

# AN OPTIMAL LOWER CURVATURE BOUND FOR CONVEX HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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ABSTRACT. It is proved that a convex hypersurface in a Riemannian manifold of sectional curvature  $\geq \kappa$  is an Alexandrov's space of curvature  $\geq \kappa$ . This theorem provides an optimal lower curvature bound for an older theorem of Buyalo.

The purpose of this paper is to provide a reference for the following theorem:

**Theorem 1.** *Let  $M$  be a Riemannian manifold with sectional curvature  $\geq \kappa$ . Then any convex hypersurface  $F \subset M$  equipped with the induced intrinsic metric is an Alexandrov's space with curvature  $\geq \kappa$ .*

Here is a slightly weaker statement:

**Theorem 2.** [Buyalo] *If  $M$  is a Riemannian manifold, then any convex hypersurface  $F \subset M$  equipped with the induced intrinsic metric is locally an Alexandrov's space.*

In the proof of Theorem 2 in [Buyalo], the (local) lower curvature bound depends on (local) upper as well as lower curvature bounds of  $M$ . We show that the approach in [Buyalo] can be modified to give Theorem 1.

**Definition 3.** *A locally Lipschitz function  $f$  on an open subset of a Riemannian manifold is called  $\lambda$ -concave ( $\lambda \in \mathbb{R}$ ) if for any unit-speed geodesic  $\gamma$ , the function*

$$f \circ \gamma(t) - \lambda t^2/2$$

*is concave.*

**Lemma 4.** *Let  $f : \Omega \rightarrow \mathbb{R}$  be a  $\lambda$ -concave function on an open subset  $\Omega$  of a Riemannian manifold. Then there is a sequence of nested open domains  $\Omega_i$ , with  $\Omega_i \subset \Omega_j$  for  $i < j$  and  $\cup_i \Omega_i = \Omega$ , and a sequence of smooth  $\lambda_i$ -concave functions  $f_i : \Omega_i \rightarrow \mathbb{R}$  such that*

- (i) *on any compact subset  $K \subset \Omega$ ,  $f_i$  converges uniformly to  $f$ ;*
- (ii)  *$\lambda_i \rightarrow \lambda$  as  $i \rightarrow \infty$ .*

This lemma is a slight generalization of [Greene–Wu, Theorem 2] and can be proved exactly the same way.

*Proof of Theorem 1.* Without loss of generality one can assume that

- (a)  $\kappa \geq -1$ ,

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- (b)  $F$  bounds a compact convex set  $C$  in  $M$ ,
- (c) there is a  $(-2)$ -concave function  $\mu$  defined in a neighborhood of  $C$  and  $|\mu(x)| < 1/10$  for any  $x \in C$ ,
- (d) there is unique minimal geodesic between any two points in  $C$ .

(If not, rescale and pass to the boundary of the convex piece cut by  $F$  from a small convex ball centered at  $x \in F$ , taking  $\mu = -10 \operatorname{dist}_x^2$ .)

Consider the function  $f = \operatorname{dist}_F$ . By Rauch comparison (as in [Petersen, 11.4.8]), for any unit-speed geodesic  $\gamma$  in the interior of  $C$ ,  $(f \circ \gamma)''$  is bounded in the support sense by the corresponding value in the model case (where  $M = \mathbb{H}^2$  and  $F$  is a geodesic). In particular,

$$(f \circ \gamma)'' \leq f \circ \gamma.$$

Therefore  $f + \varepsilon \mu$  is  $(-\varepsilon)$ -concave in  $\Omega_\varepsilon = C \cap f^{-1}((0, \varepsilon))$ . Take  $K_\varepsilon = f^{-1}([\frac{1}{3}\varepsilon, \frac{2}{3}\varepsilon]) \cap C$ . Applying lemma 4, we can find a smooth  $(-\frac{\varepsilon}{2})$ -concave function  $f_\varepsilon$  which is arbitrarily close to  $f + \varepsilon \mu$  on  $K_\varepsilon$  and which is defined on a neighborhood of  $K_\varepsilon$ . Take a regular value  $\vartheta_\varepsilon \approx \frac{1}{2}\varepsilon$  of  $f_\varepsilon$ . (In fact one can take  $\vartheta_\varepsilon = \frac{1}{2}\varepsilon$ , but it requires a little work.) Since  $|\mu|_C < 1/10$ , the level set  $F_\varepsilon = f_\varepsilon^{-1}(\vartheta_\varepsilon)$  will lie entirely in  $K_\varepsilon$ . Therefore  $F_\varepsilon$  forms a smooth closed convex hypersurface. By the Gauss formula, the sectional curvature of the induced intrinsic metric of  $F_\varepsilon$  is  $\geq \kappa$ .  $F_\varepsilon$  bounds a compact convex set  $C_\varepsilon$ , where  $F_\varepsilon \rightarrow F$ ,  $C_\varepsilon \rightarrow C$  in Hausdorff sense as  $\varepsilon \rightarrow 0$ . By property (d), the restricted metrics from  $M$  to  $C, C_\varepsilon$  are intrinsic, and so  $C_\varepsilon$  is an Alexandrov space with  $F_\varepsilon$  as boundary, that converges in Gromov–Hausdorff sense to  $C$ . It follows from [Petrunin, Theorem 1.2] (compare [Buyalo, Theorem 1]) that  $F_\varepsilon$  equipped with its intrinsic metric converges in Gromov–Hausdorff sense to  $F$  equipped with its intrinsic metric. Therefore  $F$  is an Alexandrov space with curvature  $\geq \kappa$ .  $\square$

**Remark 5.** We are not aware of any proof of theorem 1 which is not based on the Gauss formula. (Although if  $M$  is Euclidean space, there is a beautiful purely synthetic proof in [Milka].) Finding such a proof would be interesting on its own, and also could lead to the generalization of theorem 1 to the case when  $M$  is an Alexandrov space.

## REFERENCES

- [Buyalo] Buyalo, S., Shortest paths on convex hypersurface of a Riemannian manifold (Russian), *Studies in Topology*, Zap. Nauchn. Sem. LOMI 66 (1976) 114–132; translated in *J. of Soviet Math.* 12 (1979), 73–85.
- [Greene–Wu] Greene, R. E.; Wu, H.-H., On the subharmonicity and plurisubharmonicity of geodesically convex functions. *Indiana Univ. Math. J.* 22 (1972/73), 641–653.
- [Milka] Milka, A. D. Shortest arcs on convex surfaces (Russian), *Dokl. Akad. Nauk SSSR* 248 1979, no. 1, 34–36; translated in *Soviet Math. Dokl.* 20 (1979), 949–952.
- [Petersen] Petersen, P., *Riemannian Geometry*, Springer-Verlag, New York Berlin Heidelberg, 1998.
- [Petrunin] Petrunin, A., *Applications of quasigeodesics and gradient curves. Comparison geometry* (Berkeley, CA, 1993–94), 203–219, *Math. Sci. Res. Inst. Publ.*, 30, Cambridge Univ. Press, Cambridge, 1997.

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