Introduction to Smooth Ergodic Theory

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Preface

This book is a revised and considerably expanded version of our book [**7**]. When the latter was published it became the only source of a systematic introduction to the core of Smooth Ergodic Theory. It included the general theory of Lyapunov exponents and its applications to stability theory of differential equations, nonuniform hyperbolicity theory, stable manifold theory (with emphasis on absolute continuity of invariant foliations), and the ergodic theory of dynamical systems with nonzero Lyapunov exponents, including geodesic flows. In the absence of other textbooks on the subject it was also used as a source or a supportive material for special topics courses on nonuniform hyperbolicity.

In 2007 we published the book [**9**], which contained an up to date exposition of Smooth Ergodic Theory and was meant as a primary reference source in the field. However, despite an impressive amount of literature in the field, there was until now no textbook containing a comprehensive introduction to the theory.

The present book is intended to cover this gap. It is aimed at graduate students specializing in dynamical systems and ergodic theory as well as anyone who wishes to acquire a working knowledge of smooth ergodic theory and to learn how to use its tools. While maintaining the essential of most of the material in [**7**], we made the book more student-oriented by carefully selecting the topics, reorganizing the material and substantially expanding the proofs of the core results. We also included a detailed description of essentially all known examples of conservative systems with nonzero Lyapunov exponents and throughout the book we added many exercises.

The book consists of two parts. While the first part introduces the reader to the basics of Smooth Ergodic Theory, the second part discusses more advanced topics. This gives the reader a broader view of the theory and may help stimulate further study. This also provides nonexperts with a broader perspective of the field.

We emphasize that the new book is self-contained. Namely, we only assume that the reader has a basic knowledge of real analysis, measure theory, differential equations, and topology and we provide the reader with necessary background definitions and state related results.

On the other hand, in view of the considerable size of the theory we were forced to make a selection of the material. As a result, some interesting topics are barely mentioned or not covered at all. We recommend the books [**9, 15**] and the surveys [**8, 58**] for a description of many other developments and some recent work. In particular, we do not consider random dynamical systems (see the books [**5, 51, 56**] and the survey [**52**]), dynamical systems with singularities, including "chaotic" billiards (see the book [**50**]), the theory of nonuniformly expanding maps (see the survey [**57**]) and one-dimensional "chaotic" maps (such as the logistic family, see [**42**]).

Smooth ergodic theory studies the ergodic properties of smooth dynamical systems on Riemannian manifolds with respect to "natural" invariant measures. Among these measures most important are smooth measures, i.e., measures that are equivalent to the Riemannian volume. There are various classes of smooth dynamical systems whose study requires different techniques. In this book we concentrate on systems whose trajectories are hyperbolic in some sense. Roughly speaking, this means that the behavior of trajectories near a given orbit resembles the behavior of trajectories near a saddle point. In particular, to every hyperbolic trajectory one can associate two complementary subspaces such that the system acts as a contraction along one of them (called the stable subspace) and as an expansion along the other (called the unstable subspace).

A hyperbolic trajectory is unstable—almost every nearby trajectory moves away from it with time. If the set of hyperbolic trajectories is sufficiently large (for example, has positive or full measure), this instability forces trajectories to become separated. On the other hand, compactness of the phase space forces them back together; the consequent unending dispersal and return of nearby trajectories is one of the hallmarks of chaos.

Indeed, hyperbolic theory provides a mathematical foundation for the paradigm that is widely known as "deterministic chaos" – the appearance of irregular chaotic motions in purely deterministic dynamical systems. This paradigm asserts that conclusions about global properties of a nonlinear dynamical system with sufficiently strong hyperbolic behavior can be deduced from studying the linearized systems along its trajectories.

The study of hyperbolic phenomena originated in the seminal work of Artin, Morse, Hedlund, and Hopf on the instability and ergodic properties of geodesic flows on compact surfaces (see the survey [**37**] for a detailed description of results obtained at this time and for references). Later, hyperbolic behavior was observed in other situations (for example, Smale horseshoes and hyperbolic toral automorphism).

The systematic study of hyperbolic dynamical systems was initiated by Smale (who mainly considered the problem of structural stability of hyperbolic systems; see [**82**]) and by Anosov and Sinai (who were mainly concerned with ergodic properties of hyperbolic systems with respect to smooth invariant measures; see [**3, 4**]). The hyperbolicity conditions describe the action of the linearized system along the stable and unstable subspaces and impose quite strong requirements on the system. The dynamical systems that satisfy these hyperbolicity conditions uniformly over all orbits are called Anosov systems.

In this book we consider the weakest (hence, most general) form of hyperbolicity known as nonuniform hyperbolicity. It was introduce and studied by Pesin in a series of papers [**67, 68, 69, 70, 71**]. The nonuniform hyperbolicity theory (which is sometimes referred to as Pesin theory) is closely related to the theory of Lyapunov exponents. The latter originated in works of Lyapunov [**59**] and Perron [**66**] and was developed further in [**23**]. We provide an extended excursion into the theory of Lyapunov exponents and in particular, introduce and study the crucial concept of Lyapunov–Perron regularity. The theory of Lyapunov exponents enables one to obtain many subtle results on stability of differential equations.

Using the language of Lyapunov exponents one can view nonuniformly hyperbolic dynamical systems as those systems where the set of points for which *all* Lyapunov exponents are nonzero is "large"—for example, has full measure with respect to an invariant Borel measure. In this case the Multiplicative Ergodic theorem of Oseledets [**65**] implies that almost every point is Lyapunov–Perron regular. The powerful theory of Lyapunov exponents then yields a profound description of the local stability of trajectories, which, in turn, serves as a ground for studying ergodic properties of these systems.

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