# ON MORSE-SMALE ENDOMORPHISMS

M.Brin and Ya.Pesin[1]

Abstract. A $C^1$-map $f$ of a compact manifold $M$ is a Morse–Smale endomorphism if the nonwandering set of $f$ is finite and hyperbolic and the local stable and global unstable manifolds of periodic points intersect transversally. Morse–Smale endomorphisms appear naturally in the dynamics of the evolution operator on the set of traveling wave solutions for lattice models of unbounded media. The main result of this paper is the openness of the set of Morse–Smale endomorphisms in the space $C^1(M,M)$ of $C^1$-maps of $M$ into itself. The usual order relation on $f$ (given by the intersections of local stable and global unstable manifolds) is used to describe the orbit structure of $f$ and its small $C^1$-perturbations.

## 1. Introduction

Morse–Smale endomorphisms arise naturally in lattice models of unbounded media with the evolution operator of diffusion type (see e.g. [AP]). For such systems, the dynamics of the evolution operator on the set of traveling wave solutions is completely determined by the following multi-dimensional Hénon type map

$$F_\varepsilon(x_1,\dots,x_k,\dots,x_n)=(x_2,\dots,x_{k+1},\dots,h(x_k)+\varepsilon g(x_1,\dots,x_n)),$$

where $x_i\in\mathbb{R}^d$, $h:\mathbb{R}^d\to\mathbb{R}^d$ is a $C^r$-diffeomorphism, $r\ge 1$, $g:\mathbb{R}^{dn}\to\mathbb{R}^d$ is a $C^r$-map, and $\varepsilon$ is sufficiently small. If the map $F_\varepsilon$ is *chaotic*, i.e., preserves an invariant mixing measure then the lattice system displays a *spatial-temporal chaos*, i.e., there exists a measure on the set of traveling wave solutions invariant and mixing with respect to both the evolution operator and the space translation operator. It is plausible that in several *physically interesting* situations the dynamics of the map $F_\varepsilon$ is completely determined by the map $h$ for all sufficiently small $\varepsilon$.

The first case is when the map $h$ has a locally maximal hyperbolic set. One can easily see that the map $F_0$ also has a locally maximal hyperbolic set. Note that $F_0$ is not invertible while $F_\varepsilon$ may be a diffeomorphism (this is the case if, for example, one assumes that the matrix $\frac{\partial g}{\partial x_1}$ is non-degenerate). The stability of a locally maximal

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hyperbolic set for a $C^1$-endomorphism under small perturbations by endomorphisms (or diffeomorphisms) was established in [AP].

Another case is when the map $h$ is a Morse–Smale diffeomorphism i.e. its non-wandering set is finite and hyperbolic and the global stable and unstable manifolds of periodic points intersect transversally (since $h$ acts on $\mathbb{R}^d$ one should also assume that infinity is a repelling fixed point for $h$).

From a physical point of view this situation often occurs when $h$ is one-dimensional. The map $F_0$ is (see [AP]) a *Morse–Smale endomorphism*, i.e., its nonwandering set is finite and hyperbolic and the local stable and global unstable manifolds of periodic points intersect transversally (see Section 2). Content-Length: 25376

In this context it is important to know whether Morse–Smale endomorphisms form an open set in the $C^1$-topology. The main result of this paper (see Theorem 4.1) provides a positive answer. A major still open question is whether Morse–Smale endomorphisms are structurally semi-stable, i.e., a small $C^1$-perturbation of a Morse–Smale endomorphism is topologically semiconjugate to it.

F. Przytycki [Pr1, Pr2], Przytycki studied regular Axiom A endomorphisms (i.e. those that are locally invertible). He proved that an Axiom A endomorphism is structurally stable if and only if it is expanding or is a diffeomorphism.

In Section 2 we quote the necessary properties of the stable and unstable manifolds.

In Section 3 we define Morse–Smale endomorphisms and consider the usual partial order relation $\geq$ on the set of nonwandering points of a Morse–Smale endomorphism $f$ i.e., $p \geq q$ if the unstable manifold of $p$ intersects the local stable manifold of $q$. We prove that $\geq$ is a partial order, has no cycles and that there are $\delta > 0$ and $\varepsilon > 0$ such that $p \geq q$ if and only if there is an $\varepsilon$-orbit of $f$ from the $\delta$-neighborhood of $p$ to the $\delta$-neighborhood of $q$. The last property is a major ingredient in the proof of the openness of Morse–Smale endomorphisms in Section 4.

## 2. Stable and unstable manifolds for endomorphisms

We begin with a standard stable manifold theorem for a differentiable map (see [Rob, Rue, Shu]). Let $p$ be a fixed point of a $C^1$-map $f : U \to \mathbb{R}^d$. Denote by $E^s(p)$, $E^u(p)$ the **stable** and **unstable subspaces** spanned by the generalized eigenvectors of $df(p)$ corresponding to the eigenvalues $\lambda$ with $|\lambda| < 1$ and $|\lambda| > 1$, respectively. The point $p$ is **hyperbolic** if no eigenvalue of $df(p)$ has absolute value 1, or equivalently, $E^s(p)$ and $E^u(p)$ span $\mathbb{R}^d$.

**2.1 Theorem.** *(See* [Rob, Rue, Shu]*.) Let $p$ be a hyperbolic fixed point of a $C^1$-map $f : U \to \mathbb{R}^d$. Then there exist local stable $W^s_{\text{loc}}(p)$ and unstable $W^u_{\text{loc}}(p)$ manifolds with the following properties:*

(1) *the manifolds $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ are of class $C^1$, pass through $p$, and are tangent at $p$ to the subspaces $E^s(p)$ and $E^u(p)$, respectively;*

(2) *$W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ are invariant under $f$, i.e.*

$$f(W^s_{\text{loc}}(p)) \subset W^s_{\text{loc}}(p),\ f(W^u_{\text{loc}}(p)) \supset W^u_{\text{loc}}(p);$$

(3) *there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for any $n > 0$*

$$d(f^n x, f^n y) < C\lambda^n d(x, y)$$

*if* $x, y \in W^s_{\rm loc}(p)$ *and*

$$d(f^n x, f^n y) > C\lambda^{-n} d(x,y)$$

*if* $f^k x,\ f^k y \in W^u_{\rm loc}(p)$ *for* $k = 0, 1, ..., n$*;*

(4) *there is* $\delta > 0$ *such that*

$$\begin{aligned} W^s_{\rm loc}(p) =& \{x \in \mathbb{R}^d \ : \ d(f^n x, p) \le \delta \text{ for all } n \ge 0\,\}, \\ W^u_{\rm loc}(p) =& \{x \in \mathbb{R}^d \ : \text{ there exist points } x_n \in \mathbb{R}^d \text{ such that} \\ & \quad f^n x_n = x \text{ and } d(f^k x_n, p) \le \delta \text{ for all } n \ge 0 \text{ and } k = 0, 1, \dots, n\,\}. \end{aligned}$$

The existence of the local unstable manifold $W^u_{\rm loc}(p)$ is shown in [Shu] (see Theorem 5.2). The existence of the local stable manifold $W^s_{\rm loc}(p)$ is proved in [Rob] (see Theorem 10.1).

Denote by $C^1(U, \mathbb{R}^d)$ the space of $C^1$-maps of a neighborhood $U \subset \mathbb{R}^d$ into $\mathbb{R}^d$ with the $C^1$-topology.

**2.2 Theorem.** *(See* [Shu, Rue]*.) Let* $p$ *be a hyperbolic fixed point of a* $C^1$*-map* $f : U \to \mathbb{R}^d$*. Then for any* $\varepsilon > 0$ *there exists an open neighborhood* $\mathcal{U} \ni f$ *in* $C^1(U, \mathbb{R}^d)$ *such that every* $g \in \mathcal{U}$ *has a unique hyperbolic fixed point in the* $\varepsilon$*-neighborhood of* $p$*. The local stable and unstable manifolds of this point depend continuously on* $g \in \mathcal{U}$*.*

Let $f : U \to \mathbb{R}^d$ be a $C^1$-map with a hyperbolic fixed point $p \in U$. Define the **global** unstable manifold $W^u(p)$ of $p$ by

$$W^u(p) = \bigcup_{n \ge 0} f^n(W^u_{\rm loc}(p)).$$

We will need the following lemma which follows directly from the $\lambda$-lemma (or inclination lemma) of Palis (see [PaMe]).

**2.3 Lemma.** *Let* $p \in NW(f)$*,* $R, \varepsilon > 0$*. Let* $G$ *be a submanifold of dimension* $k \ge u(p)$ *which intersects* $W^s_{\rm loc}(p)$ *transversally at a point* $x$*, i.e. the intersection of the tangent planes at* $x$ *has dimension* $\le \max(k + s(p) - d, 0)$*.*

*Then there is* $n > 0$ *such that* $f^n G$ *contains a submanifold* $\widetilde{G}$ *which is* $C^1 - \varepsilon$*-close to the ball of radius* $R$ *in* $W^u(p)$ *in the induced metric.* □

## 3. Morse–Smale endomorphisms and their orbit structure

Let $f : M \to M$ be a $C^1$-map of a compact $d$-dimensional Riemannian manifold $M$. Theorem 2.1 allows one to construct local stable and unstable manifolds $W^s_{\rm loc}(p)$ and $W^u_{\rm loc}(p)$ and the the global unstable manifold $W^u(p)$ for any hyperbolic *periodic* point $p$ of $f$.

**3.1 Definition.** A $C^1$-map $f : M \to M$ of a compact $d$-dimensional manifold $M$ is a **Morse–Smale endomorphism** if

(i) the nonwandering set $NW(f)$ is finite and hyperbolic, i.e. $NW(f)$ is the set $\mathrm{Per}(f)$ of periodic points of $f$ and all of them are hyperbolic,

(ii) the local stable and global unstable manifolds of periodic points intersect transversally, i.e. if $x \in W^s_{\mathrm{loc}}(p) \cap W^u(q)$ with $p, q \in \mathrm{Per}(f)$, then $T_x W^s_{\mathrm{loc}}(p) \oplus T_x W^u(q) = T_x M$.

Note that if $f$ is an invertible Morse–Smale endomorphism then it is a Morse–Smale diffeomorphism.

It follows immediately from the definition that any orbit of $f$ eventually enters a small neighborhood of $NW(f)$ and stays in it forever. This implies the following important property of Morse–Smale endomorphisms.

**3.2 Proposition.** *For any $x \in M$ there is $n > 0$ and $p \in NW(f)$ such that $f^n x \in W^s_{\mathrm{loc}}(p)$.*

We assume now and for the remainder of this section that $f : M \to M$ is a Morse–Smale endomorphism. Following Smale's arguments in the proof of the spectral theorem for Axiom A diffeomorphisms we define a partial order $\geq$ on $NW(f)$ by $p \geq q$ if $W^u(p) \cap W^s_{\mathrm{loc}}(q) \neq 0$. A point $x$ is called **heteroclinic** if $x \in W^u(p) \cap W^s_{\mathrm{loc}}(q)$ and **transversal heteroclinic** if the intersection is transversal.

**3.3 Proposition.** *The partial order $\geq$ is transitive and has no cycles, i.e. $p_1 \geq p_2 \geq ... \geq p_k = p_1$ implies that $p_i = p_1$, $i = 2, 3, ..., k$.*

*Proof.* For $p \in NM(f)$ denote by $s(p)$ and $u(p)$ the dimensions of $W^s_{\mathrm{loc}}(q)$ and $W^u(p)$, respectively. If $p \geq q$ then $u(p) \geq u(q)$ by the transversality of the intersections of local stable and global unstable manifolds. The transitivity of $\geq$ follows immediately from Lemma 2.3.

Assume now that $p_1 \geq p_2 \geq ... \geq p_k = p_1$. Then, by Lemma 2.3 applied $k$ times to $W^u(p_1)$, the submanifolds $W^u(p_1)$ and $W^s_{\mathrm{loc}}(p_1)$ intersect transversally at a point $x \neq p_1$. It is easy to see that $x$ is a nonwandering point of $f$ with an infinite orbit. This contradicts the fact that $f$ is a Morse–Smale endomorphism. □

For $\delta > 0$ denote by $U_\delta(A)$ the $\delta$-neighborhood of $A$ in $M$.

**3.4 Proposition.**

(1) *For every $\delta > 0$ there is $n(\delta)$ such that every finite orbit of length at least $n(\delta)$ must enter the $\delta$-neighborhood of $NW(f)$, i.e.for every $x \in M$*

$$\bigcup_{i=0}^{n(\delta)} f^i x \bigcap U_\delta(NW(f)) \neq \emptyset.$$

(2) *For every $\delta > 0$ there is $N(\delta)$ such that the total time that an orbit can spend outside of the $\delta$-neighborhood of $NW(f)$ does not exceed $N(\delta)$, i.e. for every $x \in M$*

$$\sum_{i=0}^{\infty} \mathbf{1}_{U_\delta(NW(f))}(f^i x) \leq N(\delta)\,.$$

*Proof.* Assume that there is a number $\delta > 0$ and a sequence of points $x_k$ such that $f^i x_k \notin U_\delta(NW(f))$ for all $k$ and all $i \le n_k$, where $n_k \to \infty$ as $k \to \infty$. Since $M$ is compact, the sequence $x_k$ has a limit point $x$ whose positive semiorbit obviously stays out of $U_\delta(NW(f))$. An $\omega$-limit point of $x$ is a nonwandering point of $f$ lying outside $U_\delta(NW(f))$. This is a contradiction. This proves the first statement. The second statement can be proved in a similar way. □

We call a sequence of points $z_k \in M$, $k = 1, \dots, n$ an $\varepsilon$-**orbit** if $d(fz_k, z_{k+1}) \le \varepsilon$. We formulate an analog of Proposition 3.4 for $\varepsilon$-orbits. The proof is quite similar to the proof of Proposition 3.4.

**3.5 Proposition.** *For every $\delta > 0$ there is $n(\delta) > 0$ and $\varepsilon > 0$ such that for every $\varepsilon$-orbit $\{z_k\}$, $k = 1, \dots, n$ with $n \ge n(\delta)$*

$$\bigcup_{k=0}^{n} z_k \bigcap U_\delta(NW(f)) \neq \emptyset.$$

□

We will characterize the partial order $\ge$ in terms of the behavior of the orbits of $f$. We do this under the additional assumption that the nonwandering set of $f$ consists only of fixed points which is sufficient for the proof of the perturbation Theorem 4.1. However, the corresponding arguments in the proofs of Propositions 3.8, 3.9, and 3.11 below can be easily modified to work in the general case.

Assume that any point in $NW(f)$ is a fixed point of $f$. Denote by $U_\delta(x)$ the $\delta$-neighborhood of $x \in M$.

Given $\delta > 0$ and $p$, $r \in NW(f)$ we say that $r$ $\delta$-**follows** $p$ if there exist a sequence of points $x_n \in M$ and sequences of integers $a_n$, $b_n$, $c_n \to \infty$, $a_n < b_n < c_n$ such that

(1) $x_n \to p$ and $f^{c_n} x_n \to r$;
(2) $f^k x_n \in U_\delta(p)$ for $0 \le k \le a_n$ and $f^k x_n \in U_\delta(r)$ for $b_n < k \le c_n$;
(3) $f^k x_n \notin U_\delta(NW(f))$ for $a_n < k \le b_n$.

We need the following two lemmas.

**3.6 Lemma.** *Let $p$, $q \in NW(f)$ and assume that there are sequences of points $x_n \to p$ and integers $t_n \to \infty$ such that $f^{t_n} x_n \to q$. Then there exists a point $r \in NW(f)$ such that $r$ $\delta$-follows $p$, a sequence of points $y_n \in M$ and a sequence of integers $\bar{k}_n$ such that $y_n \to r$, $f^{\bar{k}_n} y_n \to q$ and $\bar{k}_n \to \infty$ as $n \to \infty$.*

*Proof.* Given $n > 0$ we associate to each collection of points $f^i x_n$, $i = 0, \dots, t_n$ a word

$$w(n) = p_{i_1(n)}^{k_1(n)} p_{i_2(n)}^{k_2(n)} \dots p_{i_{m(n)}(n)}^{k_{m(n)}(n)}$$

in the following way. The order of points $p_{i_j(n)}$, $j = 1, \dots, m(n)$ corresponds to the order in which the trajectory $f^i x_n, i = 1, \dots, k(n)$ enters the $\delta-$neighborhoods of nonwandering points $p_1, \dots, p_s$ and the number $k_j(n)$, $j = 1, \dots, m(n)$ is the amount of time the trajectory spends in the corresponding neighborhood. Since $k_n \to \infty$ as $n \to \infty$, it follows from Proposition 3.5 that

(1) there exist $M > 0$ such that $m(n) \leq M$ for any $n > 0$
(2) $\sum_{j=1}^{m(n)} k_j(n) \to \infty$ as $n \to \infty$

We claim that there exist a point $p_s = r \in NW(f)$ and a subsequence of words $w(n_l)$ such that $k_s(n_l) \to \infty$ as $l \to \infty$ and $k_j(n_l) \leq$ const for all $j = 1, \ldots, s-1$ and all $l$. It is easy to see that $r$ $\delta$-follows $p$. To prove the claim consider the smallest index $j$ for which the sequence $k_j(n)$ is unbounded. Let $k_j(n_l) \to \infty$. Since there are finitely many possible values for the index $i_j(n_l)$, we can pass to a subsequence and assume that there is $s$ such that $i_j(n_l) = s$ for all $l$, and the claim follows. □

**3.7 Lemma.** *If $p$, $r \in NW(f)$ are two points such that $r$ $\delta$-follows $p$ then $p \geq r$.*

*Proof.* Since $r$ $\delta$-follows $p$, we have the corresponding sequences $x_n$, $a_n$, $b_n$, $c_n$. Let $y_n = f^{a_n+1}x_n$. Let $y$ be a limit point of the sequence $\{y_n\}$. Since $y_n \notin U_\delta(NW(f))$, we have that $y \notin U_\delta(NW(f))$. Clearly there is $K > 0$ such that $f^k y \in U_\delta(r)$ for all $k \geq K$. Hence for $\delta$ small enough, by Statement 4 of Theorem 2.1, $f^K y \in W^s_{\text{loc}}(r)$. Similarly, one can show that $y \in W^u(p)$. □

The following proposition is an immediately corollary of Lemmas 3.6 and 3.7.

**3.8 Proposition.** *Let $p$, $q \in NW(f)$ and assume that there are sequences of points $x_n \to p$ and integers $t_n \to \infty$ such that $f^{t_n}x_n \to q$. Then $p \geq q$.* □

**3.9 Proposition.** *There exists $\delta_0 > 0$ such that for any $\delta \leq \delta_0$ whenever $p$, $q \in NW(f)$ and there is a point $x \in U_\delta(p)$ for which $f^k x \in U_\delta(q)$ for some $k > 0$ then $p \geq q$.*

*Proof.* Assume the contrary. Then there exist numbers $\delta_n \to 0$, $k_n \to \infty$ and points $p$, $q \in NW(f)$, $x_n \in U_{\delta_n}(p)$ such that $f^{k_n}x_n \in U_{\delta_n}(q)$ and it is not true that $p \geq q$. This contradicts Proposition 3.8. □

**3.10 Remark.** One can prove the following stronger version of Proposition 3.9. For any $\alpha > 0$ there is $\delta_0 > 0$ such that for any $\delta \leq \delta_0$ whenever $p$, $q \in NW(f)$ and there is a point $x \in U_\delta(p)$ for which $f^k x \in U_\delta(q)$ for some $k > 0$ then $p \geq q$ and there is a heteroclinic point $y \in W^u(p) \cap W^s_{\text{loc}}(q)$ with $d(x, y) < \alpha$.

An analog of Proposition 3.9 holds true for $\varepsilon$-orbits.

**3.11 Proposition.** *For any positive $\delta \leq \frac{\delta_0}{4}$ there is $\varepsilon > 0$ such that whenever $p$, $q \in NW(f)$ and there is an $\varepsilon$-orbit $\{z_k\}$ with $z_1 \in U_\delta(p)$ and $z_n \in U_\delta(q)$ then $p \geq q$.*

*Proof.* Let $p$, $q$, $z_n$ be as above. One can find a point $r \in NW(f)$ and numbers $a$, $b$, and $c$ such that $z_k \in U_\delta(p)$ for $k = 1, \ldots, a$, $z_k \in U_\delta(r)$ for $k = b, \ldots, c$, and $z_k \notin U_\delta(NW(f))$ for $k = a+1, \ldots, b$. By Proposition 3.4, $0 < b - a \leq n(\frac{\delta}{2})$ for a sufficiently small $\varepsilon$. Therefore, if $\varepsilon$ is small enough then there exists a point $x \in U_\delta(p)$ for which $f^k x \in U_\delta(r)$ for some $k > 0$. Thus by Proposition 3.9, $p \geq r$.

We repeatedly apply the above argument to the $\varepsilon$-orbit, and the proposition follows. □

## 4. Perturbation theorem

Denote by $C^1(M,M)$ the space of $C^1$-maps of $M$.

**4.1 Theorem.** *Let $f: M \to M$ be a Morse-Smale endomorphism. Then there is $\delta_0 > 0$ such that for any positive $\delta \le \delta_0$ there exists an open neighborhood $\mathcal{U} \ni f$ in $C^1(M,M)$ with the property that any $g \in \mathcal{U}$ is a Morse-Smale endomorphism and*

1. *there is a bijection $\chi : NW(f) \to NW(g)$ with $d(p, \chi(p)) < \delta$ for any $p \in NW(f)$;*
2. *for $p_1, p_2 \in NW(f)$ we have $p_1 \le p_2$ if and only if $\chi(p_1) \le \chi(p_2)$;*
3. *for any $q_1$, $q_2 \in NW(g)$ we have that $q_1 \le q_2$ if and only if there is a point $x \in U_\delta(q_2)$ such that $g^k x \in U_\delta(q_1)$ for some $k \ge 0$.*

*Proof.* The following lemma allows us to reduce the theorem to the case when $NW(f)$ consists only of fixed points.

**4.2 Lemma.** *If $g$ is a $C^1$-map of $M$ such that $g^m$ is a Morse-Smale endomorphism then $g$ is a Morse-Smale endomorphism.*

*Proof of Lemma 4.2.* It is sufficient to show that any point $x \notin NW(g^m)$ is a wandering point for $g$. If $x$ is such a point then, by Proposition 3.2, $g^{mn}x \in W^s_{\text{loc}}(p)$ for some $n > 0$ and $p \in NW(g^m)$. Hence, $x$ is a wandering point under $g$. □

From now on, by switching to the corresponding power, we assume that $NW(f)$ consists only of fixed points. To show that any $g$ close enough to $f$ is a Morse-Smale endomorphism we have to prove that it satisfies properties (i) and (ii) of Definition 3.1.

Fix $\delta > 0$. By standard transversality arguments, if $g$ is close enough to $f$ then for any $p \in NW(f)$ there is a unique hyperbolic fixed point $q = \chi(p)$ of $g$ such that $d(p,\, q) < \delta$. Let $x$ be a nonwandering point of $g$. Then arbitrary close to $x$ there is a point $y$ and an arbitrarily large $k$ such that the finite orbit $\mathcal{O} = \{y, gy, \dots, g^k y\}$ is a closed $\varepsilon$-orbit of $f$. If $\delta$ and $\mathcal{U}$ are small enough, Propositions 3.8 and 3.11 imply that $\mathcal{O}$ lies in a small neighborhood of a fixed point $p \in NW(f)$. It follows that $x = \chi(p)$. This completes the proof of property (i).

To prove (ii) we assume the contrary. Then there is a sequence of $C^1$-maps $g_n$ that converges to $f$ in the $C^1$-topology and each map $g_n$ has a nontransversal heteroclinic point. To simplify the notation in the argument below we use the following convention: $p$ (possibly with an index) denotes a fixed point of $f$, $q(n)$ (possibly with an index) denotes a fixed point of $g_n$, $W^u(p)$ denotes the unstable manifold of $f$, $W^u(q(n))$ denotes the unstable manifold of $g_n$ and similarly for the local stable manifolds. By passing to a subsequence, if necessary, we can assume that for any sufficiently small $\delta > 0$

1. there are fixed points $p_0 \ge p_1 \ge \cdots \ge p_l$ of $f$ and fixed points $q_j(n)$, $j = 0, 1, \dots, l$ of $g_n$ such that $q_j(n) \to p_j$ as $n \to \infty$;
2. there are nontransversal heteroclinic points $y_n \in W^u(q_0(n)) \cap W^s_{\text{loc}}(q_l(n))$ with the common unit vectors $v_n$ of the tangent spaces such that $\kappa\delta \le d(y_n,\, q_l(n)) \le \delta$ for some $\kappa > 0$, the sequence $\{y_n\}$ converges to a point $y \in W^s_{\text{loc}}(q_l(n))$ and $v_n \to v \in T_y W^s_{\text{loc}}(p_l)$, $\|v\| = 1$;
3. there are points $x_n \in W^u(q_0(n))$ such that

$$d(x_n,\, q_0(n)) \le \delta \le d(g_n x_n,\, q_0(n))$$

and the sequence $\{x_n\}$ converges to a point $x \in W^u(p_0)$;

(4) there are sequences of integers $a_0(n) = 0 \le b_0(n) < a_1(n) \le b_1(n) < \cdots < a_l(n) \le b_l(n)$ such that for $j = 1, \ldots, l$ and $n = 1, 2, \ldots$

$$g_n^i x_n \in U_\delta(q_j(n)) \quad \text{if } a_j(n) \le i \le b_j(n)$$

and

$$g_n^i x_n \notin U_\delta(NW(g_n)) \quad \text{if } b_j(n) < i < a_{j+1}(n);$$

(5) $g_n^{a_l(n)} x_n = y_n$.

Clearly $d(x, p_0) \le \delta$, $d(fx, p_0) \ge \delta$ and $d(y, p_l) \le \delta$. Hence, $d(x, p_0) \ge C\delta$, where $C = \max_{x \in M} \|df(x)\|$.

By Proposition 3.4,

$$\sum_{j=0}^{l-1} (a_{j+1}(n) - b_j(n)) < N(\delta)\,.$$

Assume first that $a_l(n) - b_0(n)$ is bounded uniformly in $n$. Then $y = f^k x$ for some $k > 0$, and hence, $y \in W^u(p_0)$. Therefore $y$ is a nontransversal heteroclinic point of $f$. This is a contradiction.

Suppose now that $a_l(n) - b_0(n)$ is not bounded in $n$. Then, by passing to a subsequence, decreasing $\delta$ and deleting some of the fixed points if necessary, we may assume that for every $j = 1, \ldots, l-1$

(6) $b_j(n) - a_j(n) \to \infty$ as $n \to \infty$ and the difference $a_{j+1}(n) - b_j(n)$ eventually becomes constant, we denote this constant by $k_j$;

(7) the sequences of points $g_n^{a_j(n)} x_n$ and $g_n^{b_j(n)} x_n$ converge to points $z_j$ and $w_j$ respectively.

For convenience, we set $w_0 = x$ and $z_l = y$. Note that $z_j \in W^s_{\text{loc}}(p_j) \cap W^u(p_{j-1})$ is a heteroclinic point of $f$.

In the argument below we need to compare two subspaces in the tangent spaces at two different points lying in the $2\delta$-neighborhood of $p_j$ for $j = 0, \ldots, l$. For a small enough $\delta$ we identify the neighborhood with a ball in $\mathbb{R}^d$. We parallel translate any subspace at any point to 0 and calculate the distance between subspaces at 0 using, for example, the Grassman metric.

Consider the image $E_n \subset T_{y_n} M$ of the tangent space to $W^u(q_0(n))$ at $x_n$ under $dg_n^{b_l(n)}$. To obtain a contradiction we will show that for any $\varepsilon > 0$ and all sufficiently large $n$ there is a subspace $V_n \subset E_n$ which is $\varepsilon$-close to $T_y W^u(p_{l-1})$ and not transversal to $W^s_{\text{loc}}(q_l(n))$. This means that $W^u(p_{l-1})$ and $W^s_{\text{loc}}(p_l)$ are not transversal at $y$ which is impossible.

We need the following lemmas.

**4.3 Lemma.** *For any $\beta > 0$ there are $\alpha > 0$ and a neighborhood $\mathcal{V} \ni f$, $\mathcal{V} \subset \mathcal{U}$ such that for any $j = 0, \ldots, l$ the following holds true: if $x$ is a point with $d(x, w_j) \le \alpha$, $E \subset T_x M$ is a subspace $\alpha$-close to $T_{w_j} W^u(p_j)$, and $g \in \mathcal{V}$ then $d(g^{k_j} x, z_{j+1}) \le \beta$ and the subspace $dg^{k_j} E$ is $\beta$-close to $T_{z_{j+1}} W^u(p_j)$.*

*Proof of Lemma 4.3.* We have by Proposition 3.4(1) that $k_j \le n(\delta)$ for all $j$ and the lemma follows. □

**4.4 Lemma.** *For any $\gamma > 0$ there are $\beta > 0$ and a neighborhood $\mathcal{W} \ni f$, $\mathcal{W} \subset \mathcal{U}$ such that for any $j = 0, \dots, l$ the following holds true: if $x$ is a point with $d(x, z_j) \le \beta$, $E \subset T_x M$ is a subspace $\beta$-close to $T_{z_j} W^u(p_{j-1})$, and $g \in \mathcal{V}$ is such that $d(g^k x, w_j) \le \beta$ for some integer $k > 0$ then the subspace $d_x g^k E$ contains a subspace $E'$ which is $\gamma$-close to $T_{w_j} W^u(p_j)$.*

*Proof of Lemma 4.4.* Recall that $z_j$, $j = 1, \dots, l$ are transversal heteroclinic points of $f$. As before we identify the $2\delta$-neighborhoods of $p_j$'s with balls in $\mathbb{R}^d$ and use parallel translation in $\mathbb{R}^d$ to identify subspaces at different points. By Theorem 2.2, for a sufficiently small $\delta > 0$ any $g$ close enough to $f$ has the unique hyperbolic fixed point $q_j = q_j(g)$ in $U_\delta(p_j)$ which depends continuously on $g$; the local stable and unstable manifolds of $g$ at $q_j$ depend continuously on $g$ in the $C^1$-topology. Denote by $F$ the orthogonal complement to $T_{z_j} W^s_{\text{loc}}(p_j)$ in $E$ and view it as a submanifold passing through $x$. It follows from the remarks above that if $g$ is sufficiently close to $f$ and $\beta$ is small enough, then the submanifold $F$ intersects $W^s_{\text{loc}}(q_j)$ transversally at a point that is $C\beta$-close to $x$ and $z_j$, where $C > 0$ does not depend on $\beta$ and $g$. Note that $k \to \infty$ as $\beta \to 0$. Hence, by the $\lambda$-lemma of Palis (see [PaMe]) for a sufficiently small $\beta$ we have that $d_x g^k F$ is $\gamma$-close to $T_{w_j} W^u(p_j)$. □

We now complete the proof of the theorem. Recall that $z_l = y$ and $W^u(p_{l-1})$ intersects $W^s_{\text{loc}}(p_l)$ transversally at $y$. Therefore, the difference between any two unit vectors, one from $T_y W^s_{\text{loc}}(p_l)$ and another from $T_y W^u(p_{l-1})$, is uniformly bounded away from 0. We choose the first vector to be the accumulation vector $v$ of the common vectors $v_n$ for the nontransversal intersections above. By moving back from $p_l$ to $p_0$ and applying repeatedly Lemmas 4.3 and 4.4, we construct vectors $\omega_n \in T_{x_n} W^u_{\text{loc}}(q_0(n))$ such that the vector $w_n = dg^{a_l(n)} v_n$ is arbitrarily close to the space $T_y W^u(p_{l-1})$ for a sufficiently large $n$. We multiply $v_n$ by appropriate positive numbers to get $w_n$ of unit length and obtain a contradiction. □

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M.Brin, Department of Mathematics, University of Maryland, College Park, MD, 20742 USA, mbrin@math.umd.edu

Ya.Pesin, Department of Mathematics, Pennsylvania State University, University Park, PA, 16802, USA, pesin@math.psu.edu