THERMODYNAMICS OF INDUCING SCHEMES AND LIFTABILITY OF MEASURES

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ABSTRACT. We describe some recent results on thermodynamical formalism for dynamical systems admitting inducing schemes. This includes constructing equilibrium measures for certain classes of potential functions. These measure minimize the free energy of the system within the class of invariant measures that can be lifted to the tower associated with the inducing scheme. We shall discuss the liftability problem and present some examples illustrating various phenomena associated with liftability.

1. INTRODUCTION

Thermodynamical formalism is a collection of methods aimed at producing special invariant measures for dynamical systems. More precisely, let f be a continuous map of a compact topological space I and $\mathcal{M}(f, I)$ the class of all invariant Borel probability measures on I. Given a continuous real valued *potential* function φ on I, one considers the *equilibrium* measures for φ, i.e. invariant Borel probability measures μ on I for which the supremum

$$\sup_{\mu\in\mathcal{M}(f,I)}\Big\{h_\mu(f)+\int_I \varphi\, d\mu\Big\} \tag{1}$$

is attained, where $h_\mu(f)$ denotes the metric entropy of the map f. In the classical case, when f is a topologically transitive (one- or two-sided) subshift of finite type and φ is a Hölder continuous function, the equilibrium measure exists and is unique. Many dynamical systems with uniform hyperbolic structure (e.g., Anosov maps, axiom A diffeomorphisms) can be modelled by subshifts of finite type and for them thermodynamical formalism can be effected.

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However, there are classes of dynamical systems, which admit no representation by subshifts of finite type, most notoriously one- or multi-dimensional systems with nonzero Lyapunov exponents. Surprisingly, many of these systems allow representations by towers – symbolic models, which are more complicated than subshifts of finite type. They can be analyzed using some recently developed and quite sophisticated techniques in statistical physics.

For a map f, a tower is determined by a subset $W \subset I$, the *base* of the tower, a positive integer-valued function τ on W, the *inducing time*, the map $F : W \to W$, the *induced map*, and a countable partition $\mathcal{R}$ of W. The function τ is constant on each partition element J and is a return time of J to W but it is not necessarily the first return time to the base. The relation between the original map f and the induced map F is given by $F(x) = f^{\tau(J)}(x)$ for each $x \in J$. A crucial feature of the tower is that $\mathcal{R}$ is a generating Bernoulli partition for the induced map F so that it is equivalent to the full (one- or two-sided) shift on a countable set of states. See the next section for more precise description of towers.

For a positive Lebesgue measure set of parameters in a "typical" family of one-dimensional unimodal maps various constructions of towers can be found in works of Jakobson [14], Benedicks and Carleson [3], and Yoccoz and Senti [24, 23]. In particular, every unimodal map satisfying the Collet-Eckmann condition admits a tower as described above. For some multimodal maps towers were constructed by Bruin, Luzzatto and van Strien [8].

Alves, Luzzatto, and Pinheiro showed that towers can be constructed for multidimensional nonuniformly expanding maps [2].

For systems with nonzero Lyapunov exponents tower constructions were introduced by Young [25] and their existence was established for some important particular systems including Hénon maps (see for example, [4, 5]) and Sinai's billiards (see [10]).

Abstract towers in measurable spaces were studied by Zweimüller [27].

The first attempt to effect thermodynamical formalism for systems admitting towers was obtained by Bruin and Keller [7]: they established existence and uniqueness of equilibrium measures for one-dimensional unimodal maps satisfying the Collet-Eckmann condition and for the potential function $\varphi_t(x) = -t \log |df(x)|$ with t sufficiently closed to 1. We stress that they use a tower construction known as the Hofbauer-Keller tower that is different from the one described above.

In [19, 18], Pesin and Senti developed thermodynamical formalisms for general systems admitting tower constructions. In particular, they described a class of potential functions for which equilibrium measures exist and are unique; see Section 3 for detailed definitions and results. The crucial elements in constructing equilibrium measures are the following:

1) Starting with a continuous potential function φ on I, one obtains the *induced potential function* $\bar{\varphi}$ on the base W by the formula $\bar{\varphi}(x) = \sum_{k=0}^{\tau(x)-1} \varphi(f^k(x))$.

2) One finds equilibrium measures for the induced map F with respect to the induced potential function $\bar{\varphi}$. Since F is conjugate to the full shift on a countable set of states, one can apply results of Mauldin and Urbański [16] and of Sarig [22, 21] (see also Aaronson, Denker and Urbanski [1], Yuri [26], and Buzzi and Sarig [9]) to establish existence and uniqueness of equilibrium measures for F. This leads to certain requirements on potential functions and thus determines the desired class of potentials.

3) One then lifts equilibrium measures to the tower. This produces an f-invariant Borel probability measure μ_φ, which is a unique equilibrium measure in a somewhat restricted sense: it minimizes the free energy $E = -(h_\mu(f) + \int_I \varphi \, d\mu)$ among not all but only *liftable measures*, i.e. the measures that can be obtained by lifting to the tower F-invariant measures on the base.

Establishing liftability of a given invariant measure could be a challenging task and one of the goals of this article is to address the liftability problem:

Given a map f admitting an inducing scheme $\{S, \tau\}$, describe the class of all liftable measures.

Let us stress that the class of liftable measures depends on the choice of the inducing scheme $\{S, \tau\}$ in two ways: changing the inducing time may reduce the class of liftable measures while changing the base of the tower corresponding to the scheme will effect the class of measures which give positive weight to the base. One may therefore be interested in constructing inducing schemes $\{S, \tau\}$ with respect to which every f-invariant Borel probability measure is liftable (see [17]).

We present various characterizations of liftability as well as some criteria for liftability. We also present an example due to Bruin, which demonstrates that there are nonliftable measures (see also [27] for a similar construction).

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2. Inducing Schemes

We describe the class of systems admitting inducing schemes, which were introduced in [19, 18]. Let $f : I \to I$ be a continuous map of a compact topological space I, S a countable collection of disjoint Borel subsets of I, and $\tau : S \to \mathbb{N}$ a positive integer-valued function. Let also $W = \bigcup_{J \in S} J$ be the *inducing domain* and $\tau : I \to \mathbb{N}$,

$$\tau(x) = \begin{cases} \tau(J), & x \in J \in S, \\ 0, & x \neq W \end{cases}$$

the *inducing time*.

Let $\overline{J}$ denote the closure of the set J. We assume that the following conditions hold:

(H1) for each $J \in S$ there exists an open connected set $U_J \supseteq J$ such that $f^{\tau(J)}|U_J$ is a homeomorphism onto its image and $f^{\tau(J)}(J) \subseteq W$;

(H2) the partition $\mathcal{R}$ of W induced by the sets $J \in S$ is "one-sided" generating: for any countable collection of elements $\{J_k\}_{k \in \mathbb{N}}$, the intersection

$$\overline{J_1} \cap \Big(\bigcap_{k \geq 2} f^{-T_k}(\overline{J_k}) \Big)$$

is nonempty and consists of a single point, where $T_k = \sum_{m=1}^{k-1} \tau(J_m)$.

Define the *induced map* $F : W \to W$ by $F(x) = f^{\tau(x)}(x)$ and set

$$X = \bigcup_{J \in S} \bigcup_{k=0}^{\tau(J)-1} f^k(J). \tag{2}$$

The set X is forward invariant under f.

In view of (H2), the induced map $F : W \to W$ is conjugate to the one-sided Bernoulli shift σ on a countable set of states S. More precisely, this means the following (see [19, 18]). Define the coding map $h : S^{\mathbb{N}} \to \overline{W}$ by

$$h : \omega = (a_0, a_1, \cdots) \mapsto x_\omega, \tag{3}$$

where $x_\omega = \bigcap_{n \geq 0} F^n(\overline{J_{a_n}})$.

Proposition 2.1. *The following statements hold:*

(1) *the map h is well-defined, continuous, and $h(S^{\mathbb{N}}) \supset W$;*
(2) *h is one-to-one on $h^{-1}(W)$;*
(3) *the induced map $F : W \to W$ is topologically conjugate to the one-sided Bernoulli shift $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ via h, i.e., $h \circ \sigma | h^{-1}(W) = F \circ h | h^{-1}(W)$.*

In what follows we assume that the following condition holds:

(H3) the set $S^{\mathbb{N}} \setminus h^{-1}(W)$ supports no shift invariant measures, which give positive weight to any open subset.

If ν is a Gibbs measure for the shift σ (see the next section), then Condition (H3) allows one to transfer it via the conjugacy map h to a measure, which gives full weight to the base W and is invariant under the induced map F.

Inducing schemes satisfying Conditions (H1)–(H3) can be constructed for many one-dimensional maps (including some unimodal and multimodal maps) and some multidimensional expanding maps. In these cases one has $f^{\tau(J)}(J) = W$. However, in the case of (nonuniformly) hyperbolic dynamical systems admitting tower constructions Condition (H1) holds with $f^{\tau(J)}(J)$ strictly inside of W and Condition (H2) should be replaced with the following one:

(H2') the partition $\mathcal{R}$ of W induced by the sets $J \in S$ is "two-sided" generating: for any countable collection of elements $\{J_k\}_{k \in \mathbb{Z}}$, the intersection
$$\overline{J_0} \cap \Big(\bigcap_{k \geq 1} f^{-T_k}(\overline{J_k}) \Big) \cap \Big(\bigcap_{k \leq -1} f^{T'_k}(\overline{J_k}) \Big)$$
is nonempty and consists of a single point, where
$$T_k = \sum_{m=0}^{k-1} \tau(J_m), \quad T'_k = \sum_{m=k+1}^{0} \tau(J_m).$$

In this case the induced map $F : W \to W$ is conjugate to the two-sided Bernoulli shift σ on a countable set of states S.

3. Thermodynamics associated with inducing schemes

Following [19, 18] we describe a class of potential functions $\varphi : I \to \mathbb{R}$, which admit unique equilibrium measures, i.e. for which the supremum (1) is achieved with respect to a certain class of invariant measures. For simplicity we consider inducing schemes satisfying Conditions (H1)–(H3) and we always assume that the topological entropy $h(f) < \infty$.

3.1. **Thermodynamics of the induced map.** We begin with a construction of equilibrium measures for the induced map F. Given a continuous function $\phi: W \to \mathbb{R}$, define its *n-variation* by

$$V_n(\phi) = \sup_{[b_1,\dots,b_n]} \sup_{x,x' \in [b_1,\dots,b_n]} \{|\phi(x) - \phi(x')|\},$$

where

$$[b_1, \dots, b_n] = \overline{J_{b_1}} \cap f^{-\tau(J_{b_1})}(\overline{J_{b_2}}) \cap \dots \cap f^{-\hat{T}}(\overline{J_{b_n}}),$$

is the *cylinder set* and

$$\hat{T} = \sum_{m=1}^{n-1} \tau(J_{b_m}).$$

Further, we define the *Gurevich pressure* of ϕ by

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{F^n(x)=x} \exp \phi_n(x) 1_{[b]}(x),$$

where $b \in S$ and

$$\phi_n(x) = \sum_{k=0}^{n-1} \phi(F^k(x)).$$

One can show that provided $\sum_{n\geq 2} V_n(\phi) < \infty$, the above limit exists and is independent of the choice of the element b.

We call a measure $\nu = \nu_\phi$ a *Gibbs measure* for ϕ if there exist constants C_1 and C_2 such that for any cylinder set $[b_1, \dots, b_n]$ and any $x \in [b_1, \dots, b_n]$ we have

$$C_1 \leq \frac{\nu([b_1, \dots, b_n])}{\exp(-nP_G(\phi) + \phi_n(x))} \leq C_2. \tag{4}$$

Denote by $\mathcal{M}(F, W)$ the class of all F-invariant Borel probability measures on W and by

$$\mathcal{M}_\phi(F, W) = \{\nu \in \mathcal{M}(F, W) : -\int_W \phi \, d\nu < \infty\}.$$

We call an F-invariant measure $\nu = \nu_\phi$ an *equilibrium measure for* ϕ if

$$\sup_{\nu \in \mathcal{M}_\phi(F,W)} \{h_\nu(F) + \int_W \phi \, d\nu\} = h_{\nu_\phi}(F) + \int_W \phi \, d\nu_\phi. \tag{5}$$

The following result establishes the variational principle, existence, and uniqueness of equilibrium measures for the induced map. In view of Proposition 2.1 and Condition (H3), it follows immediately from the corresponding results for the Bernoulli shift on a countable set of states (see for example, [21]).

Theorem 3.1 (see [19]). *The following statements hold.*

(1) *Assume that* $\sup_{x\in W}\phi < \infty$ *and that* ϕ *has summable variations, i.e.,*

$$\sum_{n\geq 1} V_n(\phi) < \infty. \tag{6}$$

Then the variational principle for ϕ *holds:*

$$P_G(\phi) = \sup_{\nu\in\mathcal{M}_\phi(F,W)} \{h_\nu(F) + \int_W \phi\, d\nu\}.$$

(2) *Assume that* $\sup_{x\in W}\phi < \infty$, $P_G(\phi) < \infty$ *and* (6) *holds. Then there exists an ergodic* F*-invariant Gibbs measure* ν_ϕ *for* ϕ*. If in addition, the entropy* $h_{\nu_\phi}(F) < \infty$*, then* $\nu_\phi \in \mathcal{M}_\phi(F,W)$ *and is a unique Gibbs and equilibrium measure for* φ*.*

3.2. **Lifted and induced measures. Abramov's and Kac's formulas.** Our next step is to describe some relations between invariant measures for f and those for F. For a Borel probability measure ν on W set

$$Q_\nu = \sum_{J\in S} \tau(J)\,\nu(J) = \int_W \tau(x)\, d\nu(x).$$

Define the measure $\pi(\nu)$ on the set X (see (2)) as follows: for any Borel subset $E \subset X$,

$$\pi(\nu)(E) = \frac{1}{Q_\nu} \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E)\cap J).$$

For the following result see, for example [11].

Proposition 3.2. *Let* $\nu \in \mathcal{M}(F,W)$ *and* $Q_\nu < \infty$*. Then* $\pi(\nu) \in \mathcal{M}(f,I)$*,* $\pi(\nu)(X) = 1$*, and* $\pi(\nu)|_W \ll \nu$*. If* ν *is ergodic, so is* $\pi(\nu)$*.*

Consider a Borel function $\varphi : I \to \mathbb{R}$ that we call a *potential function.* Define the *induced potential function* $\bar{\varphi} : W \to \mathbb{R}$ by

$$\bar{\varphi}(x) = \sum_{k=0}^{\tau(J)-1} \varphi(f^k(x)), \quad x \in J. \tag{7}$$

Although the function φ may not be continuous we shall require that the induced potential function $\bar{\varphi}$ is continuous in the topology of W.

The induced map F may *not* be the first return map, however, Abramov's formula, connecting the entropies of F and f, and Kac's formula, connecting the integrals of φ and $\overline{\varphi}$, still hold (see [19, 27], for related results see also Keller [15]).

Theorem 3.3. *Let $\nu \in \mathcal{M}(F, W)$. If $Q_\nu < \infty$ then*

$$h_\nu(F) = Q_\nu h_{\pi(\nu)}(f) < \infty.$$

If, in addition, $\int_W \bar{\varphi} d\nu$ is finite then

$$-\infty < \int_W \bar{\varphi}\, d\nu = Q_\nu \int_X \varphi\, d\pi(\nu) < \infty.$$

We call a measure $\mu \in \mathcal{M}(f, I)$ *liftable* if $\mu(W) > 0$ and there exists a measure $i(\mu) \in \mathcal{M}(F, W)$ such that $i(\mu) \ll \mu$ and $\mu = \pi(i(\mu))$. We call $i(\mu)$ the *induced measure* for μ. The following result is proved in [27].

Proposition 3.4. *For any liftable ergodic measure $\mu \in \mathcal{M}(f, I)$ the measure $i(\mu)$ is unique, ergodic, and $Q_{i(\mu)} < \infty$.*

In Section 4 we describe some conditions on the measure that guarantee its liftability.

3.3. **Equilibrium measures for the original map.** We shall now proceed with a description of the thermodynamical formalism for the original map f. Denote by $\mathcal{M}_L(f, X)$ the class of all liftable measures. Given a potential function φ, we call a measure μ_φ an *equilibrium measure* (with respect to the class of measures $\mathcal{M}_L(f, X)$) if

$$s_\varphi = \sup_{\mathcal{M}_L(f,X)} \{h_\mu(f) + \int_X \varphi\, d\mu\} = h_{\mu_\varphi}(f) + \int_X \varphi d\, \mu_\varphi.$$

We stress that the definition of equilibrium measure introduced here differs from the classical one as only measures in $\mathcal{M}_L(f, X)$ are allowed.

Let $\bar{\varphi}$ be the induced potential function (see (7)).

Theorem 3.5 (see [19, 18])**.** *Assume that $\bar{\varphi}$ is continuous, has summable variations (see* (6)*) and finite Gurevich pressure. Then*

$$-\infty < s_\varphi < \max\{0, P_G(\bar{\varphi})\} < \infty.$$

Consider the induced potential function for the *normalized* potential $\varphi - s_\varphi$, i.e., $\varphi^+ = \overline{\varphi - s_\varphi} = \bar{\varphi} - s_\varphi \tau$.

As an immediate corollary of Theorem 3.1 we obtain the following result.

Theorem 3.6 (see [19, 18])**.** *Assume that the induced potential function $\bar{\varphi}$ is continuous, has summable variations (see* (6)*) and finite Gurevich pressure. Assume also that the function φ^+ has finite Gurevich pressure and $\sup_{x\in W} \varphi^+ < \infty$. Then*

(1) *there exists an F-invariant ergodic Gibbs measure ν_{φ^+} for φ^+;*

(2) *if* $Q_{\nu_{\varphi^+}} < \infty$*, then* $\nu_{\varphi^+} \in \mathcal{M}_{\varphi^+}(F, W)$ *and*

$$\sup_{\nu \in \mathcal{M}_{\varphi^+}(F,W)} \{h_\nu(F) + \int_W \varphi^+ d\nu\} = h_{\nu_{\varphi^+}}(F) + \int_W \varphi^+ d\nu_{\varphi^+}.$$

(3) *if* $Q_{\nu_{\varphi^+}} < \infty$*, then* $\mu_\varphi = \pi(\nu_{\varphi^+}) \in \mathcal{M}_L(f, X)$*.*

The measure μ_φ is a *natural candidate* for the equilibrium measure for φ. It is indeed the equilibrium measure for φ provided the following *positive recurrence* condition holds: there exist $\varepsilon_0 > 0$ such that for any $0 \le \varepsilon < \varepsilon_0$ the function

$$\varphi^+_\varepsilon := \overline{\varphi - s_\varphi + \varepsilon} = \varphi^+ + \varepsilon\tau$$

has finite Gurevich pressure.

Theorem 3.7 (see [19, 18])**.** *Assume that* $\bar\varphi$ *is continuous, has summable variations (see* (6)*), finite Gurevich pressure, and is positively recurrent. Assume also that* $\sup_{x\in W} \varphi^+ < \infty$ *and* $Q_{\nu_{\varphi^+}} < \infty$*. Then* $\mu_\varphi = \pi(\nu_{\varphi^+})$ *is the unique equilibrium measure for* φ*, i.e.,*

$$s_\varphi = h_{\mu_\varphi}(f) + \int_X \varphi\, d\mu_\varphi = \sup_{\mathcal{M}_L(f,X)} \{h_\mu(f) + \int_X \varphi\, d\mu\}. \tag{8}$$

Verifying conditions of this theorem may not be an easy task and in [18] some stronger requirements on the potential function φ are given that ensure existence and uniqueness of equilibrium measures. For the sake of completeness we shall briefly describe these requirements.

We say that the induced potential function $\bar\varphi$ is *locally Hölder continuous* if there exists $A > 0$ and $0 < \gamma < 1$ such that

$$V_n(\bar\varphi) \le A\gamma^n, \quad n \ge 1.$$

If $\bar\varphi$ is locally Hölder continuous then it has summable variations.

We say that the potential function φ satisfies:

(1) the (FGP)*-condition* if

$$\sum_{J\in S} \sup_{x\in J} \exp \bar\varphi(x) < \infty;$$

(2) the (INT)*-condition* if

$$\sum_{J\in S} \tau(J) \sup_{x\in J} \exp(\varphi^+(x)) < \infty.$$

Theorem 3.8 (see [19])**.** *Assume that the potential function* φ *satisfies:*

(1) *the (FGP)-condition, then* $\bar\varphi$ *has finite Gurevich pressure;*
(2) *the (INT)-condition, then the function* φ^+ *satisfies the (FGP)-condition, and* $\sup_{x\in W} \varphi^+ < \infty$*;*

(3) *the (INT)-conditions, that $\bar{\varphi}$ is locally Hölder continuous and has finite Gurevich pressure, then $Q_{\nu_{\varphi^+}} < \infty$.*

As we saw above Theorem 3.7 establishes existence and uniqueness of equilibrium measures within the class of liftable measures $\mathcal{M}_L(f, X)$. If we allow the class of all f-invariant ergodic Borel probability measures $\mathcal{M}(f, I)$, then depending on the potential function φ the equilibrium measure μ_φ may be non-liftable or be supported outside of the tower, i.e, $\mu_\varphi(X) = 0$.

In [20], an example of a one-dimensional map of a compact interval is given, which admits an inducing scheme $\{S, \tau\}$, satisfying Conditions (H1) and (H2), and a potential function φ such that there exists a unique equilibrium measure μ_φ for φ (with respect to the class of measures $\mathcal{M}(f, I)$) with $\mu_\varphi(X) = 0$ (see also Section 5).

4. The liftability property

In this section we shall discuss various aspects of and present some general results on the liftability property of invariant measures.

The class of liftable measures $\mathcal{M}_L(f, X)$ depends on the choice of the inducing scheme $\{S, \tau\}$. In this regards the following problem is of interest:

Given a map f, construct an "optimal" inducing scheme with respect to which: (1) every f-invariant Borel probability measure is liftable; (2) if φ is a potential function satisfying the conditions of Theorem 3.7, then the unique equilibrium measure μ_φ for φ is also the unique equilibrium measure with respect to the class $\mathcal{M}(f, I)$ of all invariant Borel probability measures on I.

For one-dimensional maps Hofbauer and Keller constructed a different type of inducing schemes (see [13, 12, 15]). In the case of one-dimensional S-unimodal maps satisfying the Collet-Eckmann condition the liftability problem has an affirmative solution obtained by Keller [15], i.e. every invariant ergodic Borel measure with positive entropy is liftable. Furthermore, building on a recent result by Bruin [6], Pesin and Senti [19] have shown that for any unimodal map satisfying the Collet-Eckmann condition every measure $\mu \in \mathcal{M}(f, I)$ with positive metric entropy is liftable. A similar result is obtained for some multimodal maps (see [19, 17]). In this paper we do not consider the liftability property for particular classes of dynamical systems but rather discuss liftability in a general setting. We shall also present an example showing that nonliftable equilibrium measures do exist.

If the inducing time τ is the first return time to the base by Kac's theorem every f-invariant measure μ has integrable inducing time and hence, it is liftable. However, for many inducing schemes to satisfy Condition (H2) the inducing time may to be chosen bigger than the first return time and the liftability problem becomes nontrivial. Furthermore, by artificially increasing the inducing time, one may produce inducing schemes with nonliftable measures (see Example 5.2).

4.1. **Characterizations of liftability.** We describe the liftability property in terms of the abstract representation of f via the tower construction. We shall exploit the techniques from [15] and [27]. Set

$$\tilde{I} = \{(x, k) : x \in J,\ k = 0, \cdots, \tau(J) - 1,\ J \in S\},$$

and define the map $\tilde{f} : \tilde{I} \to \tilde{I}$ by

$$\tilde{f}(x, k) = \begin{cases} (x, k+1), & x \in J,\ 0 \le k < \tau(J) - 1 \\ (f^{\tau(J)}(x), 0), & x \in J,\ k = \tau(J) - 1 \end{cases}.$$

In what follows we shall denote by $\mathscr{B}$ the Borel σ-algebra in the corresponding measure space. We define the projection map $\tilde{p} : \tilde{I} \to X$ by $\tilde{p}(x, k) = f^k(x)$ whenever $x \in J$. It is easy to see that $f \circ \tilde{p} = \tilde{p} \circ \tilde{f}$. We also use the notations: $\tilde{x} = (x, k)$, $J(\tilde{x}) = J \in S$ if $x \in J$ and $\tau(\tilde{x}) = \tau(J(\tilde{x}))$.

Liftability of an f-invariant measure μ is equivalent to its liftability to the abstract tower $\tilde{I}$. More precisely, the following statement holds. Define the set $\tilde{W} \subset \tilde{I}$ by $\tilde{W} = \{(x, 0) : x \in W\}$.

Proposition 4.1 (see [27])**.** *For an f-invariant Borel probability measure μ on X there exists an induced measure ν on W such that $\pi(\nu) = \mu$ if and only if there exists a $\tilde{f}$-invariant finite Borel probability measure $\tilde{\mu}$ on $\tilde{I}$ for which $\mu = \tilde{\mu} \circ \tilde{p}^{-1}$. In this case, $\nu = \tilde{\mu}|_{\tilde{W}} \circ \tilde{p}^{-1}$.*

Let $\mu_0 = \mu|_W$. Define a measure $\tilde{\mu}_0$ on $\tilde{I}$ by setting $\tilde{\mu}_0(E, k) = \mu_0(E)$. Note that $\mu = \tilde{\mu} \circ \tilde{p}^{-1}$ implies that $\tilde{\mu} \ll \tilde{\mu}_0$ and hence, one can try to construct the lift of μ using its density with respect to $\tilde{\mu}_0$.

Define the operator $\tilde{\mathcal{L}} : L^1(\tilde{I}, \mathscr{B}, \tilde{\mu}_0) \to L^1(\tilde{I}, \mathscr{B}, \tilde{\mu}_0)$ by the following relation

$$\int (\tilde{\mathcal{L}} g_1) g_2 \, d\tilde{\mu}_0 = \int g_1 (g_2 \circ f) \, d\tilde{\mu}_0$$

for all $g_1 \in L^1(\tilde{I}, \mathscr{B}, \tilde{\mu}_0)$ and $g_2 \in L^\infty(\tilde{I}, \mathscr{B}, \tilde{\mu}_0)$.

We say that a subset $\tilde{E} \subset \tilde{I}$ is bounded if the inducing time of any point in $\tilde{E}$ is bounded, i.e. there exists $N \in \mathbb{N}$ such that $\tau(x) \le N$ for all $(x, k) \in \tilde{E}$.

Consider the sequence of functions

$$\tilde{\mathcal{A}}_n = \frac{1}{n}\sum_{k=0}^{n-1} \tilde{\mathcal{L}}^k 1_{\tilde{W}}.$$

There exists an increasing sequence of numbers $\{n_k\}$ and a function $\tilde{h} : \tilde{I} \to [0,1]$ such that for any bounded set $\tilde{E}$, we have

$$\tilde{\mathcal{A}}_{n_k}|_{\tilde{E}} \xrightarrow{w} \tilde{h}|_{\tilde{E}}.$$

Proposition 4.2 (see [27]). *There exist a lift $\tilde{\mu}$ for μ if and only if $\tilde{h}$ is not identically 0 for $\tilde{\mu}_0$-almost every point. In this case, $\tilde{\mu} = \tilde{h}d\tilde{\mu}_0$ is the lift of μ to the tower.*

4.2. **Criteria for liftability.** Using the above result we shall obtain the following criterion for liftability. Given a Borel set $A \subset X$ and $J \in S$, define

$$\epsilon(J, A) = \frac{1}{\tau(J)}\mathrm{Card}\{0 \le k \le \tau(J) - 1; \quad f^k(J) \cap A \ne \emptyset\},$$

where Card E denotes the cardinality of the set E.

Theorem 4.3. *For any f-invariant Borel ergodic probability measure μ such that $\mu(W) > 0$, if there exists a number $N \ge 0$ and a subset $A \subset I$ such that*

$$\mu(A) > \sup_{\tau(J)>N} \epsilon(J, A), \tag{9}$$

then there exists a lift $\tilde{\mu}$ for μ.

Proof. Assume that $\tilde{h} = 0$ for $\tilde{\mu}_0$-almost every point. Consider the sets $\tilde{E}_N \subset \tilde{I}$ given as

$$\tilde{E}_N = \{\tilde{x} \in \tilde{I} : \tau(\tilde{x}) \le N\}.$$

Clearly, $\tilde{E}_N$ are bounded sets and we have

$$\begin{aligned}
0 &= \int_{\tilde{E}_N} \tilde{h}\, d\tilde{\mu}_0(\tilde{x}) = \lim_{k\to\infty} \int_{\tilde{E}_N} \tilde{\mathcal{A}}_{n_k} d\tilde{\mu}_0(\tilde{x}) \\
&= \lim_{k\to\infty} \int_{\tilde{I}} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \tilde{\mathcal{L}}^j 1_{\tilde{W}} 1_{\tilde{E}_N} d\tilde{\mu}_0(\tilde{x}) \\
&= \lim_{k\to\infty} \int_{\tilde{I}} \frac{1}{n_k} \sum_{j=0}^{n_k-1} 1_{\tilde{W}} 1_{f^{-j}(\tilde{E}_N)} d\tilde{\mu}_0(\tilde{x}) \\
&= \lim_{k\to\infty} \int_{\tilde{W}} \delta_k^N(\tilde{x}) d\tilde{\mu}_0(\tilde{x}), \qquad (10)
\end{aligned}$$

where for $\tilde{x} \in \tilde{W}$,

$$\delta_k^N(\tilde{x}) = \frac{1}{n_k}\mathrm{Card}\{0 \leq j \leq n_k - 1 :\ \tilde{f}^j(\tilde{x}) \in \tilde{E}_N\}.$$

It follows that the sequence δ_k^N converges to zero in measure (with respect to $\tilde{\mu}_0$) as $k \to \infty$ and hence, by passing to a proper subsequence if necessary, we may assume that it converges $\tilde{\mu}_0$-almost everywhere. Define:

$$\begin{aligned} A_n^N(x) = \{0 \leq i \leq n-1 :\ \tilde{f}^i((x,0)) \notin \tilde{E}_N\} \\ = \{0 \leq i \leq n-1 :\ \tau(\tilde{f}^i((x,0))) > N\}. \end{aligned}$$

It follows from (10) that for μ-almost every x,

$$\lim_{k\to\infty} \frac{\mathrm{Card}A_{n_k}^N(x)}{n_k} = 1. \tag{11}$$

By Birkhoff's Ergodic Theorem, the following limit

$$r(x) := \lim_{n\to\infty} \frac{1}{n}\mathrm{Card}\{0 \leq i \leq n-1, \quad f^i(x) \in A\} \tag{12}$$

exists almost everywhere and is equal to $\mu(A)$.

Fix some x such that both (11) and (12) hold. Let $x_0 = x$, $\tau_0 = \tau(x)$ and define $x_n = f^{\tau_{n-1}}(x_{n-1})$, $\tau_n = \tau(x_n)$ for $n \geq 0$. Note that we can rewrite the trajectory of x in the following form

$$x_0, \ldots, f^{\tau_0 - 1}(x_0), x_1, \ldots, f^{\tau_1}(x_1), \ldots$$

We also set $m_0 = 0$ and $m_j = \sum_{i=0}^{j-1} \tau_i$. For any k, there exists $l(k)$ such that

$$m_{l(k)-1} < n_k \leq m_{l(k)}.$$

Lemma 4.4. *We have that*

$$\lim_{k\to\infty} \frac{m_{l(k)} - CardA_{m_{l(k)}}^N(x)}{m_{l(k)}} = 0.$$

Proof. Note that the inducing time of the point $f^{m_{l(k)-1}}(x)$ is $\tau_{l(k)-1} = m_{l(k)} - m_{l(k)-1}$.

If $\tau_{l(k)-1} > N$ the point $\tilde{f}^j(x,0)$ for $j \in \{m_{l(k)-1}, \ldots, m_{l(k)} - 1\}$ has inducing time larger than N and therefore $\{m_{l(k)-1}, \ldots, m_{l(k)} - 1\} \subset A_{m_{l(k)}}^N$ and $\{m_{l(k)-1}, \ldots, n_k - 1\} \subset A_{n_k}^N$. It follows that

$$\{0, \ldots, m_{l(k)} - 1\} \setminus A_{m_{l(k)}}^N = \{0, \ldots, n_k - 1\} \setminus A_{n_k}^N,$$

and hence,

$$\frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} = \frac{n_k - \mathrm{Card}A^N_{n_k}}{m_{l(k)}} \le \frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k}.$$

If $\tau_{l(k)-1} \le N$, we have

$$\frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} \le \frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{n_k} \le \frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k} + \frac{N}{n_k}.$$

In either case we obtain

$$0 \le \lim_{k\to\infty} \frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} \le \lim_{k\to\infty} \left(\frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k} + \frac{N}{n_k} \right) = 0,$$

where the last equality follows from (11). $\square$

By Lemma 4.4, we have

$$\begin{aligned} r(x) &= \lim_{k\to\infty} \frac{1}{m_{l(k)}} \mathrm{Card}\{0 \le i \le m_{l(k)} - 1, \quad f^i(x) \in A\} \\ &= \lim_{k\to\infty} \frac{1}{\mathrm{Card}A^N_{m_{l(k)}}} \mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\}. \end{aligned}$$

Finally, we claim that

$$\frac{1}{\mathrm{Card}A^N_{m_{l(k)}}} \mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\} \le \sup_{\tau(J)>N} \epsilon(J, A).$$

Observe that $A^N_{m_{l(k)}}$ is the set of those $0 \le i \le m_{l(k)} - 1$ for which $\tau(\tilde{f}^i(x, 0)) > N$. It follows that

$$A^N_{m_{l(k)}} = \bigcup_{j=0,\, \tau_j>N}^{l(k)-1} \{m_j, \cdots, m_{j+1} - 1\}.$$

This implies that

$$\begin{aligned}
&\frac{1}{\mathrm{Card} A^N_{m_{l(k)}}} \mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\} \\
&= \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{m_j \le i \le m_{j+1}-1,\ f^i x \in A\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&= \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{0 \le i \le \tau_j-1,\ f^i(x_j) \in A\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&\le \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{0 \le i \le \tau_j-1,\ f^i(J(x_j)) \cap A \neq \emptyset\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&= \frac{1}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j \epsilon(J(x_j), A) \\
&\le \sup_{0\le j\le l(k)-1,\,\tau_j>N} \epsilon(J(x_j), N) \le \sup_{\tau(J)>N} \epsilon(J, A).
\end{aligned}$$

It follows that

$$\mu(A) = r(x) \le \sup_{\tau(J)>N} \epsilon(J, A)$$

and it contradicts to our assumption. □

As immediate corollaries of the above result we obtain the following statements.

Corollary 4.5. *Assume that*

$$\sup_{\tau(J)>N} \epsilon(J, W) \to 0$$

as $N \to \infty$. Then every measure μ with $\mu(W) > 0$ is liftable.

Corollary 4.6. *Assume that the inducing time τ is the nth return time to the base, i.e. for $x \in J$,*

$$\tau(x) = \sum_{j=1}^{n} \rho(f^j(x))$$

(where ρ is the first return time to the base). Then any invariant measure μ with $\mu(W) > 0$ is liftable.

Proof. It is easy to see that

$$\sup_{\tau(J)>N} \epsilon(J, W) = \frac{n}{N+1} \to 0 \quad \text{as } N \to \infty$$

and Corollary 4.5 applies. □

A substantial generalization of the last result is the criterion for liftability due to Zweimüller, see [27].

Proposition 4.7. *An f-invariant Borel probability measure μ is liftable provided that $\int_X \tau d\mu < \infty$.*

5. Examples

Example 5.1.

We shall construct an example of a one-dimensional map of a compact interval possessing inducing schemes, which illustrate various phenomena associated with the liftability property. Namely,

1) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f, X)$), which has integrable inducing time;

2) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f, X)$), which is liftable but has non-integrable inducing time; in fact, all the invariant ergodic measures for the map in the example are liftable;

3) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f, I)$), which is supported outside the tower.

Our example is built upon a construction by Zweimüller [27] for abstract towers.

The map f is defined on the unit interval I. Set $I^{(1)} = [0, \frac{1}{2}]$, $I^{(2)} = (\frac{1}{2}, 1]$. We choose f such that:

- it is continuous on I;
- it maps $I^{(1)}$ diffeomorphically onto $[0, 1]$ and it maps $I^{(2)}$ diffeomorphically onto $(0, 1]$;
- $|f'(x)| > a > 1$ for $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$;
- $f(0) = 0$, $f(\frac{1}{2}) = 1$.

Let S be the set of intervals I_n such that $I_0 = I^{(2)}$ and $I_n = f^{-1}(I_{n-1}) \cap I^{(1)}$ for $n \geq 1$. Let also $\tau(I_n) = n+1, n \geq 0$. Then $\{S, \tau\}$ is an inducing scheme for f. As a result we have the map $f : X \to X$ (see (2)) and the induced map $F = f^\tau : W \to W$. Note that both X and W differ from I by a countable set.

For $\alpha > 1$, define

$$p_0(\alpha) = \frac{1}{1 + \sum_{k\geq 1} k^{-(1+\alpha)}}$$

and

$$p_n(\alpha) = \frac{n^{-(1+\alpha)}}{1 + \sum_{k\geq 1} k^{-(1+\alpha)}}, \quad n \geq 1,$$

and the potential function $\varphi_\alpha : I \to \mathbb{R}$ given as follows:

$$\varphi_\alpha(x) = \begin{cases} \log p_0(\alpha), & x \in I_0 \\ \log \frac{p_n(\alpha)}{p_{n-1}(\alpha)}, & x \in I_n,\, n \geq 1. \end{cases}$$

Although the function $\varphi_\alpha(x)$ is not continuous on I the induced potential function $\bar{\varphi}_\alpha(x)$ is continuous on W. We claim that the following statements hold:

(1) *For the above inducing scheme $\{S, \tau\}$ every measure in $\mathcal{M}(f, I)$ with $\mu(W) > 0$ is liftable, and there exists a unique equilibrium measure μ_α for φ_α with respect to $\mathcal{M}(f, X)$.*
(2) *For the potential φ_α with $\alpha > 2$, the equilibrium measure μ_α has integrable inducing time.*
(3) *For $1 < \alpha \leq 2$ the inducing time is not integrable with respect to the equilibrium measure μ_α.*
(4) *Suppose f is $C^{1+\epsilon}$ on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, and set*

$$\psi(x) = \begin{cases} -\log |f'(x)|, & x \neq 0, \frac{1}{2} \\ a_0, & x = 0 \\ a_1, & x = 1/2 \end{cases}.$$

Then for a_0 large enough, the Dirac measure at 0 is the equilibrium measure for ψ (note that this measure is supported outside the tower as $0 \in I \setminus X$).

To prove the first statement observe that for any interval I_n,

$$\epsilon(I_n, I_0) = \frac{1}{n+1}.$$

Hence,

$$\sup_{\tau(I_n)\geq N} \{\epsilon(I_n, I_0)\} = \frac{1}{N} \to 0$$

as $N \to \infty$. It is easy to see that any invariant Borel probability measure μ on X must have $\mu(I_0) > 0$ and hence, by Theorem 4.3, every measure μ with $\mu(W) > 0$ is liftable.

To establish existence and uniqueness of equilibrium measures observe that the partition $\{I_n\}$ is a countable Markov partition for the map f on the tower, so that f is topologically conjugate to a subshift of countable type via the coding map $h_r : \Sigma_A \to X$. Here the transition matrix A is such that $a_{0,n} = 1$, $n \geq 0$ and $a_{n,n-1} = 1$, $n \geq 1$ while $a_{ij} = 0$ in all other cases. The subshift of countable type σ_r given by this matrix, is called the *renewal shift*. We also have that the induced map F is topologically conjugate to the full countable Bernoulli shift σ via the coding map $h_i : S^{\mathbb{N}} \to W$.

We start by considering the Bernoulli measure κ on $S^{\mathbb{N}}$ given by $\kappa_\alpha([n]) = p_n(\alpha)$. Note that $\sum_{n\geq 0} p_n(\alpha) = 1$ and that κ is invariant under the shift map σ. The measure $\nu_\alpha = (h_i)_*\kappa_\alpha$ is invariant under the induced map F. Since

$$Q_{\nu_\alpha} = 1 + \sum_{k\geq 1} kp_k(\alpha) < \infty,$$

we can consider $\mu_\alpha = \pi(\nu_\alpha)$, which is a Markov measure on the shift space Σ_A.

We claim that $\int_X \tau d\mu_\alpha < \infty$ when $\alpha > 2$ and $\int_X \tau d\mu_\alpha = \infty$ when $1 < \alpha \leq 2$. In fact,

$$\mu_\alpha(I_i) = \frac{1}{Q_{\nu_\alpha}} \sum_{k\geq i} p_k(\alpha) = \frac{\sum_{k\geq i} p_k(\alpha)}{1 + \sum_{k\geq 1} kp_k(\alpha)}.$$

Furthermore,

$$\int_X \tau d\mu_\alpha = \sum_{k\geq 0}(1+k)\mu_\alpha(I_k) = 1+\frac{1}{Q_{\nu_\alpha}} \sum_{k\geq 1} \frac{k(k-1)}{2} p_k(\alpha) \asymp \sum_{k\geq 1} k^2 p_k(\alpha)$$

and the desired claim follows.

We shall now show that the measure μ_α is the unique equilibrium measure for φ_α. Observe that the induced potential

$$\overline{\varphi}_\alpha(x) = \sum_{k=0}^{n-1} \varphi_\alpha(f^k(x)) = \log p_n(\alpha), \quad x \in I_n.$$

Set $\phi_\alpha = \varphi_\alpha \circ h_r$ and $\overline{\phi_\alpha} = \overline{\varphi}_\alpha \circ h_i$. It suffices to show that $\mu_\alpha \circ h_r$ is the unique equilibrium measure for ϕ_α. We fix α and in what follows we simplify our notations by writing $\phi = \phi_\alpha$, $\varphi = \varphi_\alpha$, $p_n = p_n(\alpha)$, etc.

To obtain our result we shall apply some methods in the theory of countable Markov shifts, see for example, [22, 9]. Observe that $V_n(\phi) = 0$ for any n. Define

$$Z_0(\phi, [0]) = 1, \quad Z_n(\phi, [0]) = \sum_{\sigma^n x = x} \exp \phi_n(x) 1_{[0]}(x)$$

and

$$Z_n^*(\phi, [0]) = \sum_{\sigma^n x = x, \rho(x) = n} \exp \phi_n(x) 1_{[0]}(x),$$

where $\phi_n(x) = \sum_{k=0}^{n-1} \phi(\sigma^k x)$ and $\rho(x)$ is the first return time to $[0]$. Note that

$$P_G(\phi) = P_G(\phi, [0]) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, [0]).$$

We first show that $P_G(\phi) = 0$. Set

$$T(z) = \sum_{n\geq 0} z^n Z_n(\phi, [0]), \quad R(z) = \sum_{n\geq 1} z^n Z_n^*(\phi, [0]).$$

The fact that $V_n(\phi) = 0$ for all n implies that

$$\begin{aligned} Z_n(\phi, [0]) = Z_1^*(\phi, [0]) Z_{n-1}(\phi, [0]) + \cdots \\ + Z_{n-1}^*(\phi, [0]) Z_1(\phi, [0]) + Z_n^*(\phi, [0]). \end{aligned}$$

This in turn implies the following equation of formal power series known as the *renewal equation*:

$$T(z) = \frac{1}{1 - R(z)}.$$

Observe that $\exp P_G(\phi, [0])$ is exactly the radius of convergence for $T(z)$, and the latter is exactly the solution of the equation $R(z) = 1$. Notice that $Z_n^*(\phi, [0]) = p_{n-1}$, and the fact that $\sum_{k\geq 0} p_k = 1$ implies that $R(1) = 1$. We conclude that $P_G(\phi) = 0$.

By the variational principle (see Theorem 3.1), to show that $\mu \circ h_r$ is a equilibrium measure, it suffices to show that

$$h_{\mu\circ h_r}(\sigma_r) + \int_{\Sigma_A} \phi \, d(\mu \circ h_r) = 0. \tag{13}$$

In fact, if ν is the induced measure for μ, by Abramov's and Kac's formulas (see Proposition 3.3),

$$h_\mu(f) + \int_X \varphi \, d\mu = \frac{1}{Q_\nu}\Big(h_\nu(F) + \int_W \overline{\varphi} \, d\nu\Big).$$

Since $\nu \circ h_i = \kappa$ is a Bernoulli measure, we find that

$$h_\nu(F) = -\sum_{k\geq 0} p_k \log p_k, \quad \int_W \overline{\varphi} d\nu = \sum_{k\geq 0} p_k \log p_k.$$

It follows that

$$h_\mu(f) + \int_X \varphi \, d\mu = \frac{1}{Q_\nu}\Big(h_\nu(F) + \int_W \overline{\varphi} \, d\nu\Big) = 0.$$

This implies (13). We conclude that $\mu \circ h_r$ is an equilibrium measure for ϕ on Σ_A. As the potential ϕ has summable variations, is bounded from above, and has finite Gurevich pressure, Theorem 1.1 from [9] implies that the equilibrium measure is unique. By the topological conjugacy, μ is the unique equilibrium measure for φ on X.

To prove the last statement note that $0 \notin X$. Pick any

$$a_0 > \sup_{\mu \in \mathcal{M}(f,X)} \{h_\mu(f) + \int_X \psi d\mu\}$$

(observe that the function ψ is bounded from above and hence, the supremum is finite). Since $\mathcal{M}(f,X)$ only contains measures supported on X, it is clear that in this case the Dirac measure at 0 is the equilibrium measure among all the measures supported on I.

Example 5.2.

We describe an example due to Bruin (oral communication) that shows that there are inducing schemes which allow nonliftable measures. Indeed, we show that such a measure can be a unique equilibrium measures for an appropriately chosen potential function. A similar construction was used by Zweimüller [27].

Consider the map $f = 2x \pmod 1$ of the unit interval I and the countable partition of I by intervals I_n constructed in the previous example. This partition codes f into the renewal shift. Now subdivide any interval I_n into 2^{2^n} intervals of equal length and call them I_n^j, $j = 1, \dots, 2^{2^n}$. Define the inducing time $\tau(I_n^j) = 2^n + n$. We claim that the Lebesgue measure λ, which is invariant under f, is not liftable to this inducing scheme. Observe that λ is an equilibrium measure for f (for the potential $\varphi = 0$). We have that

$$\int_I \tau \, d\lambda = \sum_{n,j} \tau(I_n^j)\lambda(I_n^j) = \sum_n (2^n + n)\lambda(I_n) = \infty.$$

To show the λ is not liftable let us note that any induced measure $i(\lambda)$ of λ must be F-invariant and absolute continuous with respect to λ. Since in our case $W = X$ and the measure λ is F-invariant and ergodic, λ itself is the only candidate, i.e., $i(\lambda) = \lambda$. However, this is impossible as $\int_I \tau \, d\lambda = \infty$. Hence, λ is not liftable.

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