

# PHASE TRANSITIONS FOR UNIFORMLY EXPANDING MAPS

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ABSTRACT. Given a uniformly expanding map of two intervals we describe a large class of potentials admitting unique equilibrium measures. This class includes all Hölder continuous potentials but goes far beyond them. We also construct a family of continuous but not Hölder continuous potentials for which we observe phase transitions. This provides a version of the example in [9] for uniformly expanding maps.

## 1. INTRODUCTION

We consider a uniformly piecewise expanding map  $f$  of the interval  $I = [0, 1]$ , i.e., we assume that  $I = I^{(1)} \cup I^{(2)} \cup \dots \cup I^{(p)}$  and  $\text{int}I^{(k)} \cap \text{int}I^{(j)} = \emptyset$  for  $1 \leq k, j \leq p$ ,  $j \neq k$  and that  $f : I^{(j)} \rightarrow I$ ,  $j = 1, \dots, p$  is a  $C^1$  diffeomorphism with  $|f'| > 1$ . Given a potential function  $\varphi : I \rightarrow \mathbb{R}$ , a Borel invariant measure  $\mu_\varphi$  is called an *equilibrium measure* if

$$h_{\mu_\varphi}(f) + \int_I \varphi d\mu_\varphi = \sup\{h_\mu(f) + \int_I \varphi d\mu\},$$

where the supremum is taken over all Borel invariant probability measures. The classical thermodynamical formalism states that if  $\varphi$  is (piecewise) Hölder continuous, then  $f$  admits a unique equilibrium measure. The proof is based on representing the system as a Bernoulli shift on  $p$  symbol alphabet.

The goal of this paper is to show that the class of potentials, admitting the unique equilibrium measure, is substantially larger than the class of Hölder continuous potentials. We restrict ourselves to the case of uniformly piecewise expanding maps of two intervals although some results can easily be extended to any finite number of intervals. Our approach is based on constructing a countable Markov partition for  $f$  and representing it as the “renewal shift” – a subshift of a special

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type on a countable set of states. We describe a broad class of potentials which admit unique equilibrium measures and show that this class includes some continuous but not Hölder continuous potentials. Moreover, building on results of Sarig [9], we construct a family of continuous (but not Hölder continuous) potentials  $\varphi_c$  exhibiting phase transitions: there exists a critical value  $c_0 > 0$  such that for every  $0 < c < c_0$  there is a unique equilibrium measure for  $\varphi_c$  which is supported on  $(0, 1]$  and for  $c < c_0$  the equilibrium measure is the Dirac measure at 0. This phase transition phenomenon can be characterized in terms of the behavior of the pressure function (see Section 4).

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## 2. THERMODYNAMICS OF COUNTABLE MARKOV SUBSHIFTS

We briefly describe some results on the thermodynamics of countable Markov subshifts (see [1], [5], [8], [10], [2], [12], [13]). Let  $S = \{1, 2, \dots\}$  be an infinite alphabet and  $\Sigma_A$  the *Markov shift* with the set of states  $S$  and a transition matrix  $A = (a_{ij}), i, j \in \mathbb{N}$ . This means that  $\Sigma_A$  is the set of all one-sided infinite sequences  $\omega \in S^{\mathbb{N}}$  which are admissible by  $A$ , i.e.,  $a_{\omega_i \omega_{i+1}} = 1$  for all  $i \in \mathbb{Z}$ . Assume that  $(\Sigma_A, \sigma)$  is topologically mixing. We write  $[a_1, \dots, a_n]$  for the cylinder set  $\{(x_k) : x_i = a_i, 1 \leq i \leq n\}$ .

Given a continuous function  $\phi : \Sigma_A \rightarrow \mathbb{R}$ , called *potential*, we define the  $n$ -*variation* of  $\phi$  to be

$$V_n(\phi) = \sup_{x, y \in [a_1, \dots, a_n]} |\phi(x) - \phi(y)|.$$

We say that the potential  $\phi$  has *summable variations* if

$$(1) \quad \sum_{n \geq 2} V_n(\phi) < +\infty.$$

Furthermore, we say that  $\phi$  is *weakly Hölder continuous* if there exists  $A > 0$  and  $0 < \gamma < 1$  such that for all  $n \geq 1$ ,

$$(2) \quad V_n(\phi) \leq A\gamma^n.$$

If  $\phi$  is weakly Hölder continuous then it has summable variations.

Set

$$Z_n(\phi, a) = \sum_{\sigma^n(x)=x} \exp \phi_n(x) 1_{[a]}(x),$$

where  $\phi_n(x) = \sum_{i=0}^{n-1} \phi(\sigma^i(x))$ . Define the *Gurevich pressure* by

$$P_G(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi, a).$$

If the potential  $\phi$  has summable variations then the limit exists and does not depend on the choice of the symbol  $a$ .

A Borel  $\sigma$ -invariant probability measure  $\mu_\phi$  on  $\Sigma_A$  is called an *equilibrium measure* for  $\phi$  if

$$h_{\mu_\phi}(\sigma) + \int_{\Sigma_A} \phi d\mu_\phi = \sup\{h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu\},$$

where the supremum is taken over all Borel  $\sigma$ -invariant probability measures on  $\Sigma_A$  for which  $\int_{\Sigma_A} \phi d\mu_\phi > -\infty$ .

We consider the induced map on the cylinder  $[a]$ . For  $x \in [a]$ , define

$$r_a(x) = \inf\{n \geq 1; \sigma^n(x) \in [a]\}.$$

It is the first return time of  $x$  to  $[a]$ . We have

$$[a] = \bigcup_{n \geq 0, \xi_i \neq a} [a, \underline{\xi}, a] \cup N,$$

where  $\underline{\xi} = \xi_1, \dots, \xi_{n-1}$ ,  $r_a(x) = n$  for  $x \in [a, \underline{\xi}, a]$ , and  $N$  is a null set for any  $\sigma$ -invariant probability measure. Let  $\bar{\sigma} = \sigma^{r_a(x)} : [a] \rightarrow [a]$  be the *induced map*. The map  $\bar{\sigma}|_{[a]}$  is topologically conjugate to a countable Bernoulli shift with symbols corresponding to  $[a, \underline{\xi}, a]$ .

Given a potential  $\phi$ , we define the *induced potential* by  $\bar{\phi}(x) = \phi_{r_a(x)}(x)$ . In order to describe thermodynamics of  $\sigma$  with respect to the potential  $\phi$  we introduce the following crucial property.

Define the *discriminant* of  $\phi$  by

$$\Delta[\phi] = \sup_{p \in \mathbb{R}} \{P_G(\overline{\phi + p}); P_G(\overline{\phi + p}) < +\infty\}.$$

**Proposition 2.1** (see [9]).  $\Delta[\phi] \geq 0$  if and only if there exists  $p$  such that  $P_G(\overline{\phi + p}) = 0$ .

Set

$$Z_n^*(\phi, a) = \sum_{\sigma^n(x)=x, r_a(x)=n} e^{\phi_n(x)} 1_{[a]}(x).$$

**Proposition 2.2** (see [9]). *We have*

$$\left| \Delta[\phi] - \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, a) \Big|_{\xi=rad.} \right| \leq \sum_{n \geq 2} V_n(\phi),$$

where  $\xi = rad.$  is the radius of convergence for the power series

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a).$$

Furthermore, if we define  $p^*(\phi) = \sup\{p : P_G(\overline{\phi + p}) < \infty\}$ , then  $\log \xi = -p^*(\phi)$ .

Set  $\lambda = \exp P_G(\phi)$ . Following [9], we call the potential  $\phi$  *recurrent* if the sum

$$\sum_{n \geq 1} \lambda^{-n} Z_n(\phi, a)$$

diverges. Otherwise we call  $\phi$  *transient*. We call  $\phi$  *positive recurrent* if it is recurrent and

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, a) < +\infty$$

and *null recurrent* if

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, a) = \infty.$$

By [9], we have that  $\phi$  is recurrent if and only if  $\Delta[\phi] \geq 0$ .  $\phi$  is positive recurrent when  $\Delta[\phi] > 0$  and is either positive recurrent or null recurrent if  $\Delta[\phi] = 0$ .

**Proposition 2.3** (see [11], [9]). *Assume that the potential  $\phi$  has summable variations (see (1)),  $P_G(\phi) < +\infty$  and  $\sup \phi < \infty$ . Then the variational principle holds:*

$$P_G(\phi) = \sup \left\{ h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu : \mu \circ \sigma^{-1} = \mu, \int_{\Sigma_A} \phi d\mu > -\infty \right\}.$$

If  $\phi$  is positive recurrent then  $P_G(\phi) = -p(\phi)$ , where  $p(\phi)$  is the solution of the equation  $P_G(\overline{\phi + p}) = 0$ ; if  $\phi$  is transient then  $P_G(\phi) = -p^*(\phi)$ , where

$$p^*(\phi) = \sup\{p : P_G(\overline{\phi + p}) < +\infty\}.$$

The following proposition illustrates that if  $\phi$  is transient, there is no equilibrium measure for  $\phi$ .

**Proposition 2.4** (see [2]). *Assume that  $\phi$  has summable variations, finite Gurevich pressure and  $\sup \phi < \infty$ . If  $\phi$  admits an equilibrium measure then it is positive recurrent.*

In general, positive recurrence does not imply existence of equilibrium measure. We therefore, consider a particular case of the *renewal shift*  $(\Sigma_A, \sigma)$  where the entries of the transition matrix  $A$  are 1 at  $(0, n)$  and  $(n, n - 1)$ ;  $n \geq 1$  and 0 otherwise.

To study existence and uniqueness of equilibrium measures for the renewal shift one can use results on thermodynamical formalism for general subshifts of countable type (see for example, [1, 2, 5, 8, 13, 12])

and reexamine the conditions in this particular case. We however, exploit another approach which is based on inducing schemes, see [7]. Consider the inducing of the renewal shift on the symbol 0. The induced partition is formed by the cylinders  $J_n = [0, n-1, \dots, 1, 0]$  for  $n \geq 1$ . The collection of intervals  $\{J_n\}_{n \geq 1}$  and the inducing time  $\tau(J_n) = r_{[0]}(x) = n$ ,  $x \in J_n$  form an *inducing scheme* (see [6, 7]) with the inducing domain  $[0]$  and the induced map  $\bar{\sigma}(x) = \sigma(x)^n$ ,  $x \in J_n$ . Following [6, 7], we write

$$W = \bigcap_{n \geq 1} \bar{\sigma}^{-n} \left( \bigcup_{n \geq 1} J_n \right), \quad X = \bigcup_{n \geq 1} \bigcup_{k=0}^{n-1} \sigma^k(W \cap J_n).$$

Given a continuous function  $\phi : X \rightarrow \mathbb{R}$ , define the *induced function*  $\bar{\phi} : \bigcup_{n \geq 1} J_n \rightarrow \mathbb{R}$  by

$$\bar{\phi}(x) = \sum_{k=0}^{n-1} \phi(\sigma^k(x)) \quad \text{for } x \in J_n.$$

Denote by  $\mathcal{M}(\sigma)$  the set of all  $\sigma$ -invariant Borel ergodic probability measures  $\mu$  on  $X$  and by

$$s_\phi = \sup \left( h_\mu(\sigma) + \int_X \phi d\mu \right),$$

where the supremum is taken over all  $\mu \in \mathcal{M}(\sigma)$  with  $\int_X \phi d\mu > -\infty$ . The following result can be derived from [7].

**Proposition 2.5.** *Assume that the induced potential function  $\bar{\phi}$  has summable variations,  $-\infty < s_\phi < +\infty$ , and  $P_G(\bar{\phi} - s_\phi) = 0$ . Assume also that  $\sup(\overline{\phi - s_\phi}) < \infty$  and that*

$$(3) \quad \sum_{n=1}^{\infty} n \sup_{x \in J_n} \exp(\overline{\phi - s_\phi}) < \infty.$$

*Then there is a unique equilibrium measure  $\mu_\phi \in \mathcal{M}(\sigma)$ , i.e., a measure for which*

$$s_\phi = h_{\mu_\phi}(\sigma) + \int_I \phi d\mu_\phi = \sup \{ h_\mu(\sigma) + \int_I \phi d\mu \}.$$

Combining Propositions 2.3 and 2.5, we establish a condition, which guarantees existence and uniqueness of equilibrium measures for the renewal shift.

**Theorem 2.6.** *Let  $(\Sigma_A, \sigma)$  be the renewal shift. Assume that:*

1.  $\phi$  has summable variations;
2.  $\sup \phi < \infty$ ;

3.  $\phi$  is positive recurrent.

Then there exists a unique equilibrium measure.

*Proof.* We show that the conditions of Proposition 2.5 are satisfied. Since  $\phi$  has summable variations, so does  $\bar{\phi}$ . Due to the special structure of the renewal shift, the sum in

$$Z_n(\phi, 0) = \sum_{\sigma^n(x)=x} e^{\phi_n(x)} 1_{[0]}(x)$$

has at most  $2^n$  terms. Hence,

$$Z_n(\phi, 0) \leq 2^n e^{(\sup \phi)n}.$$

This implies that

$$P_G(\phi) \leq \log 2 + \sup \phi < \infty.$$

By Proposition 2.3,

$$P_G(\phi) = \left\{ h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu : \mu \circ \sigma^{-1} = \mu, \int_{\Sigma_A} \phi d\mu > -\infty \right\}.$$

Since the inducing time in our case is the first return time, we conclude that the right-hand side of the last equation is  $s_\phi$  and hence,  $s_\phi = P_G(\phi) < \infty$ . Taking the Dirac measure over a periodic orbit, we know that there is at least one invariant measure with  $\int \phi d\mu > -\infty$ . Hence,  $s_\phi > -\infty$ .

Using the fact that  $\phi$  is positive recurrent we have

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, 0) < +\infty.$$

Since  $\sigma$  is the renewal shift we obtain that  $Z_n^*(\phi, 0) = \exp(\overline{\phi}(x))$ , where  $x$  is the unique periodic orbit of period  $n$  in the cylinder  $[0, n-1, \dots, 0]$ . For  $\lambda = \exp P_G(\phi) = \exp s_\phi$ , we have that

$$\left| \sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, 0) - \sum_{n=1}^{\infty} n \sup_{x \in J_n} \exp(\overline{\phi} - s_\phi) \right| < \sum_{n=2}^{\infty} V_n(\phi).$$

We know that positive recurrence implies Condition (3). Note that Condition (3) in turns, implies that  $\sup(\overline{\phi} - s_\phi) < \infty$ .

Furthermore, Proposition 2.3 implies that  $P_G(\overline{\phi} - s_\phi) = 0$  and the desired result follows.  $\square$

3. SYMBOLIC REPRESENTATION OF  $f$  VIA THE RENEWAL SHIFT

Recall that  $f$  is a uniformly expanding map of the unit interval  $I = [0, 1]$ . More precisely, we assume that  $I = I^{(1)} \cup I^{(2)}$ ,  $\text{int}I^{(1)} \cap \text{int}I^{(2)} = \emptyset$  and that the map  $f : I^{(j)} \rightarrow I$ ,  $j = 1, 2$  can be extended to a  $C^1$  diffeomorphism of a small neighborhood of  $I^{(j)}$ ; finally, we assume that  $|f'(x)| > \lambda > 1$  for all  $x \in \text{int}I^{(1)} \cup \text{int}I^{(2)}$ . Consider the closed intervals  $I_0 = I^{(2)}$ ,  $I_n = f^{-1}(I_{n-1}) \cap I^{(1)}$ ,  $n \geq 1$ . They cover the semi-open interval  $(0, 1]$  and form a countable Markov partition for  $f$ .

Define the *coding map*  $h : \Sigma_A \rightarrow I \setminus \{0\}$  by

$$h(i_1, \dots, i_n, \dots) = \bigcap_{j=1}^{\infty} f^{-j}(I_{i_j}).$$

Here  $(\Sigma_A, \sigma)$  is the renewal shift.

Denote by  $Q$  the set of all end points of the partition intervals and all their preimages. Observe that the only invariant measure  $\mu$  with  $\mu(Q) = 1$  is the Dirac measure at 0,  $\delta_0$ .

**Proposition 3.1.** *The map  $h$  is onto and is one-to-one and a topological conjugacy between  $(\Sigma_A \setminus h^{-1}(Q), \sigma)$  and  $(I \setminus Q, f)$ .*

Given a potential function  $\varphi$  on  $I$ , we define  $\phi = \varphi \circ h$  to be a potential function on  $\Sigma_A$ . We denote by  $\mathcal{H}$  the class of potential functions  $\varphi$  for which the corresponding functions  $\phi$  satisfy all the conditions of Theorem 2.6, i.e.,  $\phi$  has summable variations (see (1)), is recurrent and  $\sup \phi < \infty$ .

The following result is an immediate corollary of Theorem 2.6 and Proposition 3.1.

**Proposition 3.2.** *If  $\varphi \in \mathcal{H}$  then*

- (1) *there exists a unique equilibrium measure  $\nu_\phi$  for  $\phi$ ;*
- (2) *there exists an equilibrium measure  $\mu_\varphi$  for  $\varphi$  that is either  $\delta_0$  or  $\nu_\phi \circ h^{-1}$ .*

The following example illustrates that for some potentials  $\varphi$  the measure  $\mu_\varphi$  may not be an equilibrium measure for  $\varphi$  while the Dirac measure  $\delta_0$  at 0 may be such a measure.

**Example 3.1.** *Assume that the map  $f$  is  $C^{1+\epsilon}$ , and consider the potential*

$$\varphi(x) = \begin{cases} -t \log |f'(x)|, & x \neq 0 \\ f_0, & x = 0 \end{cases}.$$

*Observe that  $\varphi$  is Hölder continuous on  $(0, 1]$ . By Theorem 3.3 below, if  $f_0 < P_G(\varphi \circ h)$ , the measure  $\mu_\varphi$  is a unique equilibrium measure for  $\varphi$  (it*

is the absolutely continuous invariant measure) and if  $f_0 > P_G(\varphi \circ h)$ , the equilibrium measure is  $\delta_0$ . Furthermore, if  $f_0 = P_G(\varphi \circ h)$  there are two equilibrium measures for  $\varphi$ .

We show that the class of functions  $\mathcal{H}$  is sufficiently large and in particular, includes all the Hölder continuous functions.

**Theorem 3.3.** *If  $\varphi$  is Hölder continuous on  $I$  then  $\varphi \in \mathcal{H}$ .*

*Proof.* Since  $\varphi$  is bounded and continuous, so is  $\phi$ . Moreover, as the map  $h$  is Hölder continuous (while  $h^{-1}$  is not), we have that  $\phi$  is Hölder continuous on  $\Sigma_A$ . This implies that  $\phi$  has summable variations (see (1)) and is bounded from above.

To establish the recurrent condition we consider inducing on the cylinder  $[0]$  as described in Section 2. Note that the only induced cylinder with return time  $n$  is  $[0, n-1, \dots, 1, 0]$ . Fix  $x_i \in [0, i-1, \dots, 1, 0]$  and consider the function  $\psi$  such that  $\psi(x) = \phi(x_i)$  whenever  $x \in [0, i-1, \dots, 1, 0]$ . For simplicity we write  $a_i = \phi(x_i)$ . The function  $\psi$  has the following properties:

- (1)  $V_n(\psi) = 0$  (this is true since  $\psi$  is constant on every cylinder);
- (2) there exist  $a, A > 0$  and  $0 < \gamma < 1$  such that  $|a_i - a| < A\gamma^i$  (this is true since  $h(x_i) \rightarrow 0$  in  $I$  and  $\phi$  is Hölder continuous);
- (3)  $|\phi(x) - \psi(x)| \leq V_n(\phi)$  for any  $x \in [0, n-1, \dots, 1, 0]$ .

We claim that

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0)|_{\xi=rad.} = +\infty.$$

To show this observe that for  $x \in [0, n-1, \dots, 1, 0]$ ,

$$Z_n^*(\psi, 0) = \exp \left\{ \sum_{i=0}^{n-1} \phi(\sigma^i x) \right\} = \exp \left\{ \sum_{i=0}^{n-1} a_i \right\}.$$

As  $n \rightarrow +\infty$ , we have that

$$Z_n^*(\psi, 0) \asymp e^{na}.$$



Hence, for  $\xi = e^{-a}$ ,

$$\begin{aligned} \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) &= \log \sum_{n=1}^{\infty} e^{-an} Z_n^*(\psi, 0) \\ &= \log \sum_{n=1}^{\infty} \exp \left\{ \sum_{i=0}^{n-1} (a_i - a) \right\} \\ &\geq \log \sum_{n=1}^{\infty} \exp \left\{ - \sum_{i=0}^{n-1} A\gamma^n \right\} \\ &\geq \log \sum_{n=1}^{\infty} \exp \left\{ -A \frac{1}{1-\gamma} \right\} = +\infty. \end{aligned}$$

Using Property 3 of the function  $\psi$ , we obtain that

$$|Z_n^*(\phi, 0) - Z_n^*(\psi, 0)| \leq \sum_{k=2}^{n+1} V_n(\phi).$$

Since  $\sum_{n \geq 2} V_n(\phi) < +\infty$ , we conclude that the power series

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a), \quad \sum_{n=1}^{\infty} z^n Z_n^*(\psi, a)$$

have the same radius of convergence. It follows that

$$\begin{aligned} \Delta[\phi] &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, 0) - \sum_{n \geq 2} V_n(\phi) \\ &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) - 2 \sum_{n \geq 2} V_n(\phi) = +\infty. \end{aligned}$$

The desired result follows.  $\square$

We now show that the class of function  $\mathcal{H}$  admits some non-Hölder continuous potentials which thus have equilibrium measures. Consider the one-parameter family of functions

$$(4) \quad \varphi_c(x) = \begin{cases} -c(1 - \log x)^{-\alpha}, & x \in (0, 1] \\ 0, & x = 0, \end{cases}$$

where  $c \in \mathbb{R}$ . Observe that  $\varphi_c$  is continuous on  $[0, 1]$  but is not Hölder continuous at zero.

**Theorem 3.4.** *For any  $\alpha > 1$  the potential  $\phi_c = \varphi_c \circ h \in \mathcal{H}$ , i.e., it satisfies all the conditions of Theorem 2.6.*

*Proof.* Set  $\beta = \min |f'|^{-1}$  and  $\gamma = \max |f'|^{-1}$ . Since  $f$  is uniformly expanding  $\beta \leq \gamma < 1$ . We write  $I_0 = [d_0, 1]$ ,  $I_1 = [d_1, d_0]$ ,  $\dots$ ,  $I_n = [d_n, d_{n-1}]$ , and  $I_{i_1, \dots, i_n} = [i_1, \dots, i_n]$  as long as  $i_1, \dots, i_n$  is admissible. Note that  $f(d_n) = d_{n-1}$  and that

- (1)  $I_n \subset [d_n, 1]$  and  $\frac{\beta}{\gamma} \leq \frac{|[0, d_n]|}{|I_n|} \leq \frac{\gamma}{\beta}$ ;
- (2)  $|I_{i_1, \dots, i_n}| \leq \gamma^n$ ;
- (3)  $d_n \geq \beta^n$ .

We first verify the summable variation condition (see (1)). Choose  $x, y \in I_{i_1, \dots, i_n}$ . Let  $a$  be an integer such that  $\gamma < \beta^{\frac{1}{a}}$ . Set  $i_1 = m$ . If  $m \geq \frac{n}{a}$ , we have that  $I_{i_1, \dots, i_n} \subset I_m \subset [d_m, 1]$  and hence,  $\beta^m \leq x, y \leq \gamma^m$ . It follows that

$$\begin{aligned}
|\varphi_c(x) - \varphi_c(y)| &= |\varphi_c'(\xi)| |x - y| \\
&= |c| \frac{\alpha(1 - \log \xi)^{-\alpha-1}}{|\xi|} |x - y| \\
&\leq |c| \frac{\alpha(1 - \log \xi)^{-\alpha-1}}{|[0, d_m]|} |I_m| \\
&\leq |c| \frac{|I_m|}{|[0, d_m]|} \alpha(1 - m \log \gamma)^{-\alpha-1} \\
&\leq C |c| |m|^{-\alpha-1} \leq C |c| \left(\frac{n}{a}\right)^{-\alpha-1}
\end{aligned}$$

If  $m < \frac{n}{a}$ , we have that

$$\begin{aligned}
|\varphi_c(x) - \varphi_c(y)| &= |\varphi_c'(\xi)| |x - y| \\
&= |c| \frac{\alpha(1 - \log \xi)^{-\alpha-1}}{|\xi|} |x - y| \\
&\leq |c| \frac{\alpha(1 - \log \xi)^{-\alpha-1}}{|[0, d_m]|} |I_{i_1, \dots, i_n}| \\
&\leq |c| \frac{|I_m|}{|[0, d_m]|} \frac{|I_{i_1, \dots, i_n}|}{|I_m|} \alpha(1 - m \log \gamma)^{-\alpha-1} \\
&\leq C' |c| \frac{\gamma^n}{\beta^{\frac{n}{a}}}
\end{aligned}$$

With  $n$  sufficient large, we can guarantee that

$$\left(\frac{\gamma}{\beta^{\frac{1}{a}}}\right)^n \leq C'' \left(\frac{n}{a}\right)^{-\alpha-1}.$$

This yields that

$$V_n(\phi_c) \asymp n^{-1-\alpha},$$

and hence,  $\phi_c$  has summable variations (see (1)). Clearly, this function is bounded from above. We shall now show that

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi_c, 0)|_{\xi=rad.} = +\infty.$$

In fact,

$$Z_n^*(\phi_c, 0) = \exp \sum_{k=0}^{n-1} \varphi_c(x_k)$$

with  $x_k \in I_k$ . Taking into account that  $\alpha > 1$  we find that

$$\begin{aligned} \exp \left\{ \sum_{k=0}^{n-1} \varphi_c(x_k) \right\} &\geq \exp \left\{ - \sum_{k=0}^{n-1} |\varphi_c(d_k)| \right\} \\ &\geq \exp \left\{ -|c| \sum_{k=0}^{n-1} (1+k|\log \gamma|)^{-\alpha} \right\} \\ &\geq \exp \left\{ -|c| (|\log \gamma|)^{-\alpha} \sum_{k=0}^{n-1} (k+1)^{-\alpha} \right\} > e^{-C}, \end{aligned}$$

where  $C > 0$  is a constant. It follows that for  $\xi = 1$ ,

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, 0) = \log \sum_{n=1}^{\infty} e^{-C} = +\infty.$$

We have  $\Delta[\phi_c] = +\infty$  and the desired result follows.  $\square$

**Remark.** *The above argument shows indeed, that the function  $\phi_c$  has summable variations and is bounded from above for all  $\alpha > 0$ .*

#### 4. PHASE TRANSITIONS

Phase transitions for the renewal shift were studied by [3], [4], [9]. Let  $(\Sigma_A, \sigma)$  be the renewal shift,  $\phi : \Sigma_A \rightarrow \mathbb{R}$  a potential function and  $\bar{\phi}(x) = \phi_{r_a(x)}(x)$  the corresponding induced potential function.

**Proposition 4.1.** *Assume that  $\phi$  satisfies Conditions 1 and 2 of Proposition 2.3 (i.e.,  $\varphi$  has summable variations and is bounded from above) and that  $\bar{\phi}$  is weakly Hölder continuous (see (2)). Then there exists a critical value  $0 < c_0 \leq +\infty$  such that the potential  $c\phi$  is positive recurrent for all  $0 < c < c_0$  and is transient for  $c > c_0$ .*

Note that this result does not exclude the case when  $c_0 = \infty$ .

In fact, if  $\varphi$  is a Hölder continuous function on  $I$  then the function  $c\varphi$  is Hölder continuous and hence, is positive recurrent for all  $c$  (see

Theorem 3.4). We provide an example of a one-parameter family of functions for which the phase transition actually occurs.

**Theorem 4.2.** *For the function  $\varphi_c(x)$ , given by (4) with  $0 < \alpha \leq 1$ , the potential function  $\phi_c = \varphi_c \circ h$  has summable variations and  $\bar{\phi}_c(x)$  is weakly Hölder continuous. There exists  $c_0 > 0$  such that*

- (1)  $\phi_c$  is positive recurrent for  $c_0 > c > 0$  and there is a unique equilibrium measure for  $\varphi_c$  among all invariant measures. This measure is supported on  $(0, 1]$ .
- (2)  $\phi_c$  is transient for  $c > c_0$  and there is no equilibrium measure for  $\varphi_c$  among the measures supported on  $(0, 1]$ , the Dirac measure at 0 is the equilibrium measure.

*Proof.* The fact that  $\phi_c$  has summable variations and is bounded from above for  $0 < \alpha < 1$  (and actually for all  $\alpha > 0$ ) was established in the proof for Theorem 3.4 (see Remark in the end of Section 3).

We shall show that the induced potential  $\bar{\phi}_c(x)$  is weakly Hölder continuous. This will imply that it has summable variations. Choose two points  $x$  and  $y$  in the interval  $[0, n_1, \dots, 0, n_2, \dots, 0, \dots, n_k, \dots, 0]$ . We have

$$\begin{aligned}
|\bar{\phi}_c(x) - \bar{\phi}_c(y)| &= \left| \sum_{i=0}^{n_1} (\phi_c(f^i(x)) - \phi_c(f^i(y))) \right| \\
&\leq \sum_{i=0}^{n_1} |c| \frac{\alpha(1 - \log \xi_i)^{-1-\alpha}}{|\xi_i|} |\phi_c(f^i(x)) - \phi_c(f^i(y))| \\
&\leq \sum_{i=0}^{n_1} |c| \frac{\alpha(1 - (n_1 - i) \log \gamma)^{-1-\alpha}}{|[0, d_{n_1-i}]|} \\
&\quad |[n_1 - i, \dots, 0, n_2, \dots, 0, \dots, n_k, \dots, 0]| \\
&\leq \sum_{i=0}^{n_1} |c| \frac{|I_{n_1-i}|}{|[0, d_{n_1-i}]|} \gamma^{k-1} |\alpha(1 - (n_1 - i) \log \gamma)^{-1-\alpha}| \\
&\leq |c| C \sum_{i=0}^{n_1} (n_1 - i)^{-\alpha-1} \gamma^{k-1} \leq A \gamma^{k-1},
\end{aligned}$$

where  $A = |c| C \sum_{n=0}^{\infty} n^{-\alpha-1}$ . Hence,  $\bar{\phi}_c(x)$  is weakly Hölder continuous.

We shall show that for  $c$  sufficiently close to zero, the potential  $\phi_c$  is positive recurrent and for  $c$  sufficiently large, it is transient. Set

$$F_n(\phi_c) = \exp \left\{ \sum_{k=0}^{n-1} \sup_{x \in I_k} \varphi_c(x) \right\}.$$

Observe that

$$\begin{aligned} \log \sum_{n \geq 1} \xi^n F_n(\phi_c)|_{\xi=rad.} - 2 \sum_{n \geq 2} V_n(\phi_c) &\leq \Delta[\phi_c] \\ &\leq \log \sum_{n \geq 1} \xi^n F_n(\phi_c)|_{\xi=rad.} \end{aligned}$$

and that

$$F_n(\phi_c) = \exp \left\{ \sum_{k=0}^{n-1} \varphi_c(d_k) \right\},$$

where  $d_k$  were introduced in the proof of Theorem 3.4 and are the endpoints of the basic intervals  $I_n$  (i.e.,  $I_n = [d_n, d_{n-1}]$ ).

We consider the two cases.

**Case I:**  $\alpha = 1$ . We have

$$\begin{aligned} F_n(\phi_c) &= \exp \left\{ \sum_{k=0}^{n-1} \varphi_c(d_k) \right\} \\ &\geq \exp \left\{ \sum_{k=0}^{n-1} -c(1 - \log \gamma^k)^{-\alpha} \right\} \\ &\geq \exp \left\{ \sum_{k=0}^{n-1} -c|\log \gamma|(k+1)^{-\alpha} \right\} \\ &\geq \exp \{-cM \log(n+1)\} \geq C(n+1)^{-cM}, \end{aligned}$$

where  $C > 0$  and  $M > 0$  are constants. Similarly, one can show that for some  $C' > 0$  and  $M' > 0$ ,

$$F_n(\phi_c) \leq C'(n+1)^{-cM'}.$$

Therefore, the radius of convergence  $\xi = 1$  and

$$\sum_{n \geq 1} F_n(\phi_c) \geq \sum_{n \geq 1} C(n+1)^{-cM} = +\infty$$

as long as  $-1 < -cM < 0$  (i.e.,  $0 < c < \frac{1}{M}$ ). For these values of  $c$  we have  $\Delta[\phi_c] = +\infty$  and hence,  $\phi_c$  is positive recurrent.

Observe that

$$\log \sum_{n \geq 1} F_n(\phi_c) \leq \log \sum_{n \geq 1} C'(n+1)^{(-cM')}.$$

The expression in the right-hand side of this inequality tends to  $-\infty$  as  $c \rightarrow \infty$ . Hence, for some  $c$  we have

$$\Delta[\phi_c] \leq \log \sum_{n \geq 1} F_n(\phi_c) < 0.$$

This implies that for  $c$  sufficiently large,  $\phi$  is transient.

**Case II:**  $0 < \alpha < 1$ . We find that

$$\exp\{-cM'(n+1)^{1-\alpha}\} \leq F_n(\phi_c) \leq \exp\{-cM(n+1)^{1-\alpha}\}.$$

The radius of convergence  $\xi = 1$  and

$$\sum_{n \geq 1} F_n(\phi_c) \geq \sum_{n \geq 1} \exp\{-cM'(n+1)^{1-\alpha}\}.$$

Therefore,

$$(5) \quad \sum_{n \geq 1} F_n(\phi_c) \geq ke^{-1}$$

as long as  $cM' < \frac{1}{(k+1)^{1-\alpha}}$ . Observe that  $\sum_{n \geq 2} V_n(\phi) \rightarrow 0$  as  $c \rightarrow 0$ . In view of (5), this implies that for  $c$  sufficiently close to zero,

$$\Delta[\phi_c] \geq \log \sum_{n \geq 1} F_n(\phi_c) - 2 \sum_{n \geq 2} V_n(\phi_c) > 0$$

and hence,  $\phi_c$  is positive recurrent.

We also have that

$$\log \sum_{n \geq 1} F_n(\phi_c) \leq \log \sum_{n \geq 1} \exp\{-cM(n+1)^{1-\alpha}\} \rightarrow -\infty$$

as  $c \rightarrow +\infty$ , and hence,  $\phi_c$  is transient for sufficiently large  $c$ .

Since  $\xi = 1$ , by Proposition 2.2,  $0 = \log \xi = p^*(\phi_c)$ . When  $c$  is small,  $\Delta[\phi_c] > 0$  implies that  $p(\phi_c) < p^*(\phi_c) = 0$ . Proposition 2.3 give that  $P_G(\phi_c) = -p(\phi_c) > 0$ . Because  $\phi_c$  is positive recurrent, by Theorem 2.6, there is a unique equilibrium measure for the renewal shift, by topological conjugacy, there is a unique equilibrium measure  $\mu_{\varphi_c}$  among the measures supported on  $(0, 1]$ . Since the only invariant measure supported outside of  $(0, 1]$  is the Dirac measure  $\delta_0$ , and from

$$h_{\delta_0}(f) + \int_I \varphi_c d\delta_0 = \varphi_c(0) < P_G(\phi_c)$$

we know that  $\delta_0$  is not an equilibrium measure, hence  $\mu_{\varphi_c}$  is the equilibrium measure among all invariant measures.

In the case when  $c$  is sufficiently large,  $\Delta[\phi_c] < 0$  and Proposition 2.3 implies that  $P_G(\phi_c) = -p^*(\phi_c) = 0$ . In this case  $\phi_c$  is transient, by Proposition 2.4, there is no equilibrium measure supported on  $(0, 1]$ .

$$h_{\delta_0}(f) + \int_I \varphi_c d\delta_0 = 0 = P_G(\phi_c)$$

implies that the Dirac measure is the equilibrium measure.  $\square$

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