-Erausquin, D.; McCann,R.; Schmuckenschlager, M., *A Riemannian interpolation inequality a la Borell, Brascamp and Lieb,* Invent. Math. 146 (2001), 219–257.

[McCann] McCann, R.J., *A convexity principle for interacting gases.* Adv. Math. 128 (1997), 153–179

[Lott–Villani] Lott, J.; Villani, C., *Ricci curvature for metric-measure spaces via optimal transport.* in press.

[Otto–Villani] Otto, F.; Villani, C., *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality.* J. Funct. Anal. 173 (2000), 361-400

[Perelman] Perelman, G., *DC Structure on Alexandrov Space,* http://www.math.psu.edu/petrunin/

[Petrunin 98] Petrunin, A., *Parallel transportation for Alexandrov space with curvature bounded below.* GAFA, Vol. 8 (1998) 123–148

[Petrunin 03] Petrunin, A., *Harmonic functions on Alexandrov space and its applications,* ERA American Mathematical Society, **9** (2003)

[Petrunin 07] Petrunin, A. *Semiconcave Functions in Alexandrov's Geometry,* Surveys in Differential Geometry XI.

[Villani] Villani, C., *Optimal transport, old and new,* Grundlehren der mathematischen Wissenschaften, Vol. 338, Springer, 2008.

[Sturm] Sturm, K.-T. *On the geometry of metric measure spaces. I–II.* Acta Math. 196, 1 (2006), 65–177.

[Sturm–v.Renesse] Sturm, K.-Th.; von Renesse, M.-K., *Transport inequalities, gradient estimates, entropy and Ricci curvature.* Comm. Pure Appl. Math. 58 (2005), 923–940