

NILPOTENCY, ALMOST NONNEGATIVE CURVATURE, AND THE GRADIENT FLOW ON ALEXANDROV SPACES

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ABSTRACT. We show that almost nonnegatively curved m -manifolds are, up to finite cover, nilpotent spaces in the sense of homotopy theory and have $C(m)$ -nilpotent fundamental groups. We also show that up to a finite cover almost nonnegatively curved manifolds are fiber bundles with simply connected fibers over nilmanifolds.

1. INTRODUCTION

Almost nonnegatively curved manifolds were introduced by Gromov in the late 70s [Gro80], with the most significant contributions to their study made by Yamaguchi in [Yam91] and Fukaya and Yamaguchi in [FY92]. Building on their ideas, in the present article we establish several new properties of these manifolds which yield, in particular, new topological obstructions to almost nonnegative curvature. Our techniques also provide simplified proofs of many results from [FY92].

A closed smooth manifold is said to be almost nonnegatively curved if it can Gromov–Hausdorff converge to a single point under a lower curvature bound. By rescaling, this definition is equivalent to the following one, which we will employ throughout this article.

Definition 1.0.1. *A closed smooth manifold M is called almost nonnegatively curved if it admits a sequence of Riemannian metrics $\{g_n\}_{n \in \mathbb{N}}$ whose sectional curvatures and diameters satisfy*

$$\sec(M, g_n) \geq -1/n \quad \text{and} \quad \text{diam}(M, g_n) \leq 1/n.$$

Almost nonnegatively curved manifolds generalize almost flat as well as nonnegatively curved manifolds. One main source of examples comes from a theorem of Fukaya and Yamaguchi. It states that if $F \rightarrow E \rightarrow B$ is a fiber bundle over an almost nonnegatively curved manifold B whose fiber F is compact and admits a nonnegatively curved metric which is invariant under the structure group, then the total space E is almost nonnegatively

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curved [FY92]. Further examples are given by closed manifolds which admit cohomogeneity one actions of compact Lie groups (compare [ST04]).

In this work we combine collapsing techniques with a non-smooth analogue of the gradient flow of concave functions on Alexandrov spaces. This notion is based on the construction of gradient curves of λ -concave functions used in [PP96] and bears many similarities to the Sharafutdinov retraction [Sha78]. Gradient flows on general metric spaces were considered in [AGS]. The gradient flow on Alexandrov spaces plays a key role in the proofs of two of the three main results in this paper, and we believe that it should also prove useful for dealing with other problems related to collapsing under a lower curvature bound.

1.1. To put the main theorems of the present work into perspective, let us first briefly recall some previously known results:

Let $M = M^m$ be an almost nonnegatively curved m -manifold.

- ◊ Gromov proved in [Gro78] that the minimal number of generators of the fundamental group $\pi_1(M)$ of M can be estimated by a constant $C_1(m)$ depending only on m , and in [Gro81] that the sum of Betti numbers of M with respect to any field of coefficients does not exceed some uniform constant $C_2 = C_2(m)$.
- ◊ Yamaguchi showed that, up to a finite cover, M fibers over a flat $b_1(M; \mathbb{R})$ -dimensional torus and that M^m is diffeomorphic to a torus if $b_1(M; \mathbb{R}) = m$ [Yam91].
- ◊ Fukaya and Yamaguchi proved that $\pi_1(M)$ is almost nilpotent, i.e., contains a nilpotent subgroup of finite index, and also that $\pi_1(M)$ is $C_3(m)$ -solvable, i.e., contains a solvable subgroup of index at most $C_3(m)$ [FY92].
- ◊ If a closed manifold has negative Yamabe constant, then it cannot volume collapse with scalar curvature bounded from below (see [Sch89, LeB01]). In particular, no such manifold can be almost nonnegatively curved.
- ◊ The \hat{A} -genus of a closed spin manifold X of almost nonnegative Ricci curvature satisfies the inequality $\hat{A}(X) \leq 2^{\dim(X)/2}$ ([Gro82], [Gal83]).

Let us now state the main results of this article.

1.2. Our first result concerns the hitherto unexplored relation between curvature bounds and the actions of the fundamental group on the higher homotopy groups.

Recall that an action by automorphisms of a group G on an abelian group V is called nilpotent if V admits a finite sequence of G -invariant subgroups

$$V = V_0 \supset V_1 \supset \dots \supset V_k = 0$$

such that the induced action of G on V_i/V_{i+1} is trivial for any i . A connected CW-complex X is called *nilpotent* if $\pi_1(X)$ is a nilpotent group that operates nilpotently on $\pi_k(X)$ for every $k \geq 2$.

Nilpotent spaces play an important role in topology since they enjoy some of the best homotopy-theoretic properties of simply connected spaces, like

a Whitehead theorem or reasonable Postnikov towers. Furthermore, unlike the category of simply connected spaces, the category of nilpotent ones is closed under many constructions such as the based loop space functor or the formation of function spaces, and group-theoretic functors, like localization and completion, have topological extensions in this category.

Theorem A (Nilpotency Theorem). *Let M be a closed almost nonnegatively curved manifold. Then a finite cover of M is a nilpotent space.*

It would be interesting to know whether the order of this covering can be estimated solely in terms of the dimension of M .

Example 1.2.1. Let $h: S^3 \times S^3 \rightarrow S^3 \times S^3$ be defined by

$$h: (x, y) \mapsto (xy, yxy).$$

This map is a diffeomorphism with inverse given by

$$h^{-1}: (u, v) \mapsto (u^2v^{-1}, vu^{-1}).$$

The induced map h_* on $\pi_3(S^3 \times S^3)$ is given by the matrix $A_h = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Notice that the eigenvalues of A_h are different from 1 in absolute value. Let M be the mapping cylinder of h . Clearly, M has the structure of a fiber bundle $S^3 \times S^3 \rightarrow M \rightarrow S^1$, and the action of $\pi_1(M) \cong \mathbb{Z}$ on $\pi_3(M) \cong \mathbb{Z}^2$ is generated by A_h . In particular, M is not a nilpotent space and hence, by Theorem A, it does not admit almost nonnegative curvature. This fact doesn't follow from any previously known results.

1.3. Our next main result provides an affirmative answer to a conjecture of Fukaya and Yamaguchi [FY92, Conjecture 0.15].

Theorem B (C -Nilpotency Theorem for π_1). *Let M be an almost nonnegatively curved m -manifold. Then $\pi_1(M)$ is $C(m)$ -nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of index at most $C(m)$.*

Notice that Theorem B is new even for manifolds of nonnegative curvature.

Example 1.3.1. For any $C > 0$ there exist prime numbers $p > q > C$ and a finite group G_{pq} of order pq which is solvable but not nilpotent. In particular, G_{pq} does not contain any nilpotent subgroup of index less than or equal to C .

Whereas none of the results mentioned so far excludes G_{pq} from being the fundamental group of some almost nonnegatively curved m -manifold, Theorem B shows that for $C > C(m)$ none of the groups G_{pq} can be realized as the fundamental group of such a manifold.

1.4. In [FY92] Fukaya and Yamaguchi also conjectured that a finite cover of an almost nonnegatively Ricci curved manifold M fibers over a nilmanifold with a fiber which has nonnegative Ricci curvature and whose fundamental group is finite. This conjecture was later refuted by Anderson [And92].

It is, on the other hand, very natural to consider this conjecture in the context of almost nonnegative *sectional* curvature. In fact, here Yamaguchi's fibration theorem ([Yam91]) and the results of [FY92] easily imply that a finite cover of an almost nonnegatively curved manifold admits a map onto a nilmanifold whose homotopy fiber is a simply connected closed manifold.

From mere topology, it is, however, not clear whether this homotopy fibration can actually always be made into a genuine fiber bundle. Our next result shows that this is indeed true, and that for manifolds of almost nonnegative sectional curvature Fukaya's and Yamaguchi's original conjecture essentially does hold.

Theorem C (Fibration Theorem). *Let M be an almost nonnegatively curved manifold. Then a finite cover \tilde{M} of M is the total space of a fiber bundle*

$$F \rightarrow \tilde{M} \rightarrow N$$

over a nilmanifold N with a simply connected fiber F . Moreover, the fiber F is almost nonnegatively curved in the sense of the following definition.

Definition 1.4.1. *A closed smooth manifold M is called almost nonnegatively curved in the generalized sense if for some nonnegative integer k there exists a sequence of complete Riemannian metrics g_n on $M \times \mathbb{R}^k$ and points $p_n \in M \times \mathbb{R}^k$ such that*

- (1) *the sectional curvatures of the metric balls of radius n around p_n satisfy*

$$\sec(B_n(p_n)) \geq -1/n;$$

- (2) *for $n \rightarrow \infty$ the pointed Riemannian manifolds $((M \times \mathbb{R}^k, g_n), p_n)$ converge in the pointed Gromov–Hausdorff distance to $(\mathbb{R}^k, 0)$;*
- (3) *the regular fibers over 0 are diffeomorphic to M for all large n .*

Due to Yamaguchi's fibration theorem [Yam91], manifolds which are almost nonnegatively curved in the generalized sense play the same central role in collapsing under a lower curvature bound as almost flat manifolds do in the Cheeger–Fukaya–Gromov theory of collapsing with bounded curvature (see [CFG92]).

It is not known whether all manifolds which are almost nonnegatively curved in the generalized sense are almost nonnegatively curved. Clearly, if $k = 0$, this definition reduces to the standard one. Moreover, it is easy to see that all results of the present article, as well as all results about almost nonnegatively curved manifolds mentioned earlier (except possibly for the ones concerning the \hat{A} -genus and Yamabe constant), hold for manifolds which are almost nonnegatively curved in the sense of Definition 1.4.1.

1.5. Let us now describe the structure of the remaining sections of this article.

In section 2, after providing some necessary background from Alexandrov geometry, we introduce the gradient flow of the square of a distance function.

It serves as one of the main technical tools in the proofs of theorem A and theorem B.

In section 3 we prove Theorem A by a direct application of the gradient flow technique.

In section 4 we prove Theorem B. The proof is also based on the gradient flow, but is more involved and employs further technical tools such as “limit fundamental groups” of Alexandrov spaces.

In section 5 we prove Theorem C. This section is completely independent from the rest of the article.

In section 6 we discuss some further open questions related to our results.

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2. ALEXANDROV GEOMETRY AND THE GRADIENT FLOW

This section provides necessary background in Alexandrov geometry. The results of sections 2.1–2.3 are mostly repeated from [PP96], [Pet95] and [Pet07]. The reader may consult [BGP92] for a general reference on Alexandrov spaces.

2.1. λ -concave functions.

Definition 2.1.1. *(for a space without boundary) Let A be an Alexandrov space without boundary. A Lipschitz function $f: A \rightarrow \mathbb{R}$ is called λ -concave if for any unit speed minimizing geodesic γ in A , the function*

$$f \circ \gamma(t) - \lambda t^2/2$$

is concave.

If A is an Alexandrov space with boundary, then its double \tilde{A} is also an Alexandrov space (see [Per91, 5.2]). Let $p: \tilde{A} \rightarrow A$ be the canonical map. Given a function f on A , set $\tilde{f} = f \circ p$.

Definition 2.1.2. *(for a space with boundary) Let A be an Alexandrov space with boundary. A Lipschitz function $f: A \rightarrow \mathbb{R}$ is called λ -concave if for any unit speed minimizing geodesic γ in \tilde{A} , the function*

$$\tilde{f} \circ \gamma(t) - \lambda t^2/2$$

is concave.

Remark 2.1.3. Notice that the restriction of a linear function on \mathbb{R}^n to a ball is not 0-concave in this sense.

Remark 2.1.4. In the above definitions, the Lipschitz condition is only technical. With some extra work, all results of this section can be extended to continuous functions.

2.2. Tangent cone and differential. Given a point p in an Alexandrov space A , we denote by $T_p = T_p(A)$ the tangent cone at p .

If d denotes the metric of an Alexandrov space A , let us denote by λA the space $(A, \lambda d)$. Let $i_\lambda : \lambda A \rightarrow A$ be the canonical map. The limit of $(\lambda A, p)$ for $\lambda \rightarrow \infty$ is the tangent cone T_p at p (see [BGP92, 7.8.1]).

Definition 2.2.1. For any function $f : A \rightarrow \mathbb{R}$ the function $d_p f : T_p \rightarrow \mathbb{R}$ such that

$$d_p f = \lim_{\lambda \rightarrow \infty} \lambda(f \circ i_\lambda - f(p))$$

is called the differential of f at p .

It is easy to see that for a λ -concave function f the differential $d_p f$ is defined everywhere, and that $d_p f$ is a 0-concave function on the tangent cone T_p .

Definition 2.2.2. Given a λ -concave function $f : A \rightarrow \mathbb{R}$, a point $p \in A$ is called critical point of f if $d_p f \leq 0$.

2.3. Gradient curves. With a slight abuse of notation we will call elements of the tangent cone T_p the “tangent vectors” at p . The origin of T_p plays the role of the zero vector and is denoted by $o = o_p$. For a tangent vector v at p we define its absolute value $|v|$ as the distance $|ov|$ in T_p . For two tangent vectors u and v at p we can define their “scalar product”

$$\langle u, v \rangle = (|u|^2 + |v|^2 - |uv|^2)/2 = |u| \cdot |v| \cos \alpha,$$

where $\alpha = \angle uov$ in T_p .

For two points $p, q \in A$ we define $\log_p q$ to be a tangent vector v at p such that $|v| = |pq|$ and such that the direction of v coincides with a direction from p to q (if such a direction is not unique, we choose any one of them). Given a curve $\gamma(t)$ in A , we denote by $\gamma^+(t)$ the right and by $\gamma^-(t)$ the left tangent vectors to $\gamma(t)$, where, respectively,

$$\gamma^\pm(t) \in T_{\gamma(t)}, \quad \gamma^\pm(t) = \lim_{\varepsilon \rightarrow +0} \frac{\log_{\gamma(t)} \gamma(t \pm \varepsilon)}{\varepsilon}.$$

For a real function $f(t)$, $t \in \mathbb{R}$, we denote by $f^+(t)$ its right derivative and by $-f^-(t)$ its left derivative. Note that our sign convention (which is chosen to agree with the notion of right and left derivatives of curves) is not quite standard. For example,

$$\text{if } f(t) = t \text{ then } f^+(t) \equiv 1 \text{ and } f^-(t) \equiv -1.$$

Definition 2.3.1. Given a λ -concave function f on A , a vector $g \in T_p(A)$ is called a gradient of f at $p \in A$ (in short: $g = \nabla_p f$) if

- (i) $d_p f(x) \leq \langle g, x \rangle$ for any $x \in T_p$, and
- (ii) $d_p f(g) = \langle g, g \rangle$.

It is easy to see that any λ -concave function has a uniquely defined gradient vector field. Moreover, if $d_p f(x) \leq 0$ for all $x \in T_p$, then $\nabla_p f = o$ (here o denotes the origin of the tangent cone T_p); otherwise,

$$\nabla_p f = d_p f(\xi)\xi$$

where ξ is the (necessarily unique) unit vector for which the function $d_p f$ attains its maximum.

Moreover, for any minimizing geodesic $\gamma : [a, b] \rightarrow U$ parameterized by arclength, the following inequality holds:

$$(2.3.1) \quad \langle \gamma^+(a), \nabla_{\gamma(a)} f \rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f \rangle \geq -\lambda(b - a).$$

Indeed,

$$\begin{aligned} \langle \gamma^+(a), \nabla_{\gamma(a)} f \rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f \rangle &\geq d_{\gamma(a)} f(\gamma^+(a)) + d_{\gamma(b)} f(\gamma^-(b)) = \\ &= (f \circ \gamma)^+|_a + (f \circ \gamma)^-|_b \geq -\lambda(b - a). \end{aligned}$$

Definition 2.3.2. A curve $\alpha : [a, b] \rightarrow A$ is called an f -gradient curve if for any $t \in [a, b]$

$$\alpha^+(t) = \nabla_{\alpha(t)} f.$$

Proposition 2.3.3. Given a λ -concave function $f : A \rightarrow \mathbb{R}$ and a point $p \in A$ there is a unique gradient curve $\alpha : [0, \infty) \rightarrow A$ such that $\alpha(0) = p$.

Moreover, if α and β are two f -gradient curves, then

$$|\alpha(t_1)\beta(t_1)| \leq |\alpha(t_0)\beta(t_0)| \exp(\lambda(t_1 - t_0)) \quad \text{for all } t_1 \geq t_0.$$

The gradient curve can be constructed as a limit of broken geodesics, made up of short segments with directions close to the gradient. The convergence, uniqueness, as well as the last inequality in Proposition 2.3.3 follow from inequality (2.3.1) above, while Corollary 2.3.5 below guarantees that the limit is indeed a gradient curve, having a unique right tangent vector at each point.

Lemma 2.3.4. Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geq k$ which Gromov–Hausdorff converges to an Alexandrov space A .

Let $f_n \rightarrow f$, where $f_n : A_n \rightarrow \mathbb{R}$ is a sequence of λ -concave functions converging to $f : A \rightarrow \mathbb{R}$.

Let $p_n \rightarrow p$, where $p_n \in A_n$ and $p \in A$.

Then

$$|\nabla_p f| \leq \liminf_{n \rightarrow \infty} |\nabla_{p_n} f_n|.$$

Corollary 2.3.5. Given a λ -concave function f on A and a sequence of points $p_n \in A$, $p_n \rightarrow p$, we have

$$|\nabla_p f| \leq \liminf_{n \rightarrow \infty} |\nabla_{p_n} f|.$$

Proof of Lemma 2.3.4. Fix an $\varepsilon > 0$ and choose q near p such that

$$\frac{f(q) - f(p)}{|pq|} > |\nabla_p f| - \varepsilon.$$

Now choose $q_n \in A_n$ such that $q_n \rightarrow q$. If $|pq|$ is sufficiently small and n is sufficiently large, the λ -concavity of f_n then implies that

$$\liminf_{n \rightarrow \infty} \frac{d_{p_n} f_n(v_n)}{|v_n|} \geq |\nabla_p f| - 2\varepsilon \quad \text{for } v_n = \log_{p_n}(q_n) \in T_{p_n}(A_n).$$

Therefore,

$$\liminf_{n \rightarrow \infty} |\nabla_{p_n} f_n| \geq |\nabla_p f| - 2\varepsilon \quad \text{for any } \varepsilon > 0,$$

i.e.,

$$\liminf_{n \rightarrow \infty} |\nabla_{p_n} f_n| \geq |\nabla_p f|.$$

□

Lemma 2.3.6. *Let f be a λ -concave function, $\lambda \geq 0$ and $\alpha(t)$ be an f -gradient curve, and let $\bar{\alpha}(s)$ be its reparameterization by arclength. Then $f \circ \bar{\alpha}$ is λ -concave.*

Proof.

$$\begin{aligned} (f \circ \bar{\alpha})^+(s_0) &= |\nabla_{\bar{\alpha}(s_0)} f| \geq \frac{d_{\bar{\alpha}(s_0)} f(\log_{\bar{\alpha}(s_0)}(\bar{\alpha}(s_1)))}{|\bar{\alpha}(s_1) \bar{\alpha}(s_0)|} \geq \\ &\geq \frac{f(\bar{\alpha}(s_1)) - f(\bar{\alpha}(s_0)) - \lambda |\bar{\alpha}(s_1) \bar{\alpha}(s_0)|^2 / 2}{|\bar{\alpha}(s_1) \bar{\alpha}(s_0)|} \geq \\ &\geq \frac{f(\bar{\alpha}(s_1)) - f(\bar{\alpha}(s_0))}{s_1 - s_0} - \lambda |\bar{\alpha}(s_1) \bar{\alpha}(s_0)| / 2. \end{aligned}$$

Since $\frac{|\bar{\alpha}(s_1) \bar{\alpha}(s_0)|}{(s_1 - s_0)} \rightarrow 1$ as $s_1 \rightarrow s_0+$, it follows that $f \circ \bar{\alpha}$ is λ -concave.

□

2.4. The Gradient Flow on Alexandrov Spaces. Let f be a λ -concave function on an Alexandrov space A . Consider the map $\Phi_f^T : A \rightarrow A$ defined as follows: $\Phi_f^T(x) = \alpha_x(T)$, where $\alpha_x : [0, \infty) \rightarrow A$ is the f -gradient curve with $\alpha_x(0) = x$. The map Φ_f^T is called *f -gradient flow at time T* . From Proposition 2.3.3 it is clear that Φ_f^T is an $\exp(\lambda T)$ -Lipschitz map. Next we want to prove that this map behaves nicely under Gromov–Hausdorff-convergence.

Theorem 2.4.1. *Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geq k$ which converges to an Alexandrov space A .*

Let $f_n \rightarrow f$, where $f_n : A_n \rightarrow \mathbb{R}$ is a sequence of λ -concave functions and $f : A \rightarrow \mathbb{R}$.

Then $\Phi_{f_n}^T \rightarrow \Phi_f^T$.

Theorem 2.4.1 immediately follows from the following Lemma:

Lemma 2.4.2. *Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geq k$ which converges to an Alexandrov space A .*

Let $f_n \rightarrow f$, where $f_n : A_n \rightarrow \mathbb{R}$ is a sequence of λ -concave functions and $f : A \rightarrow \mathbb{R}$.

Let $\alpha_n : [0, \infty) \rightarrow A_n$ be the sequence of f_n -gradient curves with $\alpha_n(0) = p_n$ and let $\alpha : [0, \infty) \rightarrow A$ be the f -gradient curve with $\alpha(0) = p$.

Then $\alpha_n \rightarrow \alpha$.

Proof. We may assume without loss of generality that f has no critical points. (Otherwise consider instead the sequence $A'_n = A_n \times \mathbb{R}$ with $f'_n(a \times x) = f_n(a) + x$.)

Let $\bar{\alpha}_n(s)$ denote the reparameterization of $\alpha_n(t)$ by arc length. Since all $\bar{\alpha}_n$ are 1-Lipschitz, we can choose a converging subsequence from any subsequence of $\bar{\alpha}_n$. Let $\bar{\beta} : [0, \infty) \rightarrow A$ be its limit.

Clearly, $\bar{\beta}$ is also 1-Lipschitz and hence $|\bar{\beta}^+| \leq 1$. Therefore, by Lemma 2.3.4,

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n \circ \bar{\alpha}_n|_a^b &= \lim_{n \rightarrow \infty} \int_a^b |\nabla_{\bar{\alpha}_n(s)} f_n| ds \geq \\ &\geq \int_a^b |\nabla_{\bar{\beta}(s)} f| \geq \int_a^b d_{\beta(t)} f(\beta^+(t)) = f \circ \beta|_a^b \end{aligned}$$

On the other hand, since $\bar{\alpha}_n \rightarrow \bar{\beta}$ and $f_n \rightarrow f$ we have $f_n \circ \bar{\alpha}_n|_a^b \rightarrow f \circ \bar{\beta}|_a^b$. Therefore, in both of these inequalities in fact equality holds.

Hence, $|\nabla_{\bar{\beta}(s)} f| = \lim_{n \rightarrow \infty} |\nabla_{\bar{\alpha}_n(s)} f_n|$, $|\bar{\beta}^+(s)| = 1$ and the directions of $\bar{\beta}^+(s)$ and $\nabla_{\bar{\beta}(s)} f$ coincide almost everywhere. This implies that $\bar{\beta}(s)$ is a gradient curve reparameterized by arc length. In other words, if $\bar{\alpha}(s)$ denotes the reparameterization of $\alpha(t)$ by arc length, then $\bar{\beta}(s) = \bar{\alpha}(s)$ for all s . It only remains to show that the original parameter $t_n(s)$ of α_n converges to the original parameter $t(s)$ of α .

Notice that $|\nabla_{\bar{\alpha}_n(s)} f_n| dt_n = ds$ or $dt_n/ds = ds/d(f_n \circ \bar{\alpha}_n)$. Likewise, $dt/ds = ds/d(f \circ \bar{\alpha})$. Then the convergence $t_n \rightarrow t$ follows from the λ -concavity of $f_n \circ \bar{\alpha}_n$ (see Lemma 2.3.6) and the convergence $f_n \circ \bar{\alpha}_n \rightarrow f \circ \bar{\alpha}$. \square

2.5. Gradient balls.

Let A be an Alexandrov space and let $S \subset A$ be a subset of A . A function $f : A \rightarrow \mathbb{R}$ which can be represented as

$$f = \sum_i \theta_i \frac{\text{dist}_{a_i}^2}{2} \quad \text{with } \theta_i \geq 0, \quad \sum_i \theta_i = 1 \quad \text{and } a_i \in S$$

will be called *cocos-function with respect to S* (where ‘‘cocos’’ stands for **convex combination of squares of distance functions**). A broken gradient curve for a collection of such functions will be called *cocos-curve with respect to S* .

For $p \in A$ and $T, r \in \mathbb{R}_+$, let us define “the gradient ball with center p and radius T with respect to $B_r(p)$ ”, $\beta_T^r(p)$, as the set of all end points of cocos-curves with respect to $B_r(p)$ that start at p with total time $\leq T$.

Lemma 2.5.1.

(I) *There exists $T = T(m) \in \mathbb{R}_+$ such that for any m -dimensional Alexandrov space A with curvature ≥ -1 and any $q \in A$ there is a point $p \in A$ such that*

(i) $|pq| \leq 1$, and

(ii) $B_1(p) \subset \beta_T^1(p)$.

(II) *There exists $T' = T'(m) \in \mathbb{R}_+$ such that the following holds. Let A be an Alexandrov space which is a quotient $A = \tilde{A}/\Gamma$ of an m -dimensional Alexandrov space \tilde{A} with curvature ≥ -1 by a discrete action of a group of isometries Γ . Let $q \in A$ and $p = p(q) \in A$ be as in part I above.*

Then for any lift $\tilde{p} \in \tilde{A}$ of p one has that $B_1(\tilde{p}) \subset \beta_{T'}^1(\tilde{p})$.

Proof. The proof is similar to the construction of a strained point in an Alexandrov space (see [BGP92]).

Set $\delta = 10^{-m}$. Take $a_1 = q$ and take b_1 to be a farthest point from a_1 in the closed ball $\bar{B}_1(a_1)$. Take a_2 to be a midpoint of a_1b_1 and let b_2 be a farthest point from a_2 such that $|a_1b_2| = |a_1a_2|$ and $|a_2b_2| \leq \delta|a_1b_1|$, etc. On the k -th step we have to take a_k to be a midpoint of $a_{k-1}b_{k-1}$ and b_k to be a farthest point from a_k such that $|a_i b_k| = |a_i a_k|$ for all $i < k$ and $|a_k b_k| \leq \delta|a_{k-1}b_{k-1}|$.

After m steps, take p to be a midpoint of $a_m b_m$. We only have to check that we can find $T = T(m)$ such that $\beta_T^1(p) \supset B_1(p)$.

Let t_i be the minimal time such that $B_{|a_i b_i|/\delta^m}(p) \subset \beta_{t_i}^1(p)$. Then one can take $T = t_1$. Therefore it is enough to give estimates for t_m and t_{k-1}/t_k only in terms of δ and m . Looking at the ends of broken gradient curves starting at p for the functions $\text{dist}_p^2/2$, $\text{dist}_{a_i}^2/2$ and $\text{dist}_{b_i}^2/2$, we easily see that $t_n \leq 1/\delta^m$. Now, looking at the ends of broken gradient curves starting at $B_{|a_{k-1}b_{k-1}|/\delta^m}(p)$ for the functions $\text{dist}_p^2/2$, $\text{dist}_{a_i}^2/2$ and $\text{dist}_{b_i}^2/2$, we have that $t_{k-1}/t_k \leq 1/\delta^m$. Therefore $t_1 \leq 1/\delta^{m^2} = 10^{-m^3}$. This finishes the proof of part (I).

For part (II), notice that

- a) for any $r, t > 0$ we have $\beta_t^r(p) \subset B_{ret}(p)$;
- b) if $\beta_t^r(p) \supset B_\rho(p)$, then $\beta_{t+\tau}^r(p) \supset B_{\rho e^\tau}(p)$;
- c) if $\rho = |px|$ and $x \in \beta_t^{r+\rho}(p)$, then $\beta_\tau^r(x) \subset \beta_{t+\tau}^{r+\rho}(p)$.

Take $\varepsilon = e^{-T}/4$ and apply part (I) of the lemma to $\frac{1}{\varepsilon}\tilde{A}$ to find a point $p' \in \tilde{A}$ such that $|\tilde{p}p'| \leq \varepsilon$ and $B_\varepsilon(p') \subset \beta_T^\varepsilon(p') \subset \tilde{A}$. Then for some deck transformation γ we have $\gamma p' \in \beta_T^\varepsilon(p) \subset B_{\varepsilon e^T}(p)$. Therefore it holds that $\gamma p' \in B_{1/2}(\tilde{p})$. Hence, taking

$$T' = 2T + 1/\varepsilon = 2T + 4e^T,$$

we obtain

$$\beta_{T'}^1(p) \supset \beta_{T+1/\varepsilon}^\varepsilon(\gamma p') \supset B_1(\tilde{p}).$$

□

2.6. Short basis. We will use the following construction due to Gromov.

Given an Alexandrov space A with a marked point $p \in A$, and a group Γ acting discretely on $A = (A, d)$ one can define a short basis of the action of Γ at p as follows:

For $\gamma \in \Gamma$ define the norm of γ by the formula $|\gamma| = d(p, \gamma(p))$. Choose $\gamma_1 \in \Gamma$ with the minimal norm in Γ . Next choose γ_2 to have minimal norm in $\Gamma \setminus \langle \gamma_1 \rangle$. On the n -th step choose γ_n to have minimal norm in $\Gamma \setminus \langle \gamma_1, \gamma_2, \dots, \gamma_{n-1} \rangle$. The sequence $\{\gamma_1, \gamma_2, \dots\}$ is called a *short basis* of Γ at p . In general, the number of elements of a short basis can be finite or infinite. In the special case of the action of the fundamental group $\pi_1(A, p)$ on the universal cover of A one speaks of the short basis of $\pi_1(A, p)$.

It is easy to see that for a short basis $\{\gamma_1, \gamma_2, \dots\}$ of the fundamental group of an Alexandrov space A the following is true:

- (1) If A has diameter d then $|\gamma_i| \leq 2d$.
- (2) If A is compact then $\{\gamma_i\}$ is finite.
- (3) For any $i > j$ we have $|\gamma_i| \leq |\gamma_j^{-1} \gamma_i|$.

The third property implies that if $\tilde{p} \in \tilde{A}$ is in the preimage of p in the universal cover \tilde{A} of A and $\tilde{p}_i = \gamma_i(\tilde{p})$, then

$$|\tilde{p}_i \tilde{p}_j| \geq \max\{|\tilde{p} \tilde{p}_i|, |\tilde{p} \tilde{p}_j|\}.$$

As was observed by Gromov, if A is an Alexandrov space with curvature $\geq \kappa$ and diameter $\leq d$, the last inequality implies that $\angle \tilde{p}_i \tilde{p} \tilde{p}_j > \delta = \delta(\kappa, d) > 0$. This yields an upper bound on the number of elements of a short basis in terms of κ, d and the dimension of A .

3. NILPOTENCY OF ALMOST NONNEGATIVELY CURVED MANIFOLDS

In this section we prove Theorem A.

3.1. Preliminary lemmas. Let M be an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$, $n \in \mathbb{N}$, a sequence of Riemannian metrics on M such that $\sec(M_n) \geq -1/n$ and $\text{diam}(M_n) \leq 1/n$. Let us denote by \tilde{M} the universal covering of M , and by $\tilde{M}_n \rightarrow M_n$ the universal Riemannian covering of M_n (i.e., \tilde{M}_n is \tilde{M} equipped with the pullback of the Riemannian metric g_n).

Key Lemma 3.1.1. *Given $\varepsilon > 0$ and $r_2 > r_1 > 0$, let $\tilde{M}_n \supset B_{r_2}(p_n) \supset B_{r_1}(p_n)$. Then, for n sufficiently large, there is a $(1 + \varepsilon)$ -Lipschitz map $\Phi_n : B_{r_2}(p_n) \rightarrow B_{r_1}(p_n)$ which is homotopic to the identity on $B_{r_2}(p_n)$.*

Proof. Fix $R \gg r_2$ (here $R > 1000(1 + 1/\varepsilon)r_2$ will suffice). Notice that as $n \rightarrow \infty$, we have that $B_R(p_n) \xrightarrow{\text{GH}} B_R \subset \mathbb{R}^a$. Choose a finite $R/1000$ -net

$\{a_i\}$ of $\partial B_R \subset \mathbb{R}^q$. Let $a_{i,n} \in M_n$ be sequences such that $a_{i,n} \rightarrow a_n$. Consider the sequence of functions $f_n : M_n \rightarrow \mathbb{R}$ with $f_n = \min_i \text{dist}_{a_{i,n}}^2$.

For large n , the functions f_n are 2-concave in $B_R(p_n)$, so that, in particular, the gradient flows $\Phi_{f_n}^T|_{B_{r_2}(p_n)}$ are e^{2T} -Lipschitz. Moreover, if ξ_x denotes the starting vector of a unit speed shortest geodesic from x to p_n , then for any $x \in B_{r_2}(p_n) \setminus B_{r_1}(p_n)$ we have $\langle \xi_x, \nabla f \rangle \geq R/2$. Therefore, if $T = 2r_2/R$, then $\Phi_{f_n}^T(B_{r_2}(p_n)) \subset B_{r_1}(p_n)$. Thus $\Phi_n = \Phi_{f_n}^{2r_2/R}$ provides a $4r_2/R$ -Lipschitz map $B_{r_2}(p_n) \rightarrow B_{r_1}(p_n)$, and it is $(1 + \varepsilon)$ -Lipschitz if one chooses R sufficiently large. \square

For $\gamma \in \pi_1(M)$, set $|\gamma|_n = |p\gamma(p)|_{\tilde{M}_n}$, see 2.6.

Corollary 3.1.2. *Let M be almost nonnegatively curved manifold. Let*

$$h : \pi_1(M) \rightarrow \text{Aut}(H^*(\tilde{M}, \mathbb{Z})/\text{tor})$$

be the natural action of $\pi_1(M)$ on $H^(\tilde{M}, \mathbb{Z})$. Then there is a sequence of norms $\|*\|_n$ on $H^*(\tilde{M}, \mathbb{Z})/\text{tor}$ such that the following holds. Given any $\varepsilon > 0$, there is $n \in \mathbb{Z}_+$ such that for any $\gamma \in \pi_1(M)$ with $|\gamma|_n \leq 2\text{diam}(M_n)$ we have $\|h(\gamma)\|_n \leq 1 + \varepsilon$.*

Proof. [FY92, theorem 0.1] and Yamaguchi's fibration theorem [Yam91] imply that if n is sufficiently large, for any fixed $r \in \mathbb{R}_+$ we have that for any $p_n \in \tilde{M}_n$ the inclusion map $B_r(p_n) \rightarrow \tilde{M}_n$ is a homotopy equivalence.

Let $\|*\|_{n,r}$ denote the L_∞ -norm on differential forms on $B_r(p_n) \subset \tilde{M}_n$.

Fix $r_2 > r_1 > 0$. If ω is a differential form on $B_{r_1}(p_n) \subset M_n$ and n is sufficiently large, Lemma 3.1.1 implies that

$$\|\Phi_n^*(\omega)\|_{n,r_2} \leq (1 + \varepsilon)\|\omega\|_{n,r_1} \quad \text{and} \quad 2\text{diam}(M_n) \leq r_2 - r_1.$$

If now ω is a form on $B_{r_2}(p_n) \in \tilde{M}_n$ and $\gamma \in \pi_1(M)$ such that

$$|\gamma|_n = |p_n \gamma(p_n)| \leq 2\text{diam}(M_n) \leq r_2 - r_1,$$

then $B_{r_1}(p_n) \subset B_{r_2}(\gamma(p_n)) \subset \tilde{M}_n$, whence

$$\|\Phi_n^*(\gamma^*(\omega))\|_{n,r_2} \leq (1 + \varepsilon)\|\gamma^*(\omega)\|_{n,r_1} \leq (1 + \varepsilon)\|\omega\|_{n,r_2}.$$

Thus, for the induced norms on the de Rham cohomology of \tilde{M} (and on its integral subspace $H^*(\tilde{M}, \mathbb{Z})/\text{tor}$) we have

$$\|[\gamma^*(\omega)]\|_{n,r_2} \leq (1 + \varepsilon)\|[\omega]\|_{n,r_2}.$$

Therefore the sequence of norms $\|*\|_n = \|*\|_{n,r_2}$ satisfies the conditions of the Corollary. \square

Lemma 3.1.3. *There exists a constant $N = N(n, k) \in \mathbb{Z}_+$ such that the following holds. If G is a subgroup of $GL(n, \mathbb{Z})$ and S is a set of generators of G with $\#(S) \leq k$ such that the eigenvalues of each element of S^N are all equal to 1 in absolute value, then the same is true for the eigenvalues of all elements of G .*

Proof. Let B be the set of all matrices in $GL(n, \mathbb{Z})$ for which all of their eigenvalues are equal to 1 in absolute value. Since the characteristic polynomials of such matrices are uniformly bounded and have integer coefficients, there are only finitely many of them. Let \bar{B} be the Zariski closure of B in the set of all real $n \times n$ matrices. By the above, all elements of \bar{B} satisfy that the absolute values of all of their eigenvalues are equal to 1.

Consider now the space $V = \mathbb{R}^{kn^2}$ of k -tuples of real $n \times n$ matrices.

Consider a collection of matrices $(M_1, M_2, \dots, M_k) \in V$, where $M_i \in GL(n, \mathbb{R})$. Let F_k be a free group on k generators, generated by $S = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$, and let $h : F_k \rightarrow GL(n, \mathbb{R})$ be the homomorphism defined by $h(\gamma_i) = M_i$. The property that for any $\gamma \in F_k$ $h(\gamma)$ be an element of \bar{B} then describes an algebraic subset $A_\gamma \subset V$.

The intersection $A = \bigcap_{\gamma \in F_k} A_\gamma$ is also algebraic, and therefore there is a finite number $N = N(n, k)$ such that for $S^N \subset F_k$, $A = \bigcap_{\gamma \in S^N} A_\gamma$. \square

Lemma 3.1.4. *Let Γ be a subgroup of $GL(n, \mathbb{Z})$ such that the eigenvalues of each element of Γ are equal to 1 in absolute value. Then Γ contains a subgroup Γ' of finite index all of whose elements have eigenvalues equal to 1.*

Proof. Let G denote the Zariski closure of Γ in $GL(n, \mathbb{R})$. Then G , being an algebraic group, is a Lie group with finitely many components. Let G_\circ be the identity component of G . By the same argument as in the proof of the previous lemma, the set of all characteristic polynomials of the elements of G is finite. Therefore the characteristic polynomial of any element of G_\circ is identically equal to $(x - 1)^n$.

Therefore, the subgroup $\Gamma' = \Gamma \cap G_\circ$ satisfies all conditions of the Lemma. \square

Remark 3.1.5. As was pointed out to us by Yu. Zarkhin, one can alternatively take Γ' to be the kernel of the composition of the homomorphisms $\Gamma \rightarrow GL(n, \mathbb{Z}) \rightarrow GL(n, \mathbb{Z}/3\mathbb{Z})$. In this way one obtains a bound

$$[\Gamma : \Gamma'] \leq 3^{n^2}.$$

To see that Γ' satisfies the conclusion of Lemma 3.1.4, one should notice that every element of Γ is a quasi-unipotent matrix since all its eigenvalues are roots of unity. The desired result then follows from the so-called Minkowski Lemma. Apply, for instance, [SZ96, Th. 7.2] for $n = 3$, $k = 1$ (so $R(1, 3) = 1$), where we take \mathcal{O} to be the ring of $n \times n$ integer matrices.

3.2. Proof of Theorem A. Let M be an almost nonnegatively curved manifold. Denote, as usual, by $M_n = (M, g_n)$, $n \in \mathbb{N}$, a sequence of Riemannian metrics on M such that $\sec(M_n) \geq -1/n$ and $\text{diam}(M_n) \leq 1/n$, by \tilde{M} the universal covering of M , and by $\tilde{M}_n \rightarrow M_n$ the universal Riemannian covering of M_n .

After passing to a finite cover of M , by [FY92] we may assume that $\pi_1(M)$ is nilpotent.

Fix $p \in M$ and let $\{\gamma_{i,n}\}$ be a short basis of $\pi_1(M_n, p)$ (see 2.6). Then, if n is sufficiently large, the short basis $\{\gamma_{i,n}\}$ has at most $k = k(\dim M)$ elements and its elements satisfy $|\gamma_{i,n}|_n \leq 2/n$ for every i . Moreover, Corollary 3.1.2 implies that given $\varepsilon > 0$, for all large n and every i we have $\|h(\gamma_{i,n})\|_n < 1 + \varepsilon$ and $\|h(\gamma_{i,n}^{-1})\|_n < 1 + \varepsilon$.

Take $N = N(k, m)$ as in Lemma 3.1.3. One can choose $\varepsilon > 0$ so small that if p is a polynomial with integer coefficients for which all of its roots have absolute values lying between $1/(1 + \varepsilon)^N$ and $(1 + \varepsilon)^N$, then all roots of p have absolute values equal to 1. This follows from the fact that the total number of integer polynomials all of whose roots are contained in a fixed bounded region is finite.

Set $S_n := \{\gamma_{i,n}\}$. Then for any $\gamma \in S_n^N$ we have $\|h(\gamma)\|_n < (1 + \varepsilon)^N$ and $\|h(\gamma^{-1})\|_n < (1 + \varepsilon)^N$. Therefore the absolute values of all eigenvalues lie between $1/(1 + \varepsilon)^N$ and $(1 + \varepsilon)^N$. Since the characteristic polynomial of $h(\gamma)$ has integer coefficients, the absolute values of all the eigenvalues of $h(\gamma)$ are in fact equal to 1.

Apply now Lemma 3.1.3. It follows that for any $\gamma \in \pi_1(M)$ the absolute values of all eigenvalues of $h(\gamma)$ are equal to 1.

Then Lemma 3.1.4 implies that after passing to a finite cover M' of M , for any $\gamma \in \pi_1(M')$ all eigenvalues of $h(\gamma)$ are equal to 1. By Engel's theorem, one can choose an integral basis of $H^*(\tilde{M}, \mathbb{R})$ such that the action of $\pi_1(M)$ on $H^*(\tilde{M}, \mathbb{Z})/tor$ is given by upper triangular matrices.

Therefore, by passing to a finite cover M'' of M' , we can assume that the action of $\pi_1(M'')$ on $H^*(\tilde{M}, \mathbb{Z})$ (and on $H_*(\tilde{M}, \mathbb{Z})$) is nilpotent.

Recall (see, e.g., [HMR75, 2.19]) that a connected CW complex with nilpotent fundamental group is nilpotent if and only if the action of its fundamental group on the homology of its universal cover is nilpotent. Thus M'' is a nilpotent space, whence the proof of Theorem A is complete. \square

4. C-NILPOTENCY OF THE FUNDAMENTAL GROUP

4.1. In this section we will prove Theorem B. It will follow from the following somewhat stronger result.

Theorem 4.1.1. *For any integer m there exist constants $\epsilon(m) > 0$ and $C(m) > 0$ such that the following holds. If M^m is a closed smooth m -manifold which admits a Riemannian metric g with $\sec(M^m, g) > -\epsilon(m)$ and $\text{diam}(M^m, g) < 1$, then the fundamental group of M^m is $C(m)$ -nilpotent, i.e., $\pi_1(M^m)$ contains a nilpotent subgroup of index $\leq C(m)$.*

Remark 4.1.2. The proofs of Theorems A and C show that corresponding versions of those results also do hold when these theorems are reformulated in a fashion similar to Theorem 4.1.1.

By an argument by contradiction, Theorem 4.1.1 follows from the following statement:

Given a sequence of Riemannian m -manifolds (M_n, g_n) with diameters $\text{diam}(M_n, g_n) \leq 1/n$ and sectional curvatures $\text{sec}(g_n) \geq -1/n$, one can find $C \in \mathbb{R}$ such that $\pi_1(M_n)$ is C -nilpotent for all sufficiently large n .

4.2. Algebraic lemmas. Recall that the group of outer automorphisms $\text{Out}(G)$ of a group G is defined as the quotient of its automorphism group $\text{Aut}(G)$ by the subgroup of inner automorphisms $\text{Inn}(G)$.

Lemma 4.2.1 (*A characterization of C -nilpotent groups*). *Let*

$$\{1\} = G_\ell \subseteq \dots \subseteq G_1 \subseteq G_0 = G$$

be a sequence of groups satisfying the following properties:

For any i we have that

- (i) $G_i \trianglelefteq G$ is normal in G ;
- (ii) the image of the conjugation homomorphism $h_i : G \rightarrow \text{Out}(G_i/G_{i+1})$ is finite of order at most C_i ;
- (iii) G_i/G_{i+1} contains an abelian subgroup E_i of index $\leq c_i$.

Then G contains a nilpotent subgroup N of index at most

$$C = C(c_1, \dots, c_\ell, C_1, \dots, C_\ell)$$

where N is of nilpotency class $\leq \ell$.

Proof. First of all, notice that property (i) assures that the objects described in parts (ii) and (iii) of the lemma are well-defined.

Set $\Gamma_i := G_i/G_{i+1}$.

Let $H = \bigcap \ker h_i$. Notice that $[G : H] \leq \prod_i C_i$ and that the image of H under the conjugation homomorphism $f_i : G \rightarrow \text{Aut}(\Gamma_i)$ lies in $\text{Inn}(\Gamma_i)$, i.e., $f_i|_H : H \rightarrow \text{Inn}(\Gamma_i)$.

By passing to a subgroup we can assume that $E_i \trianglelefteq \Gamma_i$ is normal of index $\leq C(c_i)$ (we can take $C(c_i) = c_i!$).

By increasing E_i if necessary we can assume that E_i contains the center of Γ_i .

Let Z_i be the image of E_i under the projection map $\pi : \Gamma_i \rightarrow \text{Inn}(\Gamma_i)$. Clearly $[\text{Inn}(\Gamma_i) : Z_i] \leq c_i$ and $Z_i \trianglelefteq \text{Inn}(\Gamma_i)$ is normal.

Let

$$N = H \cap \left(\bigcap_i f_i^{-1}(Z_i) \right), \quad N_i = N \cap G_i.$$

Then

$$[G : N] \leq C = C(c_1, \dots, c_\ell, C_1, \dots, C_\ell)$$

and N satisfies:

For any i

- (i') $N_i \trianglelefteq N$ is normal in N ;
- (ii') N_i/N_{i+1} is in the center of N/N_{i+1}

i.e., N is nilpotent of nilpotency length $\leq \ell$.

Condition (i') is obvious so we only need to check (ii').

To see (ii') observe that by construction the image of the conjugation action $N \rightarrow \text{Aut}(\Gamma_i)$ lies in $\text{Inn}(\Gamma_i)$. Moreover, in fact it lies in $\pi(A_i)$ and as such it acts trivially on E_i . Lastly observe that $N_i/N_{i+1} \subset E_i$.

Indeed, by construction, for any $g \in N_i/N_{i+1} \subset \Gamma_i$ there is $a \in E_i$ such that $\pi(g) = \pi(a)$. Therefore, $g = az$ for some z in the center of Γ_i . By our assumption on E_i this means that $g \in E_i$.

Thus N acts trivially on N_i/N_{i+1} which means that N is nilpotent and G is C -nilpotent. \square

Trivial Lemma 4.2.2 (*A characterization of finite actions*). *If S is a finite set of generators of a group G with $S^{-1} = S$, and $h : G \rightarrow H$ is a homomorphism with $|h(S^n)| < n$ for some $n > 0$, then $h(S^n) = h(G)$ and, in particular, $|h(G)| < n$.*

Let now Γ be a group which acts discretely by isometries on an Alexandrov space A with curvature ≥ -1 . Choose a marked point $p \in A$. Assume that $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a finite short basis of Γ at p (see 2.6), and that $\theta \leq |\gamma_i| \leq 1$, where $|\gamma| \stackrel{\text{def}}{=} |p\gamma(p)|$. Let $\#(R)$ denote the number of elements $\gamma \in \Gamma$ with $|\gamma| \leq R$. The Bishop–Gromov inequality implies that

$$\#(R) \leq v_{-1}^m(R)/v_{-1}^m(\theta),$$

where $m = \dim A$ and $v_{-1}^m(r)$ is the volume of the ball of radius r in the m -dimensional simply connected space form of curvature -1 . Therefore, if $\#\#(L)$ denotes the number of homomorphisms $h : \Gamma \rightarrow \Gamma$ with norm $\leq L$ (i.e., the number of homomorphisms for which it holds that for any $\gamma \in \Gamma$ one has that $|h(\gamma)| \leq L|\gamma|$), then

$$(4.2.1) \quad \#\#(L) \leq \#(L)^n \leq \left[\frac{v_{-1}^m(L)}{v_{-1}^m(\theta)} \right]^n.$$

4.3. The blow-up construction. For $n \rightarrow \infty$, the manifolds M_n clearly converge to a point $=: A_0$.

Set $M_{n,1} := M_n$ and $\vartheta_{n,1} := \text{diam} M_{n,1}$.

Rescale now $M_{n,1}$ by $\frac{1}{\vartheta_{n,1}}$ so that $\text{diam}(M_{n,1}/\vartheta_{n,1}) = 1$. Passing to a subsequence if necessary, one has that the manifolds $\frac{1}{\vartheta_{n,1}}M_{n,1}$ converge to A_1 , where A_1 is a compact nonnegatively curved Alexandrov space with diameter 1.

Now choose a regular point $\bar{p}_1 \in A_1$, and consider distance coordinates around $\bar{p}_1 \in U_1 \rightarrow \mathbb{R}^{k_1}$, where k_1 is the dimension of A_1 . The distance functions can be lifted to $U_{n,1} \subset \frac{1}{\vartheta_{n,1}}M_{n,1}$.

Let $M_{n,2}$ be the level set of $U_{n,1} \rightarrow \mathbb{R}^{k_1}$ that corresponds to \bar{p}_1 . Clearly, $M_{n,2}$ is a compact submanifold of codimension k_1 . Set $\vartheta_{n,2} := \text{diam} M_{n,2}$.

Passing again to a subsequence if necessary, one has that the sequence $\frac{1}{\vartheta_{n,2}}M_{n,2}$ converges to some Alexandrov space A_2 . As before, A_2 is a compact nonnegatively curved Alexandrov space with diameter 1. Set $k_2 :=$

$k_1 + \dim A_2$. If one now chooses a marked point in $M_{n,2}$, then, as $n \rightarrow \infty$, $M_n/\vartheta_{n,2}$ converges to $A_2 \times \mathbb{R}^{k_1}$, which is of some dimension $k_2 > k_1$.

We repeat this procedure until, at some step, $k_\ell = m$.

As a result one obtains a sequence $\{A_i\}$ of compact nonnegatively curved Alexandrov spaces with diameter 1 that satisfies

$$\dim A_i = k_i - k_{i-1}, \quad \text{so that} \quad \sum_{i=1}^{\ell} \dim A_i = m.$$

We also obtain a sequence of rescaling factors $\vartheta_{n,i} = \text{diam} M_{n,i}$, and a nested sequence of submanifolds

$$\{p_n\} = M_{n,\ell} \subset \cdots \subset M_{n,2} \subset M_{n,1} = M_n,$$

which in turn induces a sequence of homomorphisms

$$\{1\} = \pi_1(M_{n,\ell}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \pi_1(M_{n,2}) \xrightarrow{\cong} \pi_1(M_{n,1}) = \pi_1(M_n).$$

Let $G_i := G_i(n) := i^i \pi_1(M_{n,i})$.

For n sufficiently large, the subgroups $G_i(n)$ are those which are generated by elements of norm $\leq 3\vartheta_{n,i}$. Equivalently, if one takes a short basis $\{\gamma_i\}$ of $G(n)$, then G_i is the subgroup generated by all elements γ_i such that $|\gamma_i| \leq 3\vartheta_{n,i}$.

4.4. Limit fundamental groups of Alexandrov spaces. We will now define the “limit” or “L-fundamental groups” of the Alexandrov spaces A_i constructed above. This notion is similar to the notion of the fundamental group of an orbifold. However, we note in advance that the construction of the L-fundamental group does not only depend on the spaces A_i , but also on the chosen rescaled subsequence of M_n . In fact, the following construction shows that the limit fundamental group of A_i , $\pi_1^L(A_i)$, is isomorphic to $\pi_1(M_{n,i}, M_{n,i+1})$ for all sufficiently large n . But, unlike $\pi_1(M_{n,i})$, the groups $\pi_1^L(A_i)$ will not depend on n .

The limit fundamental groups of A_i . Consider the converging sequence $(M_n/\vartheta_{n,i}, p_n) \xrightarrow{\text{GH}} (A_i \times \mathbb{R}^{k_i-1}, \bar{p}_i \times 0)$ (here the interesting case is collapsing). Recall that $\bar{p}_i \in A_i$ is a regular point. Fix $\varepsilon > 0$ such that $\text{dist}_{\bar{p}_i}$ on A_i does not have critical values in $(0, 2\varepsilon)$. Take a sequence R_n which converges very slowly to infinity (here we will need $R_n \vartheta_{n,i}/\vartheta_{n,i-1} \rightarrow 0$ and $R_n \rightarrow \infty$).

Consider then a sequence of Riemannian coverings

$$\Pi: (\tilde{B}_n, \tilde{p}_n) \rightarrow (B_{R_n}(p_n), p_n)$$

of $B_{R_n}(p_n) \subset M_n/\vartheta_{n,i}$ with $\pi_1(\tilde{B}_n, \tilde{p}_n) = \pi_1(B_\varepsilon(p_n), p_n)$, where $B_\varepsilon(p_n) \subset M_n/\vartheta_{n,i}$.

After passing to a subsequence if necessary, the sequence $(\tilde{B}_n, \tilde{p}_n)$ converges to a nonnegatively curved Alexandrov space $\tilde{A}_i \times \mathbb{R}^{k_i-1}$, where the space \tilde{A}_i has the same dimension as A_i . Indeed, by construction it contains an isometric copy of $B_\varepsilon(p_{n,i})$, and therefore

$$\dim \tilde{A}_i + k_{i-1} = \dim \lim_{i \rightarrow \infty} B_\varepsilon(p_{n,i}) = \dim A_i + k_{i-1}.$$

Let us show that for all sufficiently large n ,

$$\iota(\pi_1(M_{n,i+1})) \trianglelefteq \pi_1(M_{n,i}).$$

Assume that $\Pi(\tilde{q}_n) = \tilde{p}_n$ and that $\tilde{q}_n \rightarrow \bar{q}_n \in \tilde{A}_i$. Connect \bar{p}_n and \bar{q}_n by a geodesic which, by [Pet98], only passes through regular points. Note that in a small neighborhood of this geodesic in M_n we have two copies of $M_{n,i+1}$, near \tilde{p}_n and \tilde{q}_n . Therefore, applying Yamaguchi's Fibration Theorem in a small neighborhood of this geodesic, we can construct a diffeomorphism from $M_{n,i+1}$ to itself. This implies that for any loop γ which after lifting connects $\tilde{p}_n\tilde{q}_n$, we have $\gamma^{-1}\iota\pi_1(M_{n,i+1})\gamma \subset \iota\pi_1(M_{n,i+1})$, i.e., $\iota\pi_1(M_{n,i+1}) \triangleleft \pi_1(M_{n,i})$ (for an alternative argument see also [FY92]).

This easily yields that $A_i = \tilde{A}_i/\Gamma_i$, where Γ_i is a group of isometries which acts discretely on \tilde{A}_i . The group Γ_i is denoted by $\pi_1^L(A_i)$ (the *limit* or *L-fundamental group* of A_i). This group is clearly isomorphic to

$$\pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\iota(\pi_1(M_{n,i+1}))$$

for all sufficiently large n , and the space \tilde{A}_i will be called the *universal covering* of A_i .

Since \tilde{A}_i is nonnegatively curved and $A_i = \tilde{A}_i/\pi_1^L(A_i)$ is compact, by Toponogov's splitting theorem \tilde{A}_i isometrically splits as $\tilde{A}_i = K_i \times \mathbb{R}^{s_i}$, where K_i is a compact Alexandrov space with $\text{curv} \geq 0$. Since $\pi_1^L(A_i)$ is a group of isometries that acts discretely on \tilde{A}_i , it follows that $\pi_1^L(A_i)$ is a virtually abelian group.

4.5. Final steps. Consider now the corresponding series

$$\{1\} = G_\ell(n) \subset \dots \subset G_1(n) \subset G_0(n) = \pi_1(M_n).$$

The theorem then follows from the following

Lemma 4.5.1. *For all sufficiently large n , the series*

$$\{1\} = G_\ell(n) \subset \dots \subset G_1(n) \subset G_0(n)$$

constructed above satisfies the assumptions of Lemma 4.2.1 for numbers C_i and c_i which do not depend on n .

We first prove the following

Sublemma 4.5.2. *Each subgroup $G_i(n)$ is normal in $G(n)$.*

Proof. We will show by reverse induction on k that $G_i(n) \trianglelefteq G_k(n)$ for any $k \leq i$. Let us assume that we already know that $G_i(n) \trianglelefteq G_{k+1}(n)$. Since

$$\iota\pi_1(M_{n,k+1}) \trianglelefteq \pi_1(M_{n,k}),$$

we know that $G_{k+1}(n) \trianglelefteq G_k(n)$. Consider the covering

$$\Pi_{k+1}: (\tilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \rightarrow (M_n, p_n)$$

with covering group $\Gamma_{k+1}(n)$.

Clearly $(\tilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \xrightarrow{\text{GH}} \mathbb{R}^{s_i}$ for some integer s_i . Applying Lemma 2.5.1, it follows that for any $a \in G$ with $|a| < 1$ there is a cocos-curve γ in $\tilde{M}_{n,k+1}$ with total time T connecting \tilde{p}_n and $a(\tilde{p}_n)$ in $\tilde{M}_{n,k+1}$. Then clearly $\gamma \sim ga$ for some $g \in G_{k+1}(n)$. Let us denote by $\Phi^T : \tilde{M}_{n,i} \rightarrow \tilde{M}_{n,i}$ the gradient flow corresponding to γ .

Let γ_j be a loop from the short basis of $G_i(n)$. As was mentioned in 4.3, if n is large, then $\text{length } \gamma_j \leq 3\vartheta_{n,i}$. Let us denote by $\tilde{\gamma}_j$ a lift of γ_j to $\tilde{M}_{n,i}$. Let $\tilde{p}_{n,j} \in \tilde{M}_{n,i}$ be its starting point. Since $[\gamma_j] \in G_i(n)$, we have that $\tilde{\gamma}_j$ is a loop in $\tilde{M}_{n,i}$. Consider then the loop $\gamma'_j = \Pi \circ \Phi^T \circ \tilde{\gamma}_j$. Clearly,

$$[\gamma_j] = a^{-1}g^{-1}[\gamma'_j]ga, \quad \text{or} \quad [\gamma'_j] = ga[\gamma_j]a^{-1}g^{-1}.$$

Now Proposition 2.3.3 implies that

$$\text{length}(\gamma'_j) \leq \exp(2T) \text{length}(\gamma_j).$$

Thus, for sufficiently large n ,

$$ga[\gamma_j]a^{-1}g^{-1} \in G_i(n),$$

and since $g \in G_i(n) \leq G_{k+1}(n)$ it follows that

$$a[\gamma_j]a^{-1} \in G_i(n),$$

i.e., $G_i(n) \leq G_k(n)$. □

Proof of Lemma 4.5.1. The group

$$\pi_1^L(A_i) = \pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\iota(\pi_1(M_{n,i}))$$

is virtually abelian. Let d_i be the minimal index of an abelian subgroup of $\pi_1^L(A_i)$. The epimorphism $i^i : \pi_1(M_{n,i}) \rightarrow G_i$ induces an epimorphism $\pi_1^L(A_i) \rightarrow G_i(n)/G_{i+1}(n)$. Therefore, $G_i(n)/G_{i+1}(n)$ is d_i -abelian for all large n .

Consider the covering $\Pi_i : \tilde{M}_{n,i} \rightarrow M_n$ with covering group $G_i(n)$, and let $\tilde{p}_{n,i}$ be a preimage of p_n . Clearly $(\tilde{M}_{n,i}, \tilde{p}_{n,i}) \xrightarrow{\text{GH}} \mathbb{R}^{s_i}$ for some integer s_i . Applying Lemma 2.5.1, it follows that for any $a \in G(n)$ with $|a| < 1$ there is a cocos-curve γ in $\tilde{M}_{n,i}$ which connects p and $a(p)$. Then clearly $\gamma \sim ga$ for some $g \in G_i(n)$. Let us denote by $\Phi^T : \tilde{M}_{n,i} \rightarrow \tilde{M}_{n,i}$ the gradient flow corresponding to γ .

Let $b \in G_i(n)$ and β be a loop representing b . Let us denote by $\tilde{\beta}$ a lift of β to $\tilde{M}_{n,i}$. Let $\tilde{p}_{n,i} \in \tilde{M}_{n,i}$ be its starting point. Since $[\beta] \in G_i(n)$, we have that $\tilde{\beta}$ is a loop in $\tilde{M}_{n,i}$.

Consider now the loop $\beta' = \Pi \circ \Phi^T \circ \tilde{\beta}$. Clearly,

$$b = [\beta] = a^{-1}g^{-1}[\beta']ga, \quad \text{or} \quad [\beta'] = gaba^{-1}g^{-1}.$$

Proposition 2.3.3 then implies that

$$\text{length}(\beta') \leq \exp(2T) \text{length}(\beta)$$

Therefore, if $h_a : G_i(n)/G_{i+1}(n) \rightarrow G_i(n)/G_{i+1}(n)$ is induced by the conjugation $b \rightarrow aba^{-1}$, then for any $a \in G(n)$ there is $g \in G_i(n)$ such that $|h_{ga}| \leq \exp(2T)$.

Let now δ_i be the minimal norm of the elements of $\pi_1^L(A_i)$, where $\pi_1^L(A_i)$ acts on \tilde{A}_i . Then (4.2.1) implies that the image of the action of $G(n)$ by conjugation in $\text{Out}(G_i(n)/G_{i+1}(n))$ is C_i -finite, where C_i depends only on c_i , T , and δ_i . \square

4.6. Remark on nonfree actions. Theorem 4.1.1 can be reformulated as follows:

There exists a constant $\epsilon(m) > 0$ such that if N^m is a Riemannian manifold which admits a free discrete isometric action by a group Γ such that $\text{sec}(N) > -\epsilon(m)$ and $\text{diam}(N/\Gamma) < 1$, then Γ is $C(m)$ -nilpotent.

As was pointed out to us by B. Wilking, in the above reformulation of Theorem 4.1.1 one can actually easily remove the assumption that the Γ action be free.

Corollary 4.6.1. *There exists a constant $\epsilon(m) > 0$ such that if N^m is a Riemannian manifold which admits a discrete isometric action by a group Γ such that $\text{sec}(N) > -\epsilon(m)$ and $\text{diam}(N/\Gamma) < 1$, then Γ is $C(m)$ -nilpotent.*

Proof. Let $\epsilon = \epsilon(m)$ be as provided by Theorem 4.1.1 and suppose N satisfies the assumptions of the corollary for this ϵ . Let F be the frame bundle of N . Then the action of Γ on N lifts to a free isometric action on F . As was observed in [FY92], using Cheeger's rescaling trick, F can be equipped with a Γ invariant metric satisfying $\text{sec}(F) > -\epsilon(m)$ and $\text{diam}(F/\Gamma) < 1$. Since the induced action of Γ on F is free, the claim of the corollary now follows from Theorem 4.1.1. \square

5. PROOF OF THE FIBRATION THEOREM

5.1. Let M be an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$ a sequence of Riemannian metrics on M such that $\text{sec}(M_n) \geq -1/n$ and $\text{diam}(M_n) \leq 1/n$.

Let us denote by \tilde{M} the universal cover of M and by $\tilde{M}_n \rightarrow M_n$ the universal Riemannian covering of M_n (i.e., \tilde{M} equipped with the pull back of the metric g_n on M).

By [FY92], passing to a finite cover we may assume that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Hence, to prove the topological part of Theorem C, it is enough to show the following:

Theorem 5.1.1. *Let M be a closed almost nonnegatively curved m -manifold such that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Then M is the total space of a fiber bundle*

$$F \rightarrow M \rightarrow N$$

where the base N is a nilmanifold and the fiber F is simply connected.

The assumption on Γ implies that we can fix a series

$$\Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \Gamma_2 \triangleright \dots \triangleright \Gamma_\ell = \{1\}$$

such that Γ_i is normal in Γ and $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$.

Let us first us give an informal proof.

5.2. An informal proof of Theorem 5.1.1. We use induction to construct the bundles $F_i \rightarrow M \xrightarrow{f_i} N_i$, where each N_i is a nilmanifold with $\pi_1(N_i) = \Gamma/\Gamma_i$ and $\pi_1(F_i) \cong \Gamma_i$. Since the base of induction is trivial, we are only interested in the induction step.

Fix $p \in N_i$, and let $F_i(p)$ be the fiber over p . For any sufficiently large n choose a subgroup $G_i = G_i(n)$ such that $\Gamma_i \triangleleft G_i \triangleleft \Gamma_{i+1}$ and $[\Gamma_i : G_i]$ is finite, but large enough so that the cover $\bar{F}_i(p)$ of $F_i(p)$ corresponding to G_i is Hausdorff close to a unit circle S^1 .

Construct now a bundle map $\varphi_p: \bar{F}_i(p) \rightarrow S^1$ by lifting distance functions from S^1 (This can be done by a slight generalization of a construction in [FY92] and [BGP92]). Let $\omega_p = d\varphi_p$.

Then ω_p is a closed integral non-degenerate one-form on $F_i(p)$. Since deck transformations are isometries, after averaging by \mathbb{Z}_a , where $a = [\Gamma_i : G_i]$, we can assume that ω_p is \mathbb{Z}_a -invariant. Thus ω_p descends to a form on $F_i(p)$ which when integrated gives a bundle map $F_i(p)$ onto a small S^1 .

Note that although this bundle is defined only up to rotations of S^1 , its fibers are well-defined.

Since Γ_{i+1} is normal in Γ , the choice of the covering $\bar{F}_i(p)$ of $F_i(p)$ is unambiguous for all $p \in N_i$. By using a partition of unity on N_i we can glue the forms ω_p into a global 1-form on M which satisfies the following properties:

- a) $\omega|_{F(p)}$ is closed and integral for any p ;
- b) $\omega|_{F(p)}$ is non-degenerate.

Integrating ω over the various $F(p)$'s we construct a continuous family of bundles $F_p \rightarrow S^1$. The level sets partition each $F(p)$ and hence the whole M into fibers of a fiber bundle, whose quotient space is then a circle bundle N_{i+1} over N_i . \square

This gives a good idea of the proof. However, to make it precise some extra work has to be done. In particular, one has to be careful with the construction of ω . To make this construction possible we have to keep track of how $F(p)$ was obtained. Namely, we have to use that the fiber $F(p)$ was obtained by a construction as in Yamaguchi's fibration theorem (see [Yam91] or [BGP92]). This makes the induction proof quite technical.

We now proceed with the rigorous proof of Theorem 5.1.1.

5.3. Proof of Theorem 5.1.1. Let us denote by $\tilde{M}_{n,i}$ the Riemannian covering of M_n with respect to Γ_i .

For any choice of marked points p_n we have that

$$(\tilde{M}_{n,i}, p_n, \pi_1(M)) \xrightarrow{\text{GH}} (\mathbb{R}^i, 0, \mathbb{R}^i)$$

in equivariant Gromov–Hausdorff convergence, where \mathbb{R}^i acts on itself by translations. Indeed, the limit space must be a nonnegatively curved simply connected Alexandrov space, and since $\text{diam}M_n \rightarrow 0$ we have that it possesses a transitive group action by a nilpotent group. Then Euclidean space, acting as a group of translations, is here the only choice, and it is easy to see that the dimension of the limit must be equal to i .

Therefore $(\tilde{M}_n, p_n, \pi_1(M)) \xrightarrow{\text{GH}} (\mathbb{R}^\ell, 0, \mathbb{R}^\ell)$, and we may also assume that for each i we have that $(\tilde{M}_n, p_n, \Gamma_i) \xrightarrow{\text{GH}} (\mathbb{R}^\ell, 0, \mathbb{R}^{\ell-i})$, where $\mathbb{R}^{\ell-i}$ is the coordinate subspace of \mathbb{R}^ℓ which corresponds to the first $\ell - i$ elements of the standard basis.

Now, let us give a technical definition:

If R is a Riemannian manifold, let us denote by $\widetilde{\text{dist}}_p$ the average of a distance function over a small ball around p . This enables us to work with the C^1 function $\widetilde{\text{dist}}_p$ instead of the Lipschitz function dist_p .

Definition 5.3.1. *Let $R_n \xrightarrow{\text{GH}} R$ be a sequence of Riemannian m -manifolds with curvature $\geq \kappa$ which Gromov–Hausdorff converges to a Riemannian m' -manifold R , where $m' \leq m$. A sequence of forms ω_n on R_n is said to ε -approximate a form ω on R , if*

- (i) *for any point $p \in R$ there is a neighborhood $U \ni p$ which admits a distance chart $f : U \rightarrow \mathbb{R}^{m'}$,*

$$f(x) = (\text{dist}_{a_1}(x), \text{dist}_{a_2}(x), \dots, \text{dist}_{a_{m'}}(x))$$

which is a smooth regular map, and

- (ii) *smooth lifts of f to $U_n \subset R_n$ give, for n large enough, regular maps*

$$f_n(x) = (\widetilde{\text{dist}}_{a_{1,n}}(x), \widetilde{\text{dist}}_{a_{2,n}}(x), \dots, \widetilde{\text{dist}}_{a_{m',n}}(x))$$

with $a_{i,n} \in M_n$, $a_{i,n} \rightarrow a_n$ such that

$$|(f_n \circ f^{-1})^*(\omega) - \omega_n|_{C^0} < \varepsilon$$

for all sufficiently large n .

Theorem 5.1.1 now easily follows from the following lemma:

Lemma 5.3.2. *Given $\varepsilon > 0$ there is a sequence of one-forms*

$$\{\omega_{1,n}, \omega_{2,n}, \dots, \omega_{k,n}\}$$

on \tilde{M}_n with the following properties:

- (i) *For each i , $\omega_{i,n}$ is a $\pi_1(M)$ -invariant form on \tilde{M}_n .*
(ii) *The forms $\omega_{i,n}$ ε -approximate the coordinate forms dx_i on \mathbb{R}^k . In particular, the forms $\{\omega_{i,n}\}$ are nowhere zero and almost orthonormal at each point.*
(iii) *If for any $j < i$ it holds that $\widetilde{\omega}_{j,n}(X) = \widetilde{\omega}_{j,n}(Y) = 0$, then $d\omega_{i,n}(X, Y) = 0$. In particular, for each i and all sufficiently large n , the distribution corresponding to the system of equations*

$$\omega_{j,n}(X) = 0 \quad \text{for all } j \leq i$$

defines on \tilde{M}_n a foliation $\mathcal{F}_{i,n}$.

- (iv) If $\tilde{F}_{i,n}(x) \subset \tilde{M}_n$ denotes the fiber of the foliation $\mathcal{F}_{i,n}$ through the point $x \in \tilde{M}_n$, then each $\tilde{F}_{i,n}(x)$ is Γ_i -invariant, i.e., for any $\gamma \in \Gamma_i$ one has that $\tilde{F}_{i,n}(x) = \tilde{F}_{i,n}(\gamma x)$. Moreover, the action of Γ_i on $\tilde{F}_{i,n}(x)$ is cocompact for each i . In particular, $\mathcal{F}_{i,n}$ induces on M_n the structure of a fiber bundle.

Proof. We will construct these forms by induction. Assume that we have already constructed one-forms $\omega_1, \omega_2, \dots, \omega_{i-1}$ which meet all the required properties. They give a $\pi_1(M)$ -invariant fibration of \tilde{M}_n by submanifolds $\tilde{F}_{i-1,n}(x)$ through each point $x \in \tilde{M}_n$, with tangent spaces defined by the equations $\omega_j(X) = 0$ for $j = 1, \dots, i-1$.

Denote by $\theta : \mathbb{R} \rightarrow [0, 1]$ a smooth monotone function which is equal to 1 before 0 and 0 after 1. Choose numbers $\delta_n > 0$ slowly converging to 0, and let $\Theta_{i,n} : \tilde{M}_n \rightarrow \mathbb{R}_+$ be the function defined by

$$\Theta_{i,n}(x) = \min_{y \in F_{i-1,n}(x)} \theta(|p_n y| / \delta_n).$$

Clearly $\Theta_{i,n}$ is a continuous Γ_{i-1} -invariant function which is constant on each $F_{i-1,n}(x)$. Moreover, for large n , $\Theta_{i,n}$ has support in some $C_i \delta_n$ -neighborhood of $F_{i-1,n}(p_n)$, and is equal to 1 in some $c_i \delta_n$ -neighborhood of $F_{i-1,n}(p_n)$.

Now let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be a smooth nondecreasing function which is 0 before $1/2$ and 1 after $3/2$. Consider the form

$$\omega'_{i,n} = \Theta_{i,n} \cdot d(\varphi \circ \widetilde{\text{dist}}_{\Gamma_i a_{i,n}}),$$

where $a_{i,n} \in \tilde{M}_n$ is a sequence of points converging to $-e_i \in \mathbb{R}^\ell$, and $\widetilde{\text{dist}}_{\Gamma_i a_{i,n}}$ is the average of $\text{dist}_{\Gamma_i x}$ for x in a small ball around $a_{i,n}$. The support of $\omega'_{i,n}$ has two components, one which contains p_n (notice here that $p_n \rightarrow 0 \in \mathbb{R}^\ell$), and another which does not. (It follows from the construction that the limit of $F_{i-1,n}(p_n)$ is a coordinate plane in \mathbb{R}^ℓ).

Set $\omega''_{i,n} := \omega'_{i,n}$ on the component of p_n , and let this form be 0 otherwise. Clearly, $\omega''_{i,n}$ is then a continuous Γ_i -invariant form whose restriction to $\tilde{F}_{i-1,n}(x)$ is exact. Moreover, each level set of its integral over $\tilde{F}_{i-1,n}(x)$ is Γ_i -invariant.

By construction, the form $\omega''/|\omega''|$ is now (in the sense of definition 5.3.1) close to dx_i at the points where $|\omega''| \neq 0$. Take

$$\omega_{i,n} = c \sum_{\gamma \in \Gamma/\Gamma_i} \gamma \omega',$$

where the coefficient c is chosen in such a way that $|\omega_{i,n}(p_n)| = 1$. As δ_n is a sequence slowly converging to zero, we may assume that $\text{diam}(M_n)/\delta_n \rightarrow 0$. Therefore, $\omega_{i,n}$ is the form we need. \square

Notice that the proof actually shows that the fibers in Theorem 5.1.1 are almost nonnegatively curved manifolds in the generalized sense with $k = \ell$. Therefore, the proof of Theorem C is complete. \square

6. OPEN QUESTIONS

We would like to conclude this work by posing a number of related open questions.

6.1. Is the torsion contained in the center? As was noted earlier, Theorem B is new even for manifolds of nonnegative curvature. For such manifolds it is known that their fundamental groups are almost abelian, and Fukaya and Yamaguchi conjectured the following (see [FY92]):

Conjecture 6.1.1 (Fukaya–Yamaguchi). *The fundamental group of a nonnegatively curved m -manifold is $C(m)$ -abelian.*

In this regard we would like to pose the following two conjectures:

Main Conjecture 6.1.2. *There is $C = C(m)$ such that if M^m is almost nonnegatively curved then there is a nilpotent subgroup $N \subset \pi_1(M)$ of index $\leq C$ whose torsion is contained in its center (or, at least, whose torsion is commutative).*

Conjecture 6.1.3. *If M^m is almost nonnegatively curved, then the action of $\pi_1(M)$ on $\pi_2(M)$ is almost trivial (or maybe even $C(m)$ -trivial), i.e., there exists a finite index subgroup of $\pi_1(M)$ (or, respectively, a subgroup of index $\leq C(m)$) which acts trivially on $\pi_2(M)$.*

Conjecture 6.1.2 implies in particular that the fundamental groups of closed positively curved m -manifolds are $C(m)$ -abelian.

In fact, as was pointed out to us by B. Wilking, if true, Conjecture 6.1.2 would also imply a positive answer to Conjecture 6.1.1. Indeed, if $\sec(M) \geq 0$, then the universal cover \tilde{M} of M is isometric to the product $\mathbb{R}^n \times K$, where K is a compact Riemannian manifold and the $\pi_1(M)$ action on $\mathbb{R}^n \times K$ is diagonal. It follows from [Wil00, Cor. 6.3] that one can deform the metric on M so that its universal cover is still isometric to $\mathbb{R}^n \times K$ and the induced action on K is finite. By passing, as in the proof of Corollary 4.6.1, to the induced action on the frame bundle of K , one reduces the statement to Conjecture 6.1.2.

We tried to prove these conjectures by studying successive blow-ups of the collapsing sequence M_n as done in Section 4.3.

We can prove Conjectures 6.1.2 and 6.1.3 in the case where all spaces A_i which appear in the construction in Section 4.3 are closed Riemannian manifolds. Moreover, we believe we have an argument to prove it if all A_i 's are Alexandrov spaces without boundary.

It seems that if we would have just a slightly better understanding of collapsing to a ray, then we could prove the conjectures. Here is the simplest related question which we cannot solve:

Question 6.1.4. Let $M_n = (S^2 \times \mathbb{R}^2, g_n)$ be a sequence of complete Riemannian manifolds with $\sec(M_n) \geq -\varepsilon_n$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Assume that for a sequence of points $p_n \in M_n$ we have that $(M_n, p_n) \xrightarrow{\text{GH}} (\mathbb{R}_+, 0)$. Let $q_n \in M_n$ be a sequence of points such that $|p_n q_n| = 1$ and such that there is a sequence of rescalings $\lambda_n \rightarrow \infty$ such that $(\lambda_n M_n, q_n) \xrightarrow{\text{GH}} S^2 \times S^1 \times \mathbb{R}$, where the latter space is equipped with the product of the canonical metrics.

- (i) Can it happen that $(\lambda_n M_n, p_n) \xrightarrow{\text{GH}} (\mathbb{R}_+, 0)$?
- (ii) Is it true that the Gromov–Hausdorff limit of $(\lambda_n M_n, p_n)$ is at least 3-dimensional?
- (iii) What are the possible limits of $(\lambda_n M_n, p_n)$?

In the case where all A_i are Riemannian manifolds, crucial to the proof is the following topological statement:

Theorem 6.1.5. Given manifolds F_1, F_2, \dots, F_n such that each F_i is either S^1 or is simply connected, if E is the total space of a tower of fiber bundles

$$E = E_n \xrightarrow{F_n} E_{n-1} \xrightarrow{F_{n-1}} \dots \xrightarrow{F_1} E_0 = \{pt\}$$

and each of the bundles $E_k \xrightarrow{F_k} E_{k-1}$ are homotopically trivial over the 1-skeleton, then $\pi_1(E)$ contains a nilpotent subgroup N such that

$$[\pi_1(E) : N] \leq C(F_1, F_2, \dots, F_n) \text{ and } \text{Tor}(N) \subset Z(N).$$

The proof of this theorem that we currently have is surprisingly nontrivial¹, and finding an easier proof will probably help to prove our conjecture in full generality.

Conjectures 6.1.1 and 6.1.2 are also related to the following conjecture of Rong (cf. [Ron96b, Ron96a]):

Conjecture 6.1.6 (Rong). *Positively curved m -manifolds have $C(m)$ -cyclic fundamental groups.*

This conjecture has been proved by Rong [Ron96b] under the additional assumption of a uniform upper curvature bound. We also believe that if one could carry out the above program for proving Conjecture 6.1.2, one would have a good shot at handling Rong’s Conjecture as well.

6.2. The simply connected case. So far we have only discussed manifolds with nontrivial fundamental groups. However, some of our arguments also work in a more general setting. We hope that it might be possible to use them to obtain new restrictions on simply connected almost nonnegatively curved manifolds as well as on collapsing with a lower curvature bound.

Let us indicate one possible approach to do so.

Let us denote by $\mathcal{M}(F)$ the space of self homotopy equivalences of a manifold F . Assume now that F is simply connected and that $\tilde{f} : S^k \times$

¹ A sketch of the proof of Theorem 6.1.5 is presently available under <http://www.math.psu.edu/petrunin/papers/alnpos/TinZ.pdf>

$F \rightarrow F$ is a map such that $\tilde{f}_u : F \rightarrow F$ is homotopic to the identity for some (and therefore ANY) $u \in S^k$. Then \tilde{f} represents an element $\alpha = [\tilde{f}] \in \pi_k(\mathcal{M}(F))$. Let g be a Riemannian metric on F . Define

$$\text{dil}_g(\tilde{f}) = \max_{u \in S^k} \text{dil}_g(\tilde{f}_u),$$

where $\text{dil}_g(\tilde{f}_u)$ stands for the optimal Lipschitz constant of \tilde{f}_u with respect to g . For any $\alpha \in \pi_k(\mathcal{M}(F))$ define

$$\text{dil}_g(\alpha) = \inf_{[h]=\alpha} \text{dil}_g(h).$$

Finally, define

$$\text{DIL}(\alpha) = \inf_g \text{dil}_g(\alpha)$$

over all Riemannian metrics on F and

$$\text{DIL}_+(\alpha) = \inf_g \text{dil}_g(\alpha)$$

over all Riemannian metrics on F with $\text{diam} \leq 1$ and $\text{sec} \geq -1$.

Clearly, both $\text{DIL}(\alpha)$ and $\text{DIL}_+(\alpha)$ are homotopy invariants of α .

Now suppose that $M_n \xrightarrow{\text{GH}} S^{k+1}$ is a sequence of Riemannian manifolds collapsing to a round sphere with $\text{sec}(M_n) \geq k$. By Yamaguchi's fibration theorem, we have that M_n is a fiber bundle over S^{k+1} with almost nonnegatively curved fiber F_n . This bundle is classified by an element α of $\pi_k(\text{Aut}(F_n))$ and by using our gradient flow technique we can estimate $\text{DIL}_+(\alpha)$ (and hence $\text{DIL}(\alpha)$) from above.

Therefore, if one could find examples of a simply connected F and an α with arbitrary big $\text{DIL}_+(\alpha)$, one would obtain new restrictions on collapsing to a sphere with curvature bounded from below, and probably more restrictions for the topological type of manifolds with lower curvature and upper diameter bounds in general. In fact, F need not be simply connected as long as the total space of the bundle $F \rightarrow M \rightarrow S^{k+1}$ is.

While we believe that finding examples with arbitrary large $\text{DIL}(\alpha)$ is very difficult (and might even be impossible), we have several candidates to produce large $\text{DIL}_+(\alpha)$.

On the other hand, the problem of finding examples of α with $\text{DIL}(\alpha) > 1$ seems quite interesting in its own right and might have other applications unrelated to collapsing.

Let us next describe some possible sources of examples with $\text{DIL}_+(\alpha) > 1$:

Example 6.2.1. Obviously, if $\text{dil}_g(h) = 1$, then h_u is a homotopy of isometries of (F, g) . Let G be the isometry group of F . Then G can be viewed as a subset of $\mathcal{M}(F)$. Therefore, if $[h] \neq 0$ in $\pi_k(\mathcal{M}(F))$, then $[h_u] \neq 0$ in $\pi_k(G)$. Now G is a compact Lie group, in particular, $\pi_2(G) = 0$ (and even more generally $\pi_{2n}(G)$ is finite). On the other hand, there are spaces F such that the space $\mathcal{M}(F)$ might have nontrivial second homotopy; for example, one can take $F = SU(6)/SU(3) \times SU(3)$, for which the canonical

metric on F has nonnegative curvature, and it follows from [OT97, Chapter 5, Example 4.14], that $\pi_2(M(F)) \otimes \mathbb{Q}$ is nontrivial. Therefore, there is an $\alpha \in \pi_2(M(F))$ such that $\text{dil}_g(\alpha) > 1$ for any metric g on F ; we believe it should be true that $\text{DIL}_+(\alpha) > 1$. Still, it might happen that $\text{DIL}(\alpha) = 1$.

Another possible source of such manifolds is provided by the following example due to D. Sullivan.

Example 6.2.2. Let N^7 be the total space of an S^3 bundle over S^4 with zero Euler class and nontrivial p_1 . Clearly N^7 is rationally equivalent to $S^4 \times S^3$. In particular, its minimal model has no nontrivial derivations of degree -1 . Therefore, by [Sul77, 13.3], there exists a diffeomorphism $f: N \rightarrow N$ which is homotopic to the identity but such that the obstruction to it being diffeotopic to the identity is a nonzero element of $H^3(N, \mathbb{Z}) \cong \mathbb{Z}$. Let M^8 be the mapping cylinder of f . Clearly M is homotopy equivalent to $N \times S^1$ and hence it is spin with signature zero. On the other hand, by construction, $p_1^2(M) \neq 0$. Since the signature of M is zero we must necessarily have that $p_2(M) \neq 0$ and hence $\hat{A}(M) \neq 0$. In particular, by the Atiyah–Hirzebruch theorem, M does not admit an S^1 action and hence the corresponding element $\alpha \in \pi_1(\mathcal{M}(M))$ has $\text{dil}_g(\alpha) > 1$ for any metric g on M .

Remark 6.2.3. As was mentioned in the introduction, it is known that a spin manifold X of almost nonnegative Ricci curvature has $\hat{A}(X) \leq 2^{\dim(X)/2}$ ([Gro82, page 41], [Gal83]). Clearly, a finite cover of the manifold M constructed above violates this restriction and therefore M does not admit almost nonnegative Ricci curvature. However, it could possibly be almost nonnegatively curved in the generalized sense.

6.3. Further questions. Recall that a simply connected space C is called *rationally elliptic* if it is homotopy equivalent to a finite CW-complex and

$$\dim \pi_*(C, \mathbb{Q}) < \infty.$$

A conjecture of Grove–Halperin [GH82] says that simply connected nonnegatively curved manifolds are rationally elliptic. This conjecture was extended by Grove to include almost nonnegatively curved manifolds [Gro02]. Later, Totaro has proposed the following definition of rationally elliptic spaces which covers manifolds with infinite fundamental groups:

A connected topological space X is *rationally elliptic* if it is homotopy equivalent to a finite CW complex, it has a finite covering which is a nilpotent space and its universal cover is rationally elliptic in the ordinary sense.

With this definition one can extend Grove’s conjecture to non simply connected manifolds as follows:

Conjecture 6.3.1. *Any almost nonnegatively curved manifold in the generalized sense is rationally elliptic.*

Theorem A reduces this conjecture to the simply connected case which is undoubtedly the most difficult part of the problem.

It has been shown in [PP06] that if M is a nilpotent closed manifold which admits a Riemannian metric with zero topological entropy, then its universal cover \tilde{M} is rationally elliptic. Coupled with Theorem A this means that to prove Conjecture 6.3.1 it would be sufficient to show that a manifold with almost nonnegative curvature in the generalized sense admits a metric with zero topological entropy. However, we think that this might be wrong in general.

As was pointed out in the discussion in the Introduction before Theorem C, it already follows from Yamaguchi's fibration theorem and [FY92] that a finite cover of an almost nonnegatively curved manifold maps onto a nilmanifold with homotopy fiber a simply connected closed manifold. While this is formally weaker than the statement of Theorem C, it would be interesting to have an answer to the following, purely topological, question:

Question 6.3.2. *Let $M \xrightarrow{f} N$ be a map from a closed manifold M to a nilmanifold N such that the homotopy fiber of f is a simply connected closed manifold. Is it true that after passing to a finite cover, the map f becomes homotopic to a fiber bundle projection?*

Question 6.3.3. *Is it true that manifolds which are almost nonnegatively curved in the generalized sense are almost nonnegatively curved?*

In view of Theorems A and B it is also reasonable to pose the following question:

Question 6.3.4. *Is it true that almost nonnegatively curved m -manifolds M^m are $C(m)$ -nilpotent spaces?*

It is clear from the proof of Theorems A and B that this is true if the universal cover of M^m has torsion free integral cohomology.

In view of Theorem B it is moreover natural to raise the following question:

Question 6.3.5. *Can one give an explicit bound on $C(m)$ in Theorem B?*

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