ENTROPY AND DIMENSION FAMILIES ASSOCIATED WITH EQUILIBRIUM MEASURES FOR HYPERBOLIC DYNAMICAL SYSTEMS

YAKOV B. PESIN

ABSTRACT. For a (one or two-sided) subshift of finite type σ and a Hölder continuous function ψ we consider the equilibrium measures $\{\mu_\beta\}_{\beta\geq 0}$ corresponding to the Hölder continuous functions $\varphi_\beta(x) = -\beta\psi$, $\beta \geq 0$. We provide a complete description of the entropy family $\mathcal{F}_E(\beta) = h_{\mu_\beta}(\sigma)$ and the dimension family $\mathcal{F}_d(\beta) = \dim_H \mu_\beta$ associated with these measures. Similar results are obtained for entropy and dimension families generated by equilibrium measures for (conformal) expanding map and (conformal) Axiom A diffeomorphisms. As a consequence we show that for a "typical" Hölder continuous function ψ the set $\{h_{\mu_\beta}(\sigma), \beta \geq 0\}$ contains all positive values of metric entropy.

1. Introduction. Entropy and Dimension Families: The Case of Subshifts of Finite Type

In this paper we study a problem which was first considered (mainly numerically) by Paladin and Vaienti in [**?**]. It deals with special entropy and dimension families associated with equilibrium measures for some hyperbolic dynamical systems.

We begin our consideration with a *one-sided subshift of finite type* (Σ_A^+, σ). Recall that Σ_A^+ is the space of one-sided sequences $\omega = (\omega_n)_{n\geq 0}$ of p symbols $\{1, \dots, p\}$ which are *admissible* with respect to the transfer matrix $A = (a_{ij})$ (whose entries $a_{ij} = 0$ or 1), i.e., $a_{\omega_n\omega_{n+1}} = 1$. σ is the *shift*, i.e., $(\sigma(\omega))_n = \omega_{n+1}$. The space Σ_A^+ is a metric space with the *standard metric*

$$d_a(\omega^{(1)}, \omega^{(2)}) = \sum_{n\geq 0} \frac{|\omega_n^{(1)} - \omega_n^{(2)}|}{a^n},$$

where $a > 1$ is a constant.

We assume that the transfer matrix A is transitive, i.e., $A^k > 0$ for some positive integer k. This ensures that the shift σ is topologically mixing.

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Let ψ be a strictly positive Hölder continuous function on Σ_A^+. Given $\beta \geq 0$, consider the Hölder continuous function on Σ_A^+ given as follows

$$\varphi_\beta(x) = -\beta\psi(x). \tag{1}$$

Since σ is topologically mixing there exists a unique *equilibrium measure* corresponding to φ_β which we denote by $\mu_\beta = \mu_{\varphi_\beta}$ (see Appendix). With the one-parameter family of measures $\{\mu_\beta\}_{\beta\geq 0}$, one can associate three functions

$$\mathcal{P}(\beta) = P_{\Sigma_A^+}(\varphi_\beta), \quad \mathcal{F}_E(\beta) = h_{\mu_\beta}(\sigma), \quad \mathcal{F}_D(\beta) = \dim_H \mu_\beta, \tag{2}$$

where $P_{\Sigma_A^+}(\varphi_\beta)$ is the *topological pressure* of φ_β on Σ_A^+ with respect to σ, $h_{\mu_\beta}(\sigma)$ is the *measure-theoretic entropy* of *sigma* with respect to μ_β, and $\dim_H \mu_\beta$ is the *Hausdorff dimension* of the measure μ_β.

The function $\mathcal{P}(\beta)$ is real analytic and monotonically decreasing (see [**?**]). The measure $\mu_0 = \mu_{\max}$ corresponding to $\beta = 0$ is the *measure of maximal entropy*, i.e.,

$$h_{\mu_0}(\sigma) = h_{\mathrm{top}}(\sigma),$$

where $h_{\mathrm{top}}(\sigma)$ is the *topological entropy* of the map σ. If the function ψ is not cohomologous to a constant function then $\mathcal{P}(\beta)$ is strictly convex, i.e.,

$$\frac{d}{d\beta}\mathcal{P}(\beta) < 0, \quad \frac{d^2}{d\beta^2}\mathcal{P}(\beta) > 0. \tag{3}$$

The derivative of the pressure can be computed by the following formula

$$\left.\frac{dP(\varphi_\beta}{d\beta}\right|_{\beta=\beta_0} = -\int_{Sigma_A^+} \psi \, d\mu_{\beta_0}\,, \tag{4}$$

where μ_{β_0} is the equilibrium measure corresponding to the function $\varphi(\beta_0)$. If the function ψ is cohomologous to a constant function then

$$\mathcal{P}(\beta) = \frac{-h_{\mathrm{top}}(\sigma)}{\beta_d}\beta + h_{\mathrm{top}}(\sigma)$$

is a linear function.

Furthermore, there exists a unique number $\beta(\psi)$ which is the root of *Bowen's equation*

$$\mathcal{P}(\beta(\psi)) = P_\Sigma(-\beta(\psi)\psi) = 0. \tag{5}$$

Our goal in this paper is to describe the functions $\mathcal{F}_E(\beta)$ and $\mathcal{F}_D(\beta)$. We call them respectively the **entropy** and **dimension families** associated with the function ψ.

First, we consider the function $\mathcal{F}_E(\beta)$.

Theorem 1.1. *(1) We have that*

$$\mathcal{F}_E(\beta) = \mathcal{P}(\beta) - \beta\frac{d}{d\beta}\mathcal{P}(\beta). \tag{6}$$

(2) Assume that the function $\psi(x)$ is not cogomologous to a constant function. Then the function $\mathcal{F}_E(\beta)$ is positive and real analytic. It attains

its maximum $h_{top}(\sigma)$ at $\beta = 0$ and $\frac{d}{d\beta}\mathcal{F}_E(\beta)|_{\beta=0} = 0$. Finally, it is strictly decreasing and hence, there exists a limit

$$\lim_{\beta\to+\infty} \mathcal{F}_E(\beta) = h^+(\psi) \geq 0. \tag{7}$$

(3) Assume that the function $\psi(x)$ is cogomologous to a constant function then

$$\mathcal{F}_E(\beta) = \mathcal{F}_E(0) = h_{top}(\sigma). \tag{8}$$

Proof. One can prove the theorem using a relation between the entropy family $\mathcal{F}_E(\beta)$ and the dimension spectrum for pointwise dimensions $f_\mu(\alpha)$ (see Remark 2 below). We present here a simple straightforward argument which does not use the multifractal analysis.

Since μ_β is the equilibrium measure corresponding to the function φ_β we obtain using formula (A1) in Appendix and (1) that

$$\mathcal{F}_E(\beta) = h_{\mu_\beta}(\sigma) = P_{\Sigma_A^+}(\varphi_\beta) - \int_{\Sigma_A^+} \varphi_\beta \, d\mu_\beta \; = P_{\Sigma_A^+}(-\beta\psi) + \beta \int_{\Sigma_A^+} \psi \, d\mu_\beta \, .$$

Now the first statement follows from formula (4) for the derivative of the pressure. Since the topological pressure $P_{\Sigma_A^+}(-\beta\psi)$ is real analytic so is the function $\mathcal{F}_E(\beta)$. Furthermore, since the function ψ is not cogomologous to a constant function in view of (3) we find that for $\beta > 0$

$$\frac{d}{d\beta}\mathcal{F}_E(\beta) = -\beta \, \frac{d^2}{d\beta^2} P_J(-\beta\psi) < 0,$$

and therefore, the function $\mathcal{F}_E(\beta)$ is strictly decreasing. In addition, $\frac{d}{d\beta}\mathcal{H}(0) = 0$. This implies the second statement. The third statement is obvious. ■

We now consider the function $\mathcal{F}_D(\beta)$. It is known (see [**?**]) that

$$\dim_H \mu_\beta = \frac{h_{\mu_\beta}(\sigma)}{\log a} \, .$$

This implies that

$$\mathcal{F}_D(\beta) = \frac{\mathcal{F}_E(\beta)}{\log a}. \tag{9}$$

As an immediate consequence of (9) we obtain the following result.

Theorem 1.2. *(1) Assume that the function $\psi(x)$ is not cogomologous to a constant function. Then the function $\mathcal{F}_D(\beta)$ is positive and real analytic. It attains its maximum $\mathcal{D}(0) = \dim_H \mu_{max} > 0$ at $\beta = 0$ and $\frac{d}{d\beta}\mathcal{F}_D(\beta)|_{\beta=0} = 0$. Finally, it is strictly decreasing and hence, there exists a limit*

$$\lim_{\beta\to+\infty} \mathcal{F}_D(\beta) = d^+(\psi) = \frac{h^+(\psi)}{\log a} \geq 0.$$

(2) Assume that the function $\psi(x)$ is cohomologous to a constant function. Then

$$\mathcal{F}_D(\beta) = \mathcal{F}_D(\beta(\psi)) = \dim_H \mu_{\beta(\psi)} = \dim_H \mu_{max}. \tag{10}$$

Remarks.

1. The functions $\mathcal{F}_E(\beta)$ and $\mathcal{F}_D(\beta)$ can be extended to $\beta < 0$ to become real analytic functions on the real line $\mathbb{R}$. If ψ is cohomologous to a constant function then $\mathcal{F}_E(\beta)$ and $\mathcal{F}_D(\beta)$ are constant (see (8) and (10)). Otherwise, the function $\mathcal{F}_E(\beta)$ is strictly decreasing for $\beta < 0$ and there exist the limits

$$\lim_{\beta\to-\infty} \mathcal{F}_E(\beta) = h^-(\psi) \geq 0, \quad \lim_{\beta\to-\infty} \mathcal{F}_D(\beta) = d^-(\psi) = \frac{h^-(\psi)}{\log a} \geq 0\,.$$

2. There are functions ψ for which $h^+(\psi)$ (see (7)) is strictly positive as well as functions ψ for which $h^+(\psi) = 0$ (see Example below and Remark 3 below). Let us notice that for functions ψ with $h^+(\psi) = 0$

$$\{h_{\mu_\beta}(f) :\, \beta \geq 0\} \cup \{0\} = [0, h_{\text{top}}(f)]$$
$$= \{h_\nu(f) :\, \nu \text{ runs over the set of all invariant measures}\}\,.$$

Example. Consider the space Σ_3 of **all** one-sided sequences $\omega = (\omega_n)$ of 3 symbols. Let us fix numbers p_i, $0 < p_i < 1$, $i = 1, 2, 3$ such that $p_1+p_2+p_3 = 1$. Define a function ψ on Σ_3 by $\psi(\omega) = -\log p_i$ if $\omega_0 = i$. The function ψ is Hölder continuous and the equilibrium measure corresponding to ψ is the Bernoulli measure with probabilities p_i. It is easy to see that the equilibrium measure μ_β corresponding to the function $\varphi_\beta = -\beta\psi$ is the Bernoulli measure with probabilities

$$p_i(\beta) = \frac{p_i^\beta}{p_1^\beta + p_2^\beta + p_3^\beta}, \quad i = 1, 2, 3.$$

It follows that

$$\mathcal{F}_E(\beta) = h_{\mu_\beta} = -\left(p_1(\beta)\log p_1(\beta) + p_2(\beta)\log p_2(\beta) + p_3(\beta)\log p_3(\beta)\right).$$

It is an easy calculation to show that if $0 < p_1 < p_2 < p_3 < 1$ then $\mathcal{F}_E(\beta) \to 0$ as $\beta \to \infty$ and if for example, $0 < p_1 = p_2 < p_3 < 1$ then $\mathcal{F}_E(\beta) \to \log 2$ ($<\log 3 = h_{\text{top}}(\sigma)$) as $\beta \to \infty$.

3. There is a simple relation between the entropy family $\mathcal{F}_E(\beta)$ (as well as the dimension family $\mathcal{F}_D(\beta)$) and the dimension spectrum for pointwise dimensions $\mathcal{D}_D(\alpha)$ (also known as the $f_\mu(\alpha)$-spectrum) where μ is the equilibrium measure corresponding to the function ψ (there is also a relation to the entropy spectrum for local entropies; see Remark 3 in Section II). Recall that (see [**?**]) the *pointwise dimension* of μ at a point $\omega \in \Sigma_A^+$ is defined as follows

$$d_\mu(\omega) = \lim_{r\to0} \frac{\log \mu(B(\omega, r))}{\log r} = \lim_{n\to\infty} \frac{\log \mu(C_{i_1\dots i_n})(\omega)}{n\log a}\,,$$

(provided the limit exists), where $B(\omega, r)$ is the ball (in the d_a-metric) centered at ω of radius r and $C_{i_1\dots i_n}(\omega)$ is the cylinder of length n containing the point ω. The *dimension spectrum for pointwise dimensions* is now defined by

$$f_\mu(\alpha) = \dim_H\{\omega :\, d_\mu(\omega) = \alpha\}.$$

Assume that the function ψ is not cogomologous to a constant function. It is shown in [**?**] that the dimension spectrum for pointwise dimensions is a real analytic and strictly convex function on an interval $[\alpha', \alpha'']$. Moreover, given a number $q \in \mathbb{R}$ define $T_D(q)$ such that

$$P_{\Sigma_A^+}(-T_D(q)\log a + q\log\chi) = 0,$$

where $\log\chi = \psi - P_{\Sigma_A^+}(\psi)$ (the function $\log\chi$ is the *normalization* of the function ψ, i.e., $P_{\Sigma_A^+}(\log\chi) = 0$). One can show that the function $T_D(q)$ is correctly defined, real analytic, strictly decreasing, and strictly convex. Moreover, the functions $f_\mu(\alpha)$ and $T_D(q)$ form a Legendre transform pair, i.e.,

$$f_\mu(\alpha(q)) = T_D(q) + q\alpha(q), \quad \alpha(q) = -\frac{d}{dq}T_D(q).$$

In particular,

$$\alpha' = \alpha(\infty) = -\lim_{q\to-\infty}\frac{d}{dq}T(q), \quad \alpha'' = \alpha(-\infty) = -\lim_{q\to+\infty}\frac{d}{dq}T(q). \tag{11}$$

We set $a = e$ (i.e., $\log a = 1$) and then write

$$T_D(\beta) = P_{\Sigma_A^+}(\beta\log\chi) = P_{\Sigma_A^+}(\beta\psi) - \beta P_{\Sigma_A^+}(\psi) = \mathcal{P}(\beta) - \beta P_{\Sigma_A^+}(\psi).$$

This implies that

$$f_\mu(\alpha(\beta)) = T_D(\beta) - \beta T_D'(\beta) = \mathcal{P}(\beta) - \beta\frac{d}{d\beta}\mathcal{P}(\beta).$$

By (6) we obtain that

$$\mathcal{F}_E(\beta) = f_\mu(\alpha(\beta)), \quad \alpha(\beta) = -\frac{d}{d\beta}P_{\Sigma_A^+}(\beta\log\chi). \tag{12}$$

This relation and the well-known properties of the $f_\mu(\alpha)$-spectrum allow one to obtain another proof of Theorem 1.1.

We use the relation (12) to establish another crucial property of the entropy family $\mathcal{F}_E(\beta)$. It follows from (11) and Theorem 1.1 that

$$h^+(\psi) = f_\mu(\alpha(\infty)), \quad h^-(\psi) = f_\mu(\alpha(-\infty)).$$

Applying now a result of Schmeling (see [**?**]) on a "typical" values of $f_\mu(\alpha(+\infty))$ and $f_\mu(\alpha(-\infty))$ we have the following statement.

Theorem 1.3. *For every $a > 1$*

(1) given $h^-, h^+ < h_{top}(\sigma)$, there exists a Hölder continuous function ψ on Σ_A^+ (with respect to the standard metric d_a) such that $h^-(\psi) = h^-$ and $h^+(\psi) = h^+$;

(2) there exists a residual set $\mathcal{A} \subset \mathcal{H}_\alpha$ (the space of Hölder continuous functions on Σ_A^+ with respect to the standard metric d_a and with a given Hölder exponent α) such that $h^-(\psi) = 0$ and $h^+(\psi) = 0$ for every $\psi \in \mathcal{A}$.

In other words, *for a "typical" Hölder continuous function ψ the set $\{h_{\mu_\beta}(\sigma),\ \beta \geq 0\}$ contains all positive values of metric entropy.*

4. By (3), there exists a limit

$$\lim_{\beta\to\infty} \frac{d}{d\beta}\mathcal{P}(\beta) \stackrel{\text{def}}{\to} = p' < 0.$$

Given $\beta > 0$, we find that

$$P_{\Sigma_A^+}(-\beta\psi) = \frac{d}{d\beta}P_{\Sigma_A^+}(-\beta\psi)(\beta - q(\beta)),$$

where $q(\beta)$ is the intercept of the line tangent to $P_{\Sigma_A^+}(-\beta\psi)$ at the point β. By (6),

$$\mathcal{F}_E(\beta) = -\frac{d}{d\beta}P_{\Sigma_A^+}(-\beta\psi)q(\beta)$$

and hence, $q(\beta) > 0$. In view of (3) the function $q(\beta)$ is decreasing and hence, there exists a limit $q = \lim_{\beta\to\infty} q(\beta) \geq 0$. As an immediate consequence we obtain the following result.

Theorem 1.4. *The topological pressure $\mathcal{P}(\beta) = P_{\Sigma_A^+}(-\beta\psi)$ has an asymptotic of the form $q + p'\beta$ with $q > 0$ and $p' < 0$.*

Similar result holds for a *two-sided subshift of finite type* (Σ_A, σ). Recall that Σ_A is the space of two-sided sequences $\omega = (\omega_n)_{n\in\mathbb{Z}}$ of p symbols $\{1, \dots, p\}$ which are admissible with respect to the transfer matrix $A = (a_{ij})$ and σ is the shift. The space Σ_A is a metric space with the standard metric

$$d_a(\omega^{(1)}, \omega^{(2)}) = \sum_{n\in\mathbb{Z}} \frac{|\omega_n^{(1)} - \omega_n^{(2)}|}{a^{|n|}},$$

where $a > 1$ is a constant. We assume that the transfer matrix A is transitive and hence, the shift σ is topologically mixing.

Consider a strictly positive Hölder continuous function ψ on Σ_A. Given $\beta \geq 0$, let φ_β be the Hölder continuous function on Σ_A defined by (1) and $\mu_\beta = \mu_{\varphi_\beta}$ the corresponding equilibrium measure. With the one-parameter family of measures $\{\mu_\beta\}_{\beta\geq 0}$, one can associate the **entropy family** $\mathcal{F}_E(\beta)$ and the **dimension family** $\mathcal{F}_D(\beta)$ defined by (2). It is known (see [**?**]) that

$$\dim_H \mu_\beta = 2\frac{h_{\mu_\beta}(f)}{\log a}.$$

This implies that

$$\mathcal{F}_D(\beta) = 2\frac{\mathcal{F}_E(\beta)}{\log a}.$$

It follows that the functions $\mathcal{F}_E(\beta)$ and $\mathcal{F}_D(\beta)$ have properties stated in Theorems 1.1, 1.2, and 1.3.

2. Dimension and Entropy Families: The Case of Conformal Expanding Maps

Consider a $C^{1+\alpha}$-map f of a smooth Riemannian manifold $\mathcal{M}$. Recall (see for example, [?]) that f is called **expanding** if there exists a compact subset $J \subset \mathcal{M}$ such that

(1) there exist $C > 0$ and $\lambda > 1$ such that $\|df_x^n v\| \geq C\lambda^n \|v\|$ for all $x \in J$, $v \in T_x\mathcal{M}$, and $n \geq 1$ (with respect to a Riemannian metric on $\mathcal{M}$);

(2) there exists an open neighborhood V of J (called a **basin**) such that $J = \{x \in V \,:\, f^n(x) \in V \text{ for all } n \geq 0\}$

The set J is called a **repeller** for f. Obviously, f is a local homeomorphism, i.e., there exists $r_0 > 0$ such that for every $x \in J$ the map $f|B(x, r_0)$ is a homeomorphism onto its image.

Consider a strictly positive Hölder continuous function ψ on J. Given $\beta \geq 0$, let φ_β be the Hölder continuous function on J defined by (1) and $\mu_\beta = \mu_{\varphi_\beta}$ the corresponding equilibrium measure. With the one-parameter family of measures $\{\mu_\beta\}_{\beta \geq 0}$, one can associate the **entropy family** $\mathcal{F}_E(\beta)$ and the **dimension family** $\mathcal{F}_D(\beta)$ defined by (2).

One can show that the function $\mathcal{F}_E(\beta)$ has properties described by Theorem 1.1 (the proof is a trivial modification of arguments in the proof of Theorem 1.1).

We now consider the function $\mathcal{F}_D(\beta)$. Recall that a smooth map $f\colon \mathcal{M} \to \mathcal{M}$ is called *conformal* if for each $x \in \mathcal{M}$ we have $df_x = a(x)\ \mathrm{Isom}_x$, where Isom_x denotes an isometry of $T_x\mathcal{M}$ and $a(x)$ is a scalar. A smooth conformal map f is expanding if $|a(x)| > 1$ for every point $x \in \mathcal{M}$. The repeller J for a conformal expanding map is called a *conformal repeller*. Since f is of class $C^{1+\alpha}$ the function $a(x)$ is Hölder continuous.

Some well-known examples of conformal expanding maps include rational maps, one-dimensional Markov maps, conformal toral endomorphisms, etc. (see [?]).

We assume that f is topologically mixing. (The general case can be reduced to this one using the Spectral Decomposition Theorem for expanding maps; see Appendix.)

The following theorem provides a description of the dimension family $\mathcal{F}_D(\beta)$.

Theorem 2.1. *(1) Assume that the function $\psi(x)$ is not cohomologous to a constant function. Then the function $\mathcal{F}_D(\beta)$ is real analytic and positive. It also has the following properties:*

$$0 < \mathcal{F}_D(\beta) \leq \dim_H J,\ \mathcal{F}_D(0) = \dim_H \mu_{max} > 0,\ \frac{d}{d\beta}\mathcal{F}_D(\beta)|_{\beta=0} > 0.$$

Finally, there exists a limit

$$\lim_{\beta \to +\infty} \mathcal{F}_D(\beta) = d^+(\psi) = \frac{h^+(\psi)}{\lambda^+(\psi)} \geq 0\,,$$

where $\lambda^+(\psi) = \lim_{\beta\to\infty} \lambda_{\mu_\beta}$ *(here* λ_{μ_β} *is the Lyapunov exponent of the measure* μ_β *and is defined by (A3); one can show that this limit exists).*

(2) Assume that the function $\psi(x)$ *is cohomologous to a constant function. Then* $\mathcal{F}_D(\beta)$ *is a constant function (see (10)).*

Proof. By the definition, the function $\mathcal{F}_D(\beta)$ is positive. Using the variational principle, (A3) and (A4) (with $\nu = \mu_\beta$) we obtain that

$$\mathcal{F}_D(\beta) = \frac{P_J(-\beta\psi) + \beta \int_J \psi \, d\mu_\beta}{\int_J \log|a| \, d\mu_\beta}.$$

Applying formula (4) for the derivative of the pressure (to the functions ψ and $-\beta \log|a|$)

$$\mathcal{F}_D(\beta) = -\frac{P_J(-\beta\psi) - \beta \frac{d}{d\beta} P_J(-\beta\psi)}{\frac{d}{d\beta} P_J(-\beta \log|a|)}.$$

Since the pressure is an analytic function (see Appendix), so is the function $\mathcal{F}_D(\beta)$. It also follows using (3) that

$$\frac{d}{d\beta}\mathcal{F}_D(\beta)|_{\beta=0} = P_J(0)\frac{d^2}{d\beta^2}P_J(-\beta\log|a|)|_{\beta=0}\left(\frac{d}{d\beta}P_J(-\beta\log|a|)|_{\beta=0}\right)^{-2} > 0.$$

Since the function $\log|a(x)|$ is continuous and hence bounded it follows from (2), (A3), and (A4) that $C^{-1}\mathcal{F}_D(\beta) \le \mathcal{F}_(\beta) \le C\mathcal{F}_D(\beta)$ where $C > 0$ is a constant. Now Theorem 1.1 implies that the function $\mathcal{F}_D(\beta)$ has a limit as $\beta \to \infty$.

The second statement of the theorem is obvious. ■

Remarks.

1. The functions $\mathcal{F}_E(\beta)$ (generated by a Hölder continuous function on a repeller) and $\mathcal{F}_D(\beta)$ (generated by a Hölder continuous function on a conformal repeller) can be extended to $\beta < 0$ to become real analytic functions on the real line $\mathbb{R}$. If ψ is cohomologous to a constant function then $\mathcal{F}_E(\beta)$ and $\mathcal{F}_D(\beta)$ are constant. Otherwise, these functions are strictly decreasing for $\beta < 0$ and there exist limits

$$\lim_{\beta\to-\infty} \mathcal{F}_E(\beta) = h^-(\psi) \ge 0, \qquad \lim_{\beta\to-\infty} \mathcal{F}_D(\beta) = d^-(\psi) = \frac{h^-(\psi)}{\lambda^-(\psi)} \ge 0,$$

where $\lambda^-(\psi) = \lim_{\beta\to-\infty} \lambda_{\mu_\beta}$ (one can show that this limit exists).

2. Let J be a conformal repeller for a conformal expanding $C^{1+\alpha}$-map f. Since the function $\log|a(x)|$ is strictly positive and Hölder continuous Theorem 2.1 applies. However, in this case one can obtain a more precise information.

Recall that there exists a unique number β_d which is the root of Bowen's equation $\mathcal{P}(\beta_d) = P_J(-\beta_d \log|a(x)|) = 0$ (see (5)). The unique equilibrium measure $\mu_d = \mu_{\beta_d}$ is known to be the *measure of maximal dimension* (see (A2) in Appendix).

Theorem 2.2. *Assume that the function* $\log|a(x)|$ *is not cohomologous to a constant function. Then the function* $\mathcal{F}_D(\beta)$ *is real analytic and positive. It attains its maximum* $\dim_H J$ *at* $\beta = \beta_d$. *It is strictly monotonically increasing for* $0 \le \beta \le \beta_d$ *and is strictly monotonically decreasing for* $\beta \ge \beta_d$ *(and hence, has a limit as* $\beta \to \infty$*).*

Proof. Following the roof of Theorem 2.1 we obtain that

$$\mathcal{F}_D(\beta) = -\frac{P_J(-\beta\log|a|)}{\frac{d}{d\beta}P_J(-\beta\log|a|)} + \beta\,.$$

It follows that

$$\frac{d}{d\beta}\mathcal{D}(\beta) = P_J(-\beta\log|a|)\frac{d^2}{d\beta^2}P_J(-\beta\log|a|)\left(\frac{d}{d\beta}P_J(-\beta\log|a|)\right)^{-2}.$$

In particular, $\frac{d}{d\beta}\mathcal{D}(\beta) = 0$ implies that $P_J(-\beta\log|a|) = 0$ and hence $\beta = \beta_d$. The desired result follows. ∎

It is an interesting open question whether $h^+(\log|a|)(= h^-(\log|a|)) = 0$ (and hence, $d^+(\log|a|)(= d^-(\log|a|)) = 0$).

Since the function $\mathcal{F}_D(\beta)$ is real analytic it may have only finitely many local extrema on any finite interval in β. We conjecture that indeed *for a residual set of Hölder continuous functions on conformal repellers the total number of local extrema of the function* $\mathcal{F}_D(\beta)$ *is finite.*

3. Consider a repeller J for an expanding map f. There is a relation between the entropy family $\mathcal{F}_E(\beta)$ and the entropy spectrum for local entropies $\mathcal{E}_E(\alpha)$ where μ is the equilibrium measure corresponding to the function ψ. Recall (see [**?**] and [**?**]) that the *local entropy* of μ at a point $x \in J$ is defined as follows. Consider a finite generating measurable partition ξ of J (for example, one can use a *Markov partition* of J; see [**?**]). For every $n > 0$, we write $\xi_n = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n}\xi$, and denote by $C_{\xi_n}(x)$ the element of the partition ξ_n that contains the point x. The local entropy at a point x is defined by

$$h_\mu(x) = -\lim_{n\to\infty}\frac{\log\mu(C_{\xi_n(x)})}{n}$$

(provided the limit exists). The *entropy spectrum for local entropies* is now defined by

$$\mathcal{E}_E(\alpha) = h_{\rm top}(\{x : h_\mu(x) = \alpha\}),$$

where $h_{\rm top}(A)$ is the topological entropy of f on the set A (which may be an arbitrary subset; see [**?**] for definitions). Assume that the function ψ is not cogomologous to a constant function. It is shown in [**?**] (see also [**?**]) that the entropy spectrum for local entropies is a real analytic and strictly convex function on an interval $[\alpha', \alpha'']$. Moreover, given a number $q \in \mathbb{R}$ define $T_E(q)$ such that

$$P_J(-T_E(q) + q\log\chi) = 0,$$

where $\log\chi = \psi - P_J(\psi)$. One can show that the function $T_E(q)$ is correctly defined, real analytic, strictly decreasing, and strictly convex. Furthermore, the functions $\mathcal{E}_E(\alpha)$ and $T_E(q)$ form a Legendre transform pair, i.e.

$$\mathcal{E}_E(\alpha(q)) = T_E(q) - q\alpha(q), \quad \alpha(q) = -\frac{d}{dq}T_E(q).$$

In particular,

$$\alpha' = \alpha(\infty) = -\lim_{q\to-\infty}\frac{d}{dq}T_E(q), \quad \alpha'' = \alpha(-\infty) = -\lim_{q\to+\infty}\frac{d}{dq}T_E(q).$$

One can show (see the argument in Remark 3 in Section I) that

$$\mathcal{F}_E(\beta) = \mathcal{E}_E(\alpha(\beta)), \quad \alpha(\beta) = -\frac{d}{d\beta}P_J(\beta\log\chi).$$

This relation and the well-known properties of the $\mathcal{E}_E(\alpha)$ allow one to obtain a complete description of the entropy family. In particular, using a result of Schmeling (see [**?**]) one can prove an analog of Theorem 1.3 for repellers.

For the function $\psi(x) = \log|a(x)|$ one can also establish a relation between the dimension family $\mathcal{F}_D(\beta)$ and the dimension spectrum for pointwise dimensions. Using this relation one can obtain a complete description of $\mathcal{F}_D(\beta)$ and in particular, prove a "genericity" result (see Theorem 1.3) and Theorems 2.1 and 2.2.

3. Dimension and Entropy Families: The Case of Conformal Axiom *A* Diffeomorphisms

We now consider a $C^{1+\alpha}$-diffeomorphism of a smooth Riemannian manifold $\mathcal{M}$ which possesses a locally maximal hyperbolic set Λ. This means (see, for example, [**?**]) that there exist a continuous splitting of the tangent bundle $T_\Lambda\mathcal{M} = E^{(s)} \oplus E^{(u)}$ and constants $C > 0$ and $0 < \lambda < 1$ such that for every $x \in \Lambda$

(1) $df E^{(s)}(x) = E^{(s)}(f(x)), \quad df E^{(u)}(x) = E^{(u)}(f(x))$;

(2) for all $n \ge 0$

$$\begin{aligned} \|df^n v\| &\le C\lambda^n\|v\| \quad \text{if } v \in E^{(s)}(x), \\ \|df^{-n} v\| &\le C\lambda^n\|v\| \quad \text{if } v \in E^{(u)}(x). \end{aligned}$$

The subspaces $E^{(s)}(x)$ and $E^{(u)}(x)$ are called *stable* and *unstable subspaces* at x respectively and they depend Hölder continuously on x.

It is well-known (see, [**?**]) that for every $x \in \Lambda$ one can construct *stable* and *unstable local manifolds*, $W^{(s)}(x)$ and $W^{(u)}(x)$. They have the following properties:

(3) $x \in W^{(s)}(x), \quad x \in W^{(u)}(x)$;

(4) $T_x W^{(s)}(x) = E^{(s)}(x), \quad T_x W^{(u)}(x) = E^{(u)}(x)$;

(5) $f(W^{(s)}(x)) \subset W^{(s)}(f(x)), \quad f^{-1}(W^{(u)}(x)) \subset W^{(u)}(f^{-1}(x))$;

(6) there exist $K > 0$ and $0 < \mu < 1$ such that for every $n \geq 0$,

$$\rho(f^n(y), f^n(x)) \leq K\mu^n \rho(y, x) \text{ for all } y \in W^{(s)}(x)$$

and

$$\rho(f^{-n}(y), f^{-n}(x)) \leq K\mu^n \rho(y, x) \text{ for all } y \in W^{(u)}(x),$$

where ρ is the distance in $\mathcal{M}$ induced by the Riemannian metric.

A hyperbolic set Λ is called *locally maximal* if there exists a neighborhood U of Λ such that for any closed f-invariant subset $\Lambda' \subset U$ we have $\Lambda' \subset \Lambda$. In this case

$$\Lambda = \underset{-\infty < n < \infty}{\to} \bigcap f^n(U).$$

The set Λ is locally maximal if and only if the following property holds:

(7) there exists $\delta > 0$ such that for all $x, y \in \Lambda$ with $\rho(x, y) \leq \delta$ the set $W^{(s)}(x) \cap W^{(u)}(y)$ consists of a single point $z \in \Lambda$, which we denote by $z = [x, y]$; moreover, the map

$$[\cdot\,,\,\cdot]\colon \{(x, y) \in \Lambda \times \Lambda : \rho(x, y) \leq \delta\} \to \Lambda$$

is continuous.

Some well-known examples of diffeomorphisms possessing locally maximal hyperbolic sets include Smale horseshoes, Smale-Williams solenoids, etc. (see [**?**]).

A diffeomorphism f is called an *Axiom A diffeomorphism* if its non-wandering set $\Omega(f)$ is a locally maximal hyperbolic set (see [**?**]).

Consider a strictly positive Hölder continuous function ψ on Λ. Given $\beta \geq 0$, let φ_β be the Hölder continuous function on Λ defined by (1) and $\mu_\beta = \mu_{\varphi_\beta}$ the corresponding equilibrium measure. With the one-parameter family of measures $\{\mu_\beta\}_{\beta \geq 0}$, one can associate the **entropy family** $\mathcal{F}_E(\beta)$ and the **dimension family** $\mathcal{F}_D(\beta)$ defined by (2).

One can show that the function $\mathcal{F}_E(\beta)$ has properties described by Theorem 1.1 (the proof is a trivial modification of arguments in the proof of Theorem 1.1).

We now consider the function $\mathcal{F}_D(\beta)$. Recall that a diffeomorphism f of a locally maximal hyperbolic set Λ is called *u-conformal* (respectively, *s-conformal*) if there exists a continuous function $a^{(u)}(x)$ (respectively, $a^{(s)}(x)$) on Λ such that $df \mid E^{(u)}(x) = a^{(u)}(x)\,\mathrm{Isom}_x$ for every $x \in \Lambda$ (respectively, $df \mid E^{(s)}(x) = a^{(s)}(x)\,\mathrm{Isom}_x$; recall that Isom_x denotes an isometry of $E^{(u)}(x)$ or $E^{(s)}(x)$). Since the subspaces $E^{(u)}(x)$ and $E^{(s)}(x)$ depend Hölder continuously on x the functions $a^{(u)}(x)$ and $a^{(s)}(x)$ are also Hölder continuous. Note that $|a^{(u)}(x)| > 1$ and $|a^{(s)}(x)| < 1$ for every $x \in \Lambda$. A diffeomorphism f on Λ is called *conformal* if it is u-conformal and s-conformal as well (see [**?**]).

We assume that f is topologically mixing. (The general case can be reduced to this one using the Spectral Decomposition Theorem for Axiom A diffeomorphisms; see Appendix.)

The following theorem provides a description of the dimension family $\mathcal{F}_D(\beta)$.

Theorem 3.1. *(1) Assume that the function $\psi(x)$ is not cohomologous to a constant function. Then the function $\mathcal{F}_D(\beta)$ is real analytic and positive. It also has the following properties:*

$$0 < \mathcal{F}_D(\beta) \le \dim_H \Lambda,\ \mathcal{F}_D(0) = \dim_H \mu_{max} > 0,\ \frac{d}{d\beta}\mathcal{F}_D(\beta)|_{\beta=0} > 0.$$

Finally, there exists a limit

$$\lim_{\beta\to+\infty} \mathcal{F}_D(\beta) = d^+(\psi) = h^+(\psi)\left(\frac{1}{\lambda^{++}(\psi)} - \frac{1}{\lambda^{+-}(\psi)}\right) \ge 0\,,$$

where $\lambda^{++}(\psi) = \lim_{\beta\to\infty} \lambda^+_{\mu_\beta}$ and $\lambda^{+-}(\psi) = \lim_{\beta\to\infty} \lambda^-_{\mu_\beta}$ (here $\lambda^+\mu_\beta$ and $\lambda^-\mu_\beta$ are the positive and negative Lyapunov exponents of the measure μ_β and are defined by (A3); one can show that this limits exist).

(2) Assume that the function $\psi(x)$ is cohomologous to a constant function. Then $\mathcal{F}_D(\beta)$ is a constant function (see (10)).

Proof. Following the argument in the proof of Theorem 2.1, using the variational principle, (A5), (A6), and (A7) (with $\nu = \mu_\beta$) we obtain that

$$\mathcal{F}_D(\beta) = \left(P_\Lambda(-\beta\psi) + \beta\int_\Lambda \psi\, d\mu_\beta\right)\left(\frac{1}{\int_\Lambda \log|a^{(u)}|\, d\mu_\beta} - \frac{1}{\int_\Lambda \log|a^{(s)}|\, d\mu_\beta}\right).$$

Applying formula (4) for the derivative of the pressure (to functions ψ, $-\beta\log|a^{(u)}|$, and $\beta\log|a^{(u)}|$) we find that

$$\mathcal{F}_D(\beta) = \left(P_\Lambda(-\beta\psi) - \beta\frac{d}{d\beta}P_\Lambda(-\beta\psi)\right) \times \left(\frac{1}{\frac{d}{d\beta}P_\Lambda(-\beta\log|a^{(u)}|)} - \frac{1}{\frac{d}{d\beta}P_\Lambda(\beta\log|a^{(s)}|)}\right).$$

The desired result follows. ■

Remarks.

1. Let Λ be a locally maximal hyperbolic set for a conformal axiom A diffeomorphism f of class $C^{1+\alpha}$. Since the functions $\log|a^{(u)}(x)|$ and $-\log|a^{(s)}(x)|$ are strictly positive and Hölder continuous the above results apply. It is also easy to see that Remarks 1 and 3 in Section II hold true for conformal axiom A diffeomorphisms.

2. Using recent results in [**?**] one can generalize some of the above results to expansive homeomorphisms with specification property (see [**?**] for definitions). In this case analyticity should be replaced by differentiability.

4. Appendix

1. Spectral Decomposition Theorem (see [**?**]). A point $x \in \mathcal{M}$ is called *non-wandering* for a map $f : \mathcal{M} \to \mathcal{M}$ if for each neighborhood U of

x there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all non-wandering points of f. It is a closed f-invariant set.

The Spectral Decomposition Theorem for expanding maps claims that the set $\Omega(f)$ can be decomposed into finitely many disjoint closed f-invariant subsets, $\Omega(f) = J_1 \cup \cdots \cup J_m$, such that $f \mid J_i$ is topologically transitive. Moreover, for each i there exist a number n_i and a set $A_i \subset J_i$ such that the sets $f^k(A_i)$ are disjoint for $0 \leq k < n_i$, their union is the set J_i, $f^{n_i}(A_i) = A_i$, and the map $f^{n_i} \mid A_i$ is topologically mixing (see [**?**] for more details).

A similar Spectral Decomposition Theorem holds true for Axiom A diffeomorphisms. It claims that the set of all non-wandering points $\Omega(f)$ can be decomposed into finitely many disjoint closed f-invariant locally maximal hyperbolic sets, $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_m$, such that $f \mid \Lambda_i$ is topologically transitive. Moreover, for each i there exist a number n_i and a set $A_i \subset \Lambda_i$ such that the sets $f^k(A_i)$ are disjoint for $0 \leq k < n_i$, their union is the set Λ_i, $f^{n_i}(A_i) = A_i$, and the map $f^{n_i} \mid A_i$ is topologically mixing.

2. Some Properties of the Topological Pressure (see [**?**], [**?**]). Let f be a continuous map of a complete metric space $\mathcal{X}$. The *Variational Principle* for topological pressure asserts that for every continuous function ψ on $\mathcal{X}$,

$$P_J(\psi) = \sup \left\{ h_\nu(f) + \int_{\mathcal{X}} \psi \, d\nu \right\},$$

where the supremum is taken over all measures on $\mathcal{X}$ invariant under f. A measure ν_ψ is called an *equilibrium measure* corresponding to the function ψ if

$$P_J(\psi) = h_{\nu_\psi}(f) + \int_{\mathcal{X}} \psi \, d\nu_\psi. \tag{A1}$$

For $\psi = 0$ the value $P_J(0)$ is called the *topological entropy* of f and is denoted by $h_{\mathrm{top}}(f)$. An equilibrium measure corresponding to $\psi = 0$ is called the *measure of maximal entropy.*

Let J be the repeller for a conformal expanding map f of a smooth Riemannian manifold $\mathcal{M}$. The pressure $P = P_J$ is a real analytic function on the space of Hölder continuous functions. Given a Hölder continuous function ψ consider the function $\varphi \colon \mathbb{R} \to C^\alpha(J, \mathbb{R})$ defined by $\varphi(\beta) = -\beta\psi$. This function is clearly real analytic. Therefore, the function $\mathcal{P}(\beta) = P_J(\varphi(\beta))$ is real analytic with respect to β.

Moreover, if the function ψ is strictly positive then $\mathcal{P}(\beta)$ is decreasing; if in addition the function ψ is not cogomologous to a constant function then $\mathcal{P}(\beta)$ is strictly decreasing and convex.

If the map f is topologically mixing then for every Hölder continuous function ψ on J an equilibrium measure corresponding to ψ is unique, has positive metric entropy, and is indeed isomorphic to a Bernoulli measure.

3. Hausdorff Dimension of Conformal Repellers and Locally Maximal Hyperbolic Sets for Conformal Axiom *A* Diffeomorphisms (see [**?**]).

The Hausdorff dimension of a conformal repeller J for a conformal expanding map f can be computed by the following formula

$$\dim_H J = \dim_H \mu_d = \beta_d = \frac{h_{\mu_d}(f)}{\lambda_{\mu_d}}, \tag{A2}$$

where μ_d is the measure of maximal dimension, β_d is the root of Bowen's equation (see (3)), and λ_{μ_d} is the *Lyapunov exponent* of the measure μ_d. Recall that the Lyapunov exponent of an invariant ergodic measure ν is defined as follows

$$\lambda_\nu = \underset{n\to\infty}{\to} \lim \frac{1}{n} \sum_{k=0}^{n} \log |a(f^k(y))| = \int_J \log |a(x)| \, d\nu(x) \tag{A3}$$

for ν almost every point y.

Furthermore, for an arbitrary ergodic measure ν on J its Hausdorff dimension can be computed by the following formula (see [**?**])

$$\dim_H \nu = \frac{h_\nu(f)}{\lambda_\nu}. \tag{A4}$$

Let Λ be a locally maximal hyperbolic set for a conformal Axiom A diffeomorphism f and ν an arbitrary ergodic measure on Λ. Its Hausdorff dimension can be computed by the following formula of Young (see [**?**])

$$\dim_H \nu = h_\nu(f) \left(\frac{1}{\lambda_\nu^+} - \frac{1}{\lambda_\nu^-} \right), \tag{A5}$$

where λ_ν^+ and λ_ν^- are positive and negative Lyapunov exponents of ν which can be computed by formulae

$$\lambda_\nu^+ = \underset{n\to\infty}{\to} \lim \frac{1}{n} \sum_{k=0}^{n} \log |a^{(u)}(f^k(y))| = \int_J \log |a^{(u)}(x)| \, d\nu(x), \tag{A6}$$

$$\lambda_\nu^- = \underset{n\to\infty}{\to} \lim \frac{1}{n} \sum_{k=0}^{n} \log |a^{(s)}(f^k(y))| = \int_J \log |a^{(s)}(x)| \, d\nu(x) \tag{A7}$$

for ν-almost every point y.

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, U.S.A.

E-mail address: pesin@math.psu.edu