NON-STATIONARY NON-UNIFORM HYPERBOLICITY: SRB MEASURES FOR DISSIPATIVE MAPS

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Abstract. We prove the existence of SRB measures for local diffeomorphisms under certain recurrence conditions on the iterates of Lebesgue measure. These conditions can be verified using a notion of "effective hyperbolicity" that is related to Lyapunov–Perron regularity; if there is a positive Lebesgue measure set of initial conditions for which trajectories have "enough" effective hyperbolicity, then the system has an SRB measure. We give examples of systems that do not admit a dominated splitting but can be shown to have SRB measures using our methods.

1. Introduction

1.1. **SRB measures and some known results.** Consider a $C^{1+\alpha}$ local diffeomorphism f of a smooth Riemannian manifold M. There are usually many different f-invariant measures μ on M, and selecting one which is "physically" meaningful is of great interest. A crucial property of such a measure is that the set of μ-generic points has positive Lebesgue measure. Then μ describes the asymptotics of orbits of points that are chosen at random with respect to Lebesgue measure.

In general, the measure μ can be quite trivial – just think of the point mass at an attracting fixed point. A substantially more interesting situation appears when:

(1) f possesses a topological attractor Λ, i.e., a compact invariant subset[1] of M for which there is an open set $U \supset \Lambda$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$;
(2) μ is a hyperbolic measure supported on Λ – that is, a measure for which almost every point has both positive and negative Lyapunov exponents, but no zero Lyapunov exponents.

In this case the standard results of nonuniform hyperbolicity theory apply (see for example, [BP07]) and in particular, for almost every $x \in \Lambda$ we

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[1]This includes the case when $\Lambda = M$.

obtain local stable $V^s(x)$ and unstable $V^u(x)$ manifolds. It is easy to see that $V^u(x) \subset \Lambda$ for every $x \in \Lambda$ such that $V^u(x)$ exists.

We say that μ is an SRB measure (after Sinai, Ruelle and Bowen) if the conditional measures generated by μ on local unstable manifolds $V^u(x)$ are equivalent to the leaf-volume $m^u(x)$ on $V^u(x)$ (i.e., the Riemannian volume on $V^u(x)$ generated by the Riemannian metric). See Definition 2.1 for details.

Given a measure μ on the attractor Λ, its *basin of attraction* $B(\mu)$ is the set of μ-generic points $x \in U$, i.e., the points such that for every continuous function φ on Λ,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi\, d\mu. \tag{1.1}$$

One can show that the basin of attraction $B(\mu)$ of an SRB measure μ has positive volume – in other words, μ is a *physical measure*. On the other hand, Figure 1 shows an example due to Bowen with a physical measure that is not an SRB measure. The attractor is the "figure eight" shape shown, and the only invariant measure on the attractor is the Dirac measure at the hyperbolic fixed point p. This measure is hyperbolic and has full basin of attraction but is not an SRB measure.

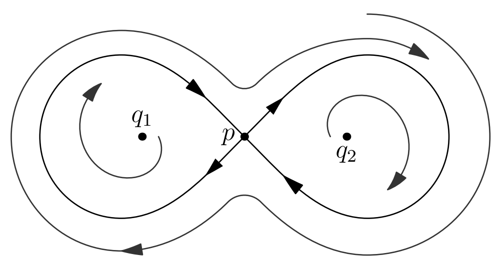

FIGURE 1. A physical measure that is not SRB.

SRB measures were introduced and studied in the 1970s by Sinai [Sin72], Ruelle [Rue76, Rue78], and Bowen [Bow75], who showed that they exist when $f|_\Lambda$ is uniformly hyperbolic.[2] The SRB measure in this case can be obtained in the following way. Let Leb denote Lebesgue measure (volume) on U, and consider its time-averaged evolutions under the map, i.e., the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \,\mathrm{Leb}\,. \tag{1.2}$$

[2]This means that $T\Lambda = E^s \oplus E^u$ where E^s and E^u are Df-invariant continuous stable and unstable distributions on Λ along which Df acts respectively as a contraction and expansion with uniform rates. In particular, this includes the case where f is an Anosov diffeomorphism.

In the uniformly hyperbolic setting, it can be shown that any limit measure (with respect to the weak* topology) of the sequence μ_n is an SRB measure.[3]

The techniques originally used in the uniformly hyperbolic setting relied on the existence of a Markov partition for the map f. A more general setting occurs when the attractor Λ is partially hyperbolic;[4] in this case we cannot expect to have a Markov partition. For every $x \in \Lambda$ one can construct local strongly stable $V^s(x)$ and strongly unstable $V^u(x)$ manifolds; furthermore, $V^u(x) \subset \Lambda$ for every $x \in \Lambda$. However, $\dim V^u(x) + \dim V^s(x) < \dim M$. In [PS82], Pesin and Sinai introduced and studied a special class of invariant measures called *u-measures*.[5] These generate absolutely continuous conditional measures on local strongly unstable manifolds but may not be hyperbolic – some or even all Lyapunov exponents in the central direction may be zero. It is shown in [PS82] that any limit measure of the sequence of measures μ_n is a u-measure.

We describe another way to construct u-measures. Given $x \in \Lambda$, one can view the leaf-volume $m^u(x)$ on $V^u(x)$ as a measure on the whole of Λ. Consider the sequence of measures on Λ given by

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m^u(x). \tag{1.3}$$

It is shown in [PS82] that any limit measure of this sequence is a u-measure. Indeed, one can prove that this remains true if we replace the local unstable manifold $V^u(x)$ with any local manifold passing through x and sufficiently close to $V^u(x)$ in the C^1 topology. The approach that we utilize in this paper is a substantial generalization of this approach, adjusted so that it applies in the non-uniformly hyperbolic setting and without any assumption of partial hyperbolicity.

Before describing this general approach, we recall some known results, beginning with partially hyperbolic systems. In order to obtain an SRB measure in this setting, one needs to impose some conditions on the Lyapunov exponents of a u-measure in the central direction. For example, suppose that there is a u-measure μ for which central Lyapunov exponents are all negative on a set of positive measure. In this case one can show (see [BDPP08]) that if orbits of unstable manifolds are dense in Λ, then μ is the unique u-measure and is the SRB measure for f. In fact, it is shown in [BDPP08] that this situation is stable under small perturbations of f: given

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[3] The same result holds if one starts with any measure which is absolutely continuous with respect to volume. One can show that the limit measure is unique if f is topologically transitive, and is the weak* limit of the sequence of measures f_*^n Leb if f is topologically mixing.

[4] This means that $T\Lambda = E^s \oplus E^c \oplus E^u$ where E^s, E^c and E^u are Df-invariant continuous stable, central and unstable distributions on Λ; Df acts along E^s and E^u respectively as a contraction and expansion with uniform rates; the amount of expansion or contraction for Df acting along E^c is strictly less than the corresponding rates along E^s and E^u.

[5] These measures were referred to in [PS82] as *Gibbs u-measures*.

$\delta > 0$, there is $\theta > 0$ such that any $C^{1+\alpha}$ diffeomorphism g that is θ-close to f in the $C^{1+\alpha}$ topology has a unique SRB measure, which is supported on a partially hyperbolic attractor Λ_g lying in the δ-neighborhood of Λ.[6]

A similar setting was considered in [BV00] where the authors assume that $f|_\Lambda$ has a dominated splitting (a weaker condition than partial hyperbolicity) and that every local unstable leaf has a positive Lebesgue measure set of points with negative central exponents (a stronger condition than the one above), and conclude that f has finitely many ergodic u-measures, which are in fact SRB measures. Under the further assumption that unstable manifolds are dense (as above), they conclude that there is exactly one such SRB measure.

The case of partially hyperbolic attractors with *positive* central exponents was addressed in [Vas09] where it was shown that if $f|_\Lambda$ has a unique u-measure μ with positive central exponents on a subset of *full* μ-measure, and for every $x \in \Lambda$ the strongly unstable manifold is dense in Λ, then μ is the unique SRB measure for f – indeed, this statement holds for any sufficiently small perturbation of f.

The case of attractors with dominated splitting and positive central exponents was considered in [ABV00], where it was shown that if f has a dominated splitting on a compact invariant set K and has positive central exponents for a subset of K with positive Lebesgue measure, then Lebesgue-a.e. point x in that subset is in the basin of attraction of an SRB measure.

All of the above results presuppose the existence of a hyperbolic structure on M which is uniform or *mixed* uniform – a combination of a uniform partial hyperbolicity and negative (positive) Lyapunov exponents in the central direction. In any case there is always a continuous splitting of the tangent bundle on a compact set, for which the angle between the stable and unstable directions is uniformly bounded away from zero, and the curvature of the stable and unstable manifolds is uniformly bounded. Thus these results do not address the existence of SRB measures for systems with non-uniformly hyperbolic attractors such as the Hénon attractor where that angle can become arbitrarily small near homoclinic tangencies.

Some strong results are now available that establish existence of SRB measures for the Hénon attractor (see [BY00]) and more general attractors with one unstable direction (see [WY08]). The underlying mechanism of constructing SRB measures in these systems is the work of L.-S. Young [LSY98] where she introduced a class of nonuniformly hyperbolic diffeomorphisms admitting a symbolic representation via a tower whose base is a hyperbolic set with direct product structure and the induced map on the base, generated by f, admits a Markov extension. Assuming that the return time R to the base is integrable, one can show that f admits an SRB measure. As in the case of uniformly hyperbolic systems, the real power of a

[6]The Lyapunov exponents of g in the central direction are all negative almost everywhere.

symbolic representation is not just to help prove existence of SRB measures but to show statistical properties such as exponential decay of correlations, Central Limit Theorem, etc.[7] Thus if all we are after is an SRB measure, it is reasonable to ask if we may achieve our goal without using the tower construction.

In the absence of a continuous splitting into stable and unstable directions or any symbolic representation via a Markov partition or tower construction, how do we get an SRB measure on a topological attractor? It seems reasonable to expect that a topological attractor for which Lebesgue almost every point in the basin of attraction has non-zero Lyapunov exponent ought to support an SRB measure; however, this does not seem to be the case. What is the obstacle? What do we need beyond non-zero exponents on a set of positive Lebesgue measure? The missing ingredient is *Lyapunov–Perron (forward) regularity*, which is a crucial tool in establishing the existence of local manifolds in non-uniform hyperbolicity theory, where it is obtained via Oseledets' Multiplicative Ergodic Theorem. However, this requires the presence of an invariant measure, which is here the very thing whose existence we want to establish!

To resolve this issue, we use an analogue of Lyapunov-Perron regularity that was introduced in [CP13] and which is suitable to study non-stationary systems. This condition enables one to reproduce some paradigms of non-uniform hyperbolicity (such as existence of stable and unstable local manifolds, absolute continuity, etc.), while at the same time being amenable to verification in particular situations.

1.2. **Description of approach.** Our strategy echoes a recurring theme in non-uniform hyperbolicity; you can have compactness (and uniformity), or you can have invariance, but you can't have both at once. Indeed, if you do, then you are back in the uniformly hyperbolic situation [HPS08].

The most familiar manifestation of this comes in the regular sets for a non-uniformly hyperbolic system (see [BP07]): one has a diffeomorphism f that is non-uniformly hyperbolic on an invariant (but non-compact) set Y, which can be written as an increasing union of compact (but non-invariant) sets Y_ℓ. Compactness of the sets Y_ℓ allows uniformity results that fail on all of Y, but a typical trajectory will occasionally escape from any given set Y_ℓ. The key is to control how quickly a typical trajectory can escape into the "bad" parts of the hierarchy of regular sets (where ℓ is very large), and how much time it spends there.

We play a similar game, not on the manifold itself, but in the space of invariant measures, with respect to the action induced by f. Building upon the idea of working with the sequence of measures (1.3), we consider a set $\mathcal{M}^{\mathrm{ac}}$ of measures which have SRB-like properties (but are not necessarily

[7] One can show that if for all $T > 0$ we have $\int R\, dm \leq C\lambda^T$, where $C > 0$, $0 < \lambda < 1$ and the integral is taken over the set $\{x \in \Gamma : R(x) > T\}$, then f has the exponential decay of correlations for an appropriately chosen class of functions.

f-invariant). The set $\mathcal{M}^{\text{ac}}$ is invariant, but not compact; however, we are able to find some compact (but not invariant) subset $\mathcal{M}^{\text{ac}}_{\mathbf{K}} \subset \mathcal{M}^{\text{ac}}$ such that the iterates of Lebesgue measure "return to $\mathcal{M}^{\text{ac}}_{\mathbf{K}}$ uniformly often". This will be made precise in Section 2, which contains the required definitions and the statements of our results. First, though, we give a brief summary of the means by which the above programme is carried out.

To accomplish the procedure outlined above, we do require some geometric structure on the manifold M that exhibits some sort of hyperbolicity in a weak sense: we only require a measurable decomposition into stable and unstable cones on a subset of positive Lebesgue measure. The expansion and contraction they exhibit need only be asymptotic (not uniform), and the angle between them may become arbitrarily small.

Using these cone families, we define admissible manifolds which are good approximations to unstable manifolds, and we consider measures with non-zero Lyapunov exponents that are absolutely continuous along these manifolds – this gives the set $\mathcal{M}^{\text{ac}}$ of SRB-like measures. We obtain the compact set $\mathcal{M}^{\text{ac}}_{\mathbf{K}}$ by placing uniform bounds on the geometry of and dynamics on the admissible manifolds supporting a measure, as well as on the densities of conditional measures along these manifolds. Then the action that f induces on the space of measures does not preserve the set $\mathcal{M}^{\text{ac}}_{\mathbf{K}}$; however, if one assumes that the averaged iterates of the Lebesgue measure have a uniformly large projection to $\mathcal{M}^{\text{ac}}_{\mathbf{K}}$, then our Theorem D yields the existence of an SRB measure.

In order to guarantee that the averaged iterates of Lebesgue measure have a uniformly large projection to $\mathcal{M}^{\text{ac}}_{\mathbf{K}}$, one must obtain information about the behaviour of admissible manifolds under the action of f. To this end, given a trajectory $\{f^n(x)\}$, a common technique is to consider neighbourhoods U_n of the origin in the tangent spaces $T_{f^n(x)}M$, and work with the sequence of local diffeomorphisms $f_n \colon U_n \to \mathbb{R}^d$ given by moving f to the tangent space via the exponential map.

A key tool for us will be precise estimates on the sizes and curvatures of the admissible manifolds for the sequence (f_n) in the fully non-uniform setting. These were obtained in [CP13] using the notion of "effective hyperbolicity", which we define here in §2.1.2. This quantity controls not only the expansion and contraction rates along unstable and stable directions for f_n, but also the change in the local curvature of admissible manifolds and in the angle between the stable and unstable directions. The key result regarding effective hyperbolicity is [CP13, Theorem A], which we state here as Theorem E: the images of an admissible manifold admit uniform bounds at "effective hyperbolic times", and so it suffices to check that a certain set of times has positive lower asymptotic density.

Our main results are Theorems A and B, which use Theorems D and E to prove the existence of SRB measures for local diffeomorphisms $f \colon U \to M$. Assuming the existence of two invariant cone families, one forward invariant (unstable) and one backward invariant (stable), we adapt the notion of effective hyperbolicity to give a quantity defined at each point where the cone families are. We consider the S of points for which the asymptotic rate of effective hyperbolicity is positive. If S has positive Lebesgue measure (or positive leaf volume on some admissible manifold), then we can use Theorems D and E to find an SRB measure. Trajectories of points $x \in S$ can be considered Lyapunov-Perron (forward) regular in the absence of invariant measures, although the requirements we impose seem somewhat stronger than the standard Lyapunov-Perron regularity.

Precise definitions and statements of all our results are given in Section 2, and we discuss applications in Section 3, including examples for which our methods give an SRB measure but no other techniques are known to work. In Section 4, we prove Theorem D on the general procedure for constructing an SRB measure. In the process, we prove a generalisation of the Lebesgue decomposition theorem. The proofs of Theorems A and B are given in Section 5.

2. Statement of results

2.1. **Effective hyperbolicity and main result.** Let M be a d-dimensional compact smooth Riemannian manifold, $U \subset M$ an open set, and $f\colon U \to M$ a $C^{1+\alpha}$ local diffeomorphism such that $\overline{f(U)} \subset U$. The simplest case is when $U = M$, so f is defined on the entire manifold, but there are many important examples in which $U \neq M$, and we write

$$\Lambda = \bigcap_{n \geq 0} f^n(U)$$

for the attractor onto which the trajectories of f accumulate.

The following definition of an SRB measure from [BP07, Definition 13.1.1] makes use of the notion of *regular set* introduced by Pesin.

Let μ be an invariant measure and suppose that μ is hyperbolic – that is, all Lyapunov exponents of μ are non-zero. Given a regular set Y_ℓ of positive measure, and sufficiently small $r > 0$, let $Q_\ell(x) = \bigcup_{w \in Y_\ell \cap B(x,r)} V^u(w)$ for every $x \in Y_\ell$, where $V^u(w)$ is the local unstable manifold through w. Denote by $\xi(x)$ the partition of $Q_\ell(x)$ by these manifolds, and let $\mu^u(w)$ be the conditional measure on $V^u(w)$ generated by μ with respect to the partition ξ.

Definition 2.1. A hyperbolic invariant measure μ is called an *SRB measure* if for any regular set Y_ℓ of positive measure and almost every $x \in Y_\ell$, $w \in Y_\ell \cap B(x,r)$, the conditional measure $\mu^u(w)$ is absolutely continuous with respect to the leaf volume $m_{V^u(w)}$ on $V^u(w)$.

2.1.1. *Measurable cone families.* Given $x \in M$, a subspace $E(x) \subset T_xM$, and $\theta(x) > 0$, the *cone* at x around $E(x)$ with angle $\theta(x)$ is

$$K(x, E(x), \theta(x)) = \{v \in T_xM \mid \measuredangle(v, E(x)) < \theta(x)\}. \tag{2.1}$$

We say that the cone $K(x, E, \theta)$ has $(\dim E)$-dimensional centre. If E is a measurable (k-dimensional) distribution on $A \subset M$ and the angle function $\theta\colon A \to \mathbb{R}^+$ is measurable, then (2.1) defines a (k-dimensional) *measurable cone family* $K(x) = K(x, E, \theta)$ on A. If in addition $\overline{Df(K(x))} \subset K(f(x))$ whenever $x, f(x) \in A$, we say that $K(x)$ is f-invariant.

Suppose that there exists a forward-invariant set $A \subset U$ of positive Lebesgue measure with two measurable cone families $K^s(x), K^u(x) \subset T_xM$ such that

(1) $\overline{Df(K^u(x))} \subset K^u(f(x))$ for all $x \in A$;
(2) $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$ for all $x \in f(A)$.
(3) K^s and K^u have d_s and d_u-dimensional centres, respectively, where $d_s + d_u = d$.

Remark. If there is a dominated splitting, then the existence of such cone families can be easily deduced, and in this case they are continuous. We emphasise, however, that in our setting the cone families $K^s(x)$ and $K^u(x)$ are not assumed to be continuous, but only measurable.

Although the families of subspaces that define the centres of these cone families are not assumed to be invariant, we can obtain invariant families $E^{s,u} \subset K^{s,u}$. For the stable direction (which is all we will need), we let $E^s(x) = \bigcap_{n\geq 0} Df^{-n}(f^n x)(K^s(f^n x))$. This intersection is a subspace whenever the trajectory of x has a larger rate of asymptotic expansion in $K^u(x)$ than it does in $K^s(x)$, which will be the case along all trajectories in which we are interested.

2.1.2. *Effective hyperbolicity.* Following [CP13], define $\lambda^u, \lambda^s\colon A \to \mathbb{R}$ by

$$\begin{aligned} \lambda^u(x) &= \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\}, \\ \lambda^s(x) &= \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}. \end{aligned} \tag{2.2}$$

The *defect from domination* at x is

$$\Delta(x) = \frac{1}{\alpha} \max(0,\ \lambda^s(x) - \lambda^u(x)), \tag{2.3}$$

where we recall that $\alpha \in (0, 1]$ is the Hölder exponent of Df. (Our techniques do not distinguish between C^2 maps and C^r maps with $r > 2$.) Write

$$\lambda(x) = \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)), \tag{2.4}$$

so that $\lambda(x) > 0$ whenever f expands vectors in $K^u(x)$ and contracts vectors in $K^s(x)$. We will be interested in points x whose forward trajectories satisfy

$$\varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k x) > 0. \tag{2.5}$$

A related (but not identical) condition is referred to in [CP13] as "effective hyperbolicity".

Given $\Gamma \subset \mathbb{N}$, the *upper asymptotic density* of Γ is

$$\overline{\delta}(\Gamma) = \varlimsup_{N\to\infty} \frac{1}{N} \#\Gamma \cap [0, N). \tag{2.6}$$

The lower asymptotic density $\underline{\delta}(\Gamma)$ is defined analogously.

Denote the angle between the boundaries of $K^s(x)$ and $K^u(x)$ by

$$\theta(x) = \inf\{\measuredangle(v, w) \mid v \in K^u(x), w \in K^s(x)\}. \tag{2.7}$$

We also want the points x that we consider to satisfy

$$\lim_{\bar{\theta}\to 0} \overline{\delta}\{n \mid \theta(f^n x) < \bar{\theta}\} = 0. \tag{2.8}$$

That is, the frequency with which the angle between the stable and unstable cones drops below a specified threshold $\bar{\theta}$ can be made arbitrarily small by taking the threshold to be small. We work with the set

$$S := \{x \in A \mid x \text{ satisfies (2.5) and (2.8)}\}. \tag{2.9}$$

Observe that whether or not a point x lies in S is determined in terms of a forward asymptotic property of the orbit of x, and hence S is forward invariant under f.

Theorem A. *If* $\operatorname{Leb} S > 0$*, then* f *has a hyperbolic SRB measure supported on* Λ*.*

A similar result can be formulated given information about the set of effectively hyperbolic points on a single admissible manifold. Given a submanifold $W \subset U$, we write m_W for the volume induced on W by the Riemannian metric. Let d_u, d_s, and A be as above, and let $W \subset U$ be an embedded submanifold of dimension d_u. Let

$$S_W = \{x \in S \cap W \mid T_x W \subset K^u(x)\}. \tag{2.10}$$

Theorem B. *If* $m_W(S_W) > 0$*, then* f *has a hyperbolic SRB measure supported on* Λ*.*

Both Theorems A and B give an alternate proof of the existence of SRB measures for uniformly hyperbolic diffeomorphisms. For such maps, with a suitably chosen metric, one has $\lambda^s < 0 < \lambda^u$ everywhere, and hence $S = M$. In Section 3, we apply these results to obtain SRB measures for new classes of maps.

Finally, we observe that by a result of Ledrappier [Led84], the number of ergodic hyperbolic SRB measures supported on Λ is at most countable, and moreover, for each such measure μ there is a number n such that the diffeomorphism f^n is Bernoulli with respect to μ.

2.1.3. *Effectively hyperbolic times.* The definition of S in (2.9) uses (2.5), which in turn relies on controlling the quantity $\lambda(x)$ along a trajectory. Positivity of $\lambda(x)$ requires simultaneous expansion in K^u and contraction in K^s. In fact it is possible to obtain the above results using a weaker condition (which is somewhat more technical).

Given $x \in A$ and $\overline{\lambda} > 0$, consider the set of *effective hyperbolic times*

$$\Gamma^e_{\overline{\lambda}}(x) = \left\{ n \mid \sum_{j=k}^{n-1} (\lambda^u - \Delta)(f^j x) \geq \overline{\lambda}(n-k) \text{ for all } 0 \leq k < n \right\}. \tag{2.11}$$

We also need a condition on the contraction in the stable direction. Given $C, \overline{\lambda} > 0$ and $q \in \mathbb{N}$, let

$$\Gamma^s_{C,\overline{\lambda},q}(x) = \big\{ n \in \mathbb{N} \mid \|Df^k(f^{n-k}x)(v)\| \leq Ce^{-\overline{\lambda}k}\|v\| \\ \text{for all } 0 \leq k \leq q \text{ and } v \in E^s(f^{n-k}x) \big\}. \tag{2.12}$$

This is similar to the condition in (2.11), but only requires control of the dynamics for the iterates $[n-q, n]$ instead of $[0, n]$. Pliss' lemma [BP07, Lemma 11.2.6] shows that if x satisfies

$$\varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k x) > \chi \tag{2.13}$$

for some $\chi > 0$, then for every $\bar{\lambda} \in (0, \chi)$ and $q \in \mathbb{N}$, we have

$$\underline{\delta}\left(\Gamma^e_{\overline{\lambda}}(x) \cap \Gamma^s_{1,\overline{\lambda},q}(x)\right) \geq \frac{\bar{\lambda} - \chi}{L - \chi},$$

where $L = \sup_x \lambda(x)$. In particular, every $x \in S$ has the property that there are $C, \overline{\lambda} > 0$ such that

$$\lim_{q\to\infty} \underline{\delta}\left(\Gamma^e_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)\right) > 0. \tag{2.14}$$

Let $\hat{S} = \{x \mid (2.8) \text{ and } (2.14) \text{ hold}\}$. Then $S \subset \hat{S}$. Moreover, we have the following.

Theorem C. *Theorems A and B continue to hold if S is replaced by $\hat{S}$.*

2.2. **General technique for constructing SRB measures.**

2.2.1. *Notation and strategy.* Let $\mathcal{M}(M)$ denote the set of all finite Borel measures on M, and let $\mathcal{M}(\Lambda)$ be the set of all finite Borel measures supported on Λ. Every f-invariant measure giving full weight to U is supported on Λ; denote the set of such measures by $\mathcal{M}(\Lambda, f)$.

Given $\mu \in \mathcal{M}(M)$, write $\|\mu\| = \mu(M)$ for the total weight of μ, and let $\mathcal{M}_1(M) = \{\mu \in \mathcal{M}(M) \mid \|\mu\| = 1\}$ be the set of Borel probability measures. Similarly, let $\mathcal{M}_1(\Lambda, f)$ be the set of f-invariant Borel probability measures. Let $\mathcal{M}^{\mathrm{h}}$ be the set of *hyperbolic* measures – that is, f-invariant Borel probability measures for which all Lyapunov exponents are non-zero almost everywhere.

To characterise SRB measures, we consider a collection $\mathcal{R}$ of submanifolds $W \subset U$ which have most of the properties of local unstable manifolds (precise definitions are in the next section). To each $W \in \mathcal{R}$ is associated a leaf volume m_W induced by the Riemannian metric. We let $\mathcal{M}^{\mathrm{ac}}$ be the set of finite measures $\mu \in \mathcal{M}(M)$ that have the following absolute continuity property: $\mu(E) = 0$ whenever $E \subset U$ is such that $m_W(E) = 0$ for every $W \in \mathcal{R}$. Then it is relatively straightforward to show that $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$ is precisely the set of SRB measures, and so we can prove existence of an SRB measure by proving that $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$ is non-empty.

To this end, we follow the well-established idea of studying the pushforwards of Lebesgue measure given by

$$(f_*^n \operatorname{Leb})(E) = \operatorname{Leb}(f^{-n}(E)). \tag{2.15}$$

As usual, we consider the Césaro averages μ_n given by (1.2). Each of these lies in $\mathcal{M}^{\mathrm{ac}}$, and weak* compactness of $\mathcal{M}_1(M)$ yields an invariant measure μ as the limit of a subsequence μ_{n_j}. However, because $\mathcal{M}^{\mathrm{ac}}$ is not necessarily compact, this does not yet show that $\mu \in \mathcal{M}^{\mathrm{ac}}$. Similarly, $\mathcal{M}^{\mathrm{h}}$ is not necessarily compact, and so we need some mechanism for preserving absolute continuity and hyperbolicity properties upon passing to the limit. In fact, μ_n is not necessarily invariant and so is not in $\mathcal{M}^{\mathrm{h}}$, and we will need to consider a different set of measures in place of $\mathcal{M}^{\mathrm{h}}$ to allow for non-invariance.

To achieve this we consider a compact (but not invariant) subset $\mathcal{M}_{\mathbf{K}}^{\mathrm{ac,h}} \subset \mathcal{M}^{\mathrm{ac}}$, defined as follows. Let $\mathcal{R}'$ be the collection of *standard pairs* (W, ρ) (see [CD09]), where $W \in \mathcal{R}$ and $\rho \in L^1(W, m_W)$ is a density function. A standard pair determines a measure $\Phi(W, \rho) \in \mathcal{M}(M)$ in the natural way, and the map $\Phi \colon \mathcal{R}' \to \mathcal{M}(M)$ induces a continuous map $\Phi^* \colon \mathcal{M}(\mathcal{R}') \to \mathcal{M}(M)$, where $\mathcal{M}(\mathcal{R}')$ is the collection of finite measures on the space of standard pairs endowed with the weak* topology (see Section 2.2.3).

Passing to a subset $\mathcal{R}'_{\mathbf{K}} \subset \mathcal{R}'$ that is compact in a certain topology, we let $\mathcal{M}_{\mathbf{K}}^{\mathrm{ac,h}} = \Phi^*(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K}}))$, where $\mathcal{M}_{\leq 1}$ denotes measures with total weight at most 1. Because Φ^* is continuous, the collection $\mathcal{M}_{\mathbf{K}}^{\mathrm{ac,h}}$ is compact. We use effective hyperbolicity to show that the measures μ_n all have uniformly positive projection to $\mathcal{M}_{\mathbf{K}}^{\mathrm{ac,h}}$, and by compactness the limiting measure μ does as well. Thus some ergodic component of μ is contained in $\mathcal{M}^{\mathrm{ac,h}}$. Membership in $\mathcal{M}^{\mathrm{h}}$ is guaranteed by effective hyperbolicity, and this completes the argument.

In fact, our argument will work for *any* sequence of measures μ_n with the property that any limiting measure is hyperbolic and the measures have a uniformly positive projection to $\mathcal{M}_{\mathbf{K}}^{\mathrm{ac}}$. For example, as in Theorem B, we may consider the averaged push-forwards of any measure supported on a single admissible manifold that is absolutely continuous with respect to leaf volume.

2.2.2. *Admissible manifolds.* Before stating the general results, we establish a notion of admissible manifolds and give a precise description of the space of measures with which we will work.

Definition 2.2. Given $x \in M$, let $\exp_x \colon T_x M \to M$ be the exponential map and Ω a neighbourhood of x on which $\exp_x^{-1}$ is well-defined. Fix $\gamma, \kappa, r > 0$

and a subspace $F \subset T_xM$. We say that a smooth embedded submanifold $V(x) \subset \Omega$ that passes through the point x is (γ,κ)-*admissible of size* r *with transversal* F if the following conditions are satisfied.

(1) $T_xM = G \oplus F$, where $G := T_xV(x)$.
(2) $V(x) = \exp_x \tilde{V}(x)$, where $\tilde{V}(x) \subset T_xM$ is the graph of a $C^{1+\alpha}$ function[8] $\psi\colon B_G(r) \to F$. (Here $B_G(r) = B(0,r) \cap G \subset T_xM$.)
(3) $D\psi$ is bounded by γ and is α-Hölder with constant κ:

$$\begin{aligned} \|D\psi\| &:= \sup_{v\in B_G(r)} \|D\psi(v)\| \leq \gamma, \\ |D\psi|_\alpha &:= \sup_{v_1,v_2\in B_G(r)} \frac{\|D\psi(v_1) - D\psi(v_2)\|}{\|v_1 - v_2\|^\alpha} \leq \kappa. \end{aligned} \tag{2.16}$$

(It follows from the definition that $\psi(0) = 0$ and $D\psi(0) = 0$.)

Remark. Our definition of admissible manifold is reminiscent of the notion of admissible manifolds in [BP07] and also of *manifolds tangent to a cone field* used in [ABV00]. There are several differences between those definitions and this one: most importantly, in (2.16) we require control not just of $\|D\psi\|$, but also of the Hölder constant of $D\psi$, so that we can bound the (Hölder) curvature of $V(x)$. Furthermore, unlike [BP07], we do not use Lyapunov coordinates, but rather work in the original Riemannian metric, and unlike [ABV00], we look at the image of the manifold in a single tangent space T_xM, rather than in all the tangent spaces T_yM for $y \in V(x)$. This reflects a crucial feature of our setting in which the relevant cones need not be defined in all of these tangent spaces.

Given $\theta, \gamma, \kappa, r > 0$, consider the following collection of "(γ,κ)-admissible manifolds of size r with transversals controlled by θ":

$$\begin{aligned} \mathcal{P}_{(\theta,\gamma,\kappa,r)} = \{&\exp_x(\operatorname{graph}\psi) \mid x \in U,\ T_xM = G \oplus F, \\ &\measuredangle(G,F) \geq \theta,\ \psi \in C^{1+\alpha}(B_G(r),F) \text{ satisfies (2.16)}\}. \end{aligned} \tag{2.17}$$

Admissible manifolds in $\mathcal{P}$ have controlled geometry. We also impose a condition on the dynamics. Fixing $C, \overline{\lambda} > 0$, consider for each $N \in \mathbb{N}$ the collection of sets

$$\begin{aligned} \mathcal{Q}_{(C,\overline{\lambda},N)} = \{&f^N(V_0) \mid V_0 \subset U, \text{ and for every } y,z \in V_0, \text{ we have} \\ &d(f^j(y), f^j(z)) \leq Ce^{-\overline{\lambda}(N-j)} d(f^N(y), f^N(z)) \text{ for all } 0 \leq j \leq N\}. \end{aligned} \tag{2.18}$$

(Note that in particular we have $W \subset f^N(U)$ for every $W \in \mathcal{Q}_{(C,\overline{\lambda},N)}$.)

The condition in (2.18) guarantees that given an admissible manifold $W \in \mathcal{Q}_{(C,\overline{\lambda},N)}$, every point in W has a backwards trajectory that contracts in the directions tangent to W. This will eventually give us positive Lyapunov exponents. We also need a condition to guarantee negative Lyapunov

[8] Recall that α is such that f is $C^{1+\alpha}$.

exponents, namely that "many" points in W have backwards trajectories that expand in the directions transverse to W.

To this end, to every $W \in \mathcal{P}_{(\theta,\gamma,\kappa,r)}$ we associate the leaf volume m_W induced on W by the Riemannian metric. Then if $W = f^N(V_0)$ for some $V_0 \subset U$, we consider the set

$$\begin{aligned} H_{(C,\overline{\lambda},N)}(W) = \{f^N(y) \mid y \in V_0, T_yM = (T_yV_0) \oplus G \\ \text{for some } G \text{ with } \|Df^{N-j}(f^j(y))|_{Df^j(y)(G)}\| \le Ce^{-\overline{\lambda}(N-j)} \\ \text{for every } 0 \le j \le N\}, \end{aligned} \tag{2.19}$$

of all points in W whose backwards trajectories of length N have expansion controlled by $(C, \overline{\lambda})$ in a direction transverse to TW.

Given $\beta > 0$, we write $\mathbf{K} = (\theta, \gamma, \kappa, r, C, \overline{\lambda}, \beta)$ for notational convenience and consider

$$\mathcal{R}_{\mathbf{K},N} = \left\{ W \in \mathcal{P}_{(\theta,\gamma,\kappa,r)} \cap \mathcal{Q}_{(C,\overline{\lambda},N)} \mid \frac{m_W\left((H_{(C,\overline{\lambda},N)}(W)\right)}{m_W(W)} \ge \beta \right\}. \tag{2.20}$$

The collection $\mathcal{R}_{\mathbf{K},N}$ may be thought of as the set of all admissible manifolds with geometry and dynamics over N backwards iterates controlled uniformly by $\mathbf{K}$. We will also have occasion to consider the collections

$$\begin{aligned} \mathcal{R}_{\mathbf{K}} &= \bigcap_{N \in \mathbb{N}} \mathcal{R}_{\mathbf{K},N}, \\ \mathcal{R} &= \bigcup_{\mathbf{K}} \mathcal{R}_{\mathbf{K}}, \end{aligned} \tag{2.21}$$

where the union in the final definition is taken over all $\theta, \gamma, \kappa, r, C, \overline{\lambda}, \beta > 0$.

Note that elements of $\mathcal{R}$ are genuine unstable manifolds (not just admissible), because we control the entire backwards trajectory. Having established the manifolds with which we work, we consider the following collection of standard pairs:

$$\mathcal{R}' = \{(W, \rho) \mid W \in \mathcal{R},\ \rho \in L^1(W, m_W)\}. \tag{2.22}$$

We will also need to consider a collection of standard pairs where the densities ρ are controlled uniformly. Fixing $L > 0$, we abuse notation slightly by implicitly including L in $\mathbf{K}$ and putting

$$\mathcal{R}'_{\mathbf{K}} = \{(W, \rho) \mid W \in \mathcal{R}_{\mathbf{K}},\ \rho \in C^\alpha(W, [1/L, L]),\ |\rho|_\alpha \le L\}, \tag{2.23}$$

where $|\rho|_\alpha = \sup_{y,z \in W} \frac{|\rho(y)-\rho(z)|}{d(y,z)^\alpha}$ and we recall that α is the Hölder exponent of Df. We make a similar definition for $\mathcal{R}'_{\mathbf{K},N}$.

2.2.3. *SRB measures.* Now we can define collections of measures on U with certain absolute continuity properties that will eventually allow us to obtain an SRB measure. First we need to consider the collection of sets that are null sets for some cover by admissible manifolds. More precisely, let

$$\mathcal{N}(\mathcal{R}) = \Big\{E \subset U \mid \text{ there is } \mathcal{W} \subset \mathcal{R} \text{ such that } E \subset \bigcup_{W\in\mathcal{W}} W \text{ and } m_W(E) = 0 \text{ for all } W \in \mathcal{W}\Big\}. \tag{2.24}$$

Then we consider the set of measures

$$\mathcal{M}^{\mathrm{ac}} = \{\mu \in \mathcal{M}(U) \mid \mu(E) = 0 \text{ for all } E \in \mathcal{N}(\mathcal{R})\}. \tag{2.25}$$

The following proposition will be proved in Section 4.

Proposition 2.1. *The intersection $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$ is precisely the set of SRB measures for f.*

Remark. The dimension of the admissible manifolds W is not specified – indeed, $\mathcal{R}$ contains manifolds at points x whose dimension is not maximal among admissible manifolds through x. However, the definition of $\mathcal{M}^{\mathrm{ac}}$ forces μ to give zero weight to any set that can be covered by admissible manifolds of any dimension such that it is a null set for each, which in particular forces μ to be absolutely continuous in all unstable directions.

As observed above, the sets $\mathcal{M}^{\mathrm{ac}}$ and $\mathcal{M}^{\mathrm{h}}$ are not necessarily weak*-compact. To obtain a suitable compact set, we associate to each standard pair (W, ρ) a linear functional $\Phi(W, \rho) \in C(M)^*$ by

$$\Phi(W, \rho)(a) = \int_W a(x)\rho(x)\, dm_W(x) \tag{2.26}$$

for $a \in C(M)$. Note that $\Phi(W, \rho)$ can be interpreted as a measure on M.

Let $\mathcal{M}(\mathcal{R}'_{\mathbf{K},N})$ denote the space of finite Borel measures on $\mathcal{R}'_{\mathbf{K},N}$. The map Φ induces a map $\Phi^*\colon \mathcal{M}(\mathcal{R}'_{\mathbf{K},N}) \to \mathcal{M}(M)$ by associating to each $\eta \in \mathcal{M}(\mathcal{R}'_{\mathbf{K},N})$ the measure $\Phi^*(\eta) \in \mathcal{M}(M)$ given by

$$\begin{aligned}\int_M a\, d\Phi^*(\eta) &:= \int_{\mathcal{R}'_{\mathbf{K},N}} \Phi(W, \rho)(a)\, d\eta(W, \rho) \\ &= \int_{\mathcal{R}'_{\mathbf{K},N}} \int_W a(x)\rho(x)\, dm_W(x)\, d\eta(W, \rho).\end{aligned} \tag{2.27}$$

(It is worth emphasising that Φ^* is not one-to-one.) Writing $\mathcal{M}_{\leq 1}$ for the collection of finite measures with total weight at most 1, we let

$$\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},N} = \Phi^*(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N})). \tag{2.28}$$

Thus $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},N}$ comprises measures that are supported on admissible manifolds with uniformly bounded size and curvature, and that are absolutely continuous on these manifolds, with well-behaved density functions; furthermore, the admissible manifolds supporting these measures must display some uniform hyperbolicity for N backwards iterates.

Given measures $\nu, \mu \in \mathcal{M}(M)$, we write $\nu \le \mu$ if $\nu(E) \le \mu(E)$ for every measurable set $E \subset M$. Given a sequence of measures on M and and a sequence of numbers $q_n \to \infty$, we say that μ_n have *uniformly large projections onto* $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},q_n}$ if there exist $\delta > 0$ and a sequence of measures $\eta_n \in \mathcal{M}(\mathcal{R}'_{\mathbf{K},q_n})$ such that $\Phi^*(\eta_n) \le \mu_n$ and $\|\Phi^*(\eta_n)\| \ge \delta$ for every n.

Theorem D. *Let M be a compact Riemannian manifold, $U \subset M$ an open set, and $f\colon U \to M$ a $C^{1+\alpha}$ local diffeomorphism such that $\overline{f(U)} \subset U$. Let also μ_n be a sequence of measures on M that converges in the weak* topology to an invariant measure μ. Suppose that there exists $\mathbf{K} = (\theta, \gamma, \kappa, r, C, \overline{\lambda}, \beta, L)$ and a sequence $q_n \to \infty$ such that the measures μ_n have uniformly large projections onto $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},q_n}$. Then μ has an ergodic component that is a hyperbolic invariant SRB measure.*

Remark. The collections $\mathcal{M}^{\mathrm{ac}}$ and $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K}}$ are built using two different notions of absolute continuity. The criterion for inclusion in $\mathcal{M}^{\mathrm{ac}}$ is that a measure gives zero weight to a certain collection of null sets defined using the reference measures m_W, while the criterion for inclusion in $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K}}$ is that a measure be defined in terms of integrating densities against measures m_W. The second criterion immediately implies the first, and if there was only a single admissible manifold W, with corresponding reference measure m_W, then the converse implication would be the Radon–Nikodym theorem. However, there are many different admissible manifolds W in $\mathcal{R}$, and so in our setting it is not clear whether the converse holds.

2.3. **A non-stationary Hadamard–Perron theorem.** In order to apply Theorem D, we need to obtain a sequence of measures μ_n that satisfies the conditions of the theorem. In Theorems A–C, these are built following the approach in (1.2) and (1.3), by pushing forward either Lebesgue measure on the manifold or leaf volume on an admissible manifold using the dynamics of f. The recurrence properties of μ_n are controlled via the notion of effective hyperbolicity (described in Section 2.1.2), using results from [CP13], which we now recall.

Fixing an initial point $x \in U$, let Ω_n be a neighborhood of $f^n(x)$ that can be identified with a neighborhood of the origin in $\mathbb{R}^d$ using the exponential map $T_{f^n(x)}M \to M$. We write $f_n\colon \Omega_n \to \mathbb{R}^d$ for the coordinate representation of f with respect to this exponential map.

We consider trajectories that do not necessarily lie on the attractor, and so are *a priori* only one-sided; thus we shall deal with a sequence of maps $\{f_n \mid n \in \mathbb{N}\}$. An invariant family of admissible manifolds can be obtained by choosing any manifold $W_0 \subset \Omega_0$, and putting $W_{n+1} = f_n(W_n) \cap \Omega_{n+1}$.

There are competing goals in the selection of the neighborhood Ω_n: on the one hand, Ω_n must be small enough so that W_n is admissible and we have some control of its (Hölder) curvature; on the other hand, Ω_n must be large enough that we can give some useful lower bound on the size of W_n.

These considerations are dealt with in [CP13]. We use Theorem A from that paper, which gives conditions on the maps f_n under which any admissible manifold W_0 has uniformly large admissible images W_n with bounded (Hölder) curvature for a set of times n with positive asymptotic frequency.

Fix a neighborhood of the origin $\Omega \subset \mathbb{R}^d$ and let $f_n \colon \Omega \to \mathbb{R}^d$ be a sequence of $C^{1+\alpha}$ diffeomorphisms onto their images such that $\|Df\|$, $\|Df^{-1}\|$, and $|Df_n|_\alpha$ are uniformly bounded by some constant e^L.

Suppose $\mathbb{R}^d = E^u_n \oplus E^s_n$ is a sequence of decompositions and $K^{u,s}_n$ is a sequence of cones around $E^{u,s}_n$ which are $Df_n(0)$-invariant – that is, we have $\overline{Df_n(0)(K^u_n)} \subset K^u_{n+1}$ and $\overline{Df_n(0)^{-1}(K^s_{n+1})} \subset K^s_n$ for each n. Let $L' \in \mathbb{R}$ be such that the angle θ_n between K^u_n and K^s_n satisfies $\theta_{n+1} \geq e^{-L'}\theta_n$ for all n.[9]

As in (2.2) and (2.3), consider

$$\begin{aligned} \lambda^u_n &= \inf\{\|Df_n(0)(v)\| \mid v \in K^u_n, \|v\| = 1\}, \\ \lambda^s_n &= \sup\{\|Df_n(0)(v)\| \mid v \in K^s_n, \|v\| = 1\}, \\ \Delta_n &= \frac{1}{\alpha}\max(0,\ \lambda^s_n - \lambda^u_n). \end{aligned} \tag{2.29}$$

Let $L'' = \max\left(\frac{L'}{\alpha}, L(1+\frac{2}{\alpha})\right)$. Fix $\bar\theta > 0$ and let

$$\lambda^e_n = \begin{cases} \lambda^u_n - \Delta_n & \theta_n \geq \bar\theta, \\ -L'' & \theta_n < \bar\theta. \end{cases} \tag{2.30}$$

The following theorem is a consequence of [CP13, Theorem A]. Further details are given in Section 5.3, including in particular the relationship between times n satisfying the condition of the theorem (which imposes a restriction on λ^e_n) and times $n \in \Gamma^e_{\bar\lambda}(x) \cap \Gamma^s_{C,\bar\lambda,q}(x)$ whose existence and frequency is guaranteed by the hypotheses of Theorem A (but which only consider the case when $\lambda^e_n = \lambda^u_n - \Delta_n$, and not the case when $\lambda^e_n = -L''$).

Theorem E. *Given $L, L', \bar\theta, \bar\chi > 0$ there are $\bar\gamma, \bar\kappa, \bar r, \delta > 0$ such that the following is true. If $\theta_0 \geq \bar\theta$ and $n \in \mathbb{N}$ is such that*

$$\sum_{j=k}^{n-1} \lambda^e_k \geq (n-k)\bar\chi \text{ for all } 0 \leq k < n, \tag{2.31}$$

then $\theta_n \geq \overline{\theta}$; moreover, if $\psi_0 \colon B_{E^u_0}(\eta) \to E^s_0$ is an arbitrary $C^{1+\alpha}$ function with $\psi_0(0) = 0$, $D\psi_0(0) = 0$, and $|D\psi_0|_\alpha \leq \bar\kappa$, then for each n satisfying (2.31) there exists a $C^{1+\alpha}$ function $\psi_n \colon B_{E^u_n}(\bar r) \to E^s_n$ satisfying the following conditions:

(1) $\psi_n(0) = 0$, $D\psi_n(0) = 0$;
(2) $\|D\psi_n\| \leq \bar\gamma$ and $|D\psi_n|_\alpha \leq \bar\kappa$;

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[9]Existence of an L' satisfying this is automatic when the sequence f_n comes from a single diffeomorphism on a compact manifold.

(3) *the graph of* ψ_n *is the connected component of* $F_n(\text{graph}(\psi_0)) \cap B_{E_n^u}(\bar{r}) \times B_{E_n^s}(\bar{\gamma}\bar{r})$ *containing the origin;*

(4) *writing* $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0$*, if* $F_n(x), F_n(y) \in \text{graph}(\psi_n)$*, then for every* $0 \le k \le n$*, we have*

$$\|F_n(x) - F_n(y)\| \ge e^{(n-k)\bar{\chi}} \|F_k(x) - F_k(y)\|. \tag{2.32}$$

3. Applications

3.1. **Large local perturbations of Axiom A systems: abstract conditions.** We will describe a class of non-uniformly hyperbolic examples to which Theorem A can be applied, establishing existence of an SRB measure. These examples are obtained by beginning with a uniformly hyperbolic system and making a large local perturbation that satisfies certain conditions. In §3.2 we describe explicitly a family of maps satisfying these conditions – these generalise the Katok map [Kat79] to a dissipative setting.

Let M be a d-dimensional smooth Riemannian manifold and $U \subset M$ an open set. Let $f\colon U \to M$ be a $C^{1+\alpha}$ diffeomorphism onto its image with $\overline{f(U)} \subset U$, and let $\Lambda = \bigcap_{n \ge 0} f^n(U)$ be the attractor for f. Assume that Λ is a hyperbolic set for f, so that for every $x \in \Lambda$ we have

$$\begin{aligned} T_x M &= E^u(x) \oplus E^s(x), \\ \|Df(x)(v^u)\| &\ge \chi \|v^u\| \text{ for all } v^u \in E^u(x), \\ \|Df(x)(v^s)\| &\le \chi^{-1} \|v^s\| \text{ for all } v^s \in E^s(x), \end{aligned} \tag{3.1}$$

where $\chi > 1$ is fixed. Note that we pass to an adapted metric if necessary. Note also that since the splitting is continuous in x, it extends to a small neighbourhood of Λ, and so in particular we may assume without loss of generality that (3.1) continues to hold for all $x \in U$.

We assume that the unstable distribution E^u is one-dimensional, and consider a map $g\colon U \to M$ that is a $C^{1+\alpha}$ diffeomorphism onto its image such that $g = f$ outside of an open set $Z \subset U$. Our conditions are formulated in terms of the action of g as trajectories pass through Z. We are most interested in the case when Z is a small neighborhood of a fixed point, so that there are some points whose g-orbits never leave Z.

Let $G\colon U \setminus Z \to U \setminus Z$ be the first return map. Given $\gamma > 0$, let $K_\gamma^{s,u}(x)$ be the stable and unstable cones of width γ for the unperturbed map f. We require the following condition:

(C1) There is $\gamma > 0$ such that $\overline{DG(K_\gamma^u(x))} \subset K_\gamma^u(G(x))$ and $DG(K_\gamma^s(x)) \supset \overline{K_\gamma^s(G(x))}$ for every $x \in U \setminus Z$.

Extend the cone families $K_\gamma^{u,s}(x)$ from $U \setminus Z$ to Z by pushing them forward with the dynamics of g. Condition **(C1)** guarantees that we obtain measurable[10] invariant cone families $K^{u,s}$ on all of U.

Let $\mathcal{A}$ be the collection of $C^{1+\alpha}$ curves $W \subset U \setminus Z$ with the property that $T_x W \subset K^u(x)$ for all $x \in W$. Given $L \in \mathbb{R}$ and $W \in \mathcal{A}$, we say that

[10]Indeed, continuous everywhere except possibly the boundary of Z.

W has *Hölder curvature bounded by* L if the unit tangent vector to W is (L, α)-Hölder with respect to the point along the curve. Fixing $L, \varepsilon > 0$, let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(L, \varepsilon)$ be the collection of curves in $\mathcal{A}$ with length between ε and 2ε and Hölder curvature bounded by L.

Given $W \in \mathcal{A}$, we say that an *admissible decomposition* for W is a (possibly infinite) collection of $W_j \subset W$ and $\tau_j \in \mathbb{N}$ such that $W \setminus \bigcup_j W_j$ is m_W-null and every W_j satisfies $g^{\tau_j}(W_j) \subset U \setminus Z$. Given an admissible decomposition, we write $t(x) = \tau_j$ for all $x \in W_j$, and $\bar{G}(x) = g^{\tau(x)}(x)$ for the induced map, so $\bar{G}(W_j) \subset U \setminus Z$. For $t \in \mathbb{N}$, we write $W(t) = \{x \in W \mid \tau(x) = t\}$.

Remark. If $g(W) \subset U \setminus Z$, then any partition yields an admissible decomposition with $\tau \equiv 1$. When $g(W)$ enters Z, the time τ_j must be taken large enough to allow $\bar{G}(W_j)$ to escape Z. We stress that $\bar{G}$ depends on W and on the choice of admissible decomposition, and need not be the first return map G. In our examples, $\bar{G}$ will be either G or $G \circ g$.

By invariance of K^u, we see that $\bar{G}(W_j) \in \mathcal{A}$. The following condition requires that there be an admissible decomposition for which we control the size and curvature of $\bar{G}(W_j)$, as well as the expansion of $\bar{G}$ on W_j.

(C2) There are $L, \varepsilon, Q > 0$ and $p \colon \mathbb{N} \to [0, 1]$ such that $\sum_{t \geq 1} tp(t) < \infty$ and every $W \in \tilde{\mathcal{A}}$ with $g(W) \cap Z \neq \emptyset$ has an admissible decomposition satisfying
- (i) $m_W(W(t)) \leq p(t) m_W(W)$ for all $t \in \mathbb{N}$;
- (ii) $\bar{G}(W_j) \in \tilde{\mathcal{A}}$ for every j; and
- (iii) if $x, y \in W_j$ then $\log \frac{|D\bar{G}(x)|_{T_x W}|}{|D\bar{G}(y)|_{T_y W}|} \leq Q d(\bar{G}x, \bar{G}y)^{\alpha}$.

Remark. Condition **(C2)** can be thought of as an analogue of the familiar inducing scheme/tower construction. The role of inducing time is played by t, which is such that at time t, each W_j returns to uniformly large scale (this is (ii)) with bounded distortion (this is (iii)). We think of the function p as a "probability envelope" that controls the probability of encountering different return times. The condition $\sum tp(t) < \infty$, together with (i), corresponds to the requirement that inducing time be integrable (expected inducing time is finite). Condition **(C3)** below will guarantee that there is a choice of inducing time at which we have uniform hyperbolicity – this is made precise in Lemma 6.1.

In our examples, g is obtained by slowing down f near a fixed point. In this case there is a natural admissible decomposition such that each $\bar{G}(W_j)$ has length between ε and 2ε, and so the challenge will be to prove an expansion estimate to verify (i), an estimate on Hölder curvature to verify (ii), and a bounded distortion estimate to verify (iii).

Finally, we need to control the defect from domination of trajectories passing through Z. Given $x \in Z$, let $\Delta(x) = \frac{1}{\alpha} \max(0, \lambda^s(x) - \lambda^u(x))$. We need to control the expansion, contraction, and defect through Z in terms of how often an orbit can enter the neighbourhood Z. We suppose that

(C3) there is $C > 0$ such that given W as in **(C2)** and $x \in W$, we have[11]

$$\sum_{j=k}^{\tau(x)} \lambda^u(g^j x) - \Delta(g^j x) \geq -C, \qquad \sum_{j=k}^{\tau(x)} \lambda^s(g^j x) \leq C \tag{3.2}$$

for every $0 \leq k \leq \tau(x)$. Moreover, we suppose that every orbit of f leaving Z takes more than $C/\log \chi$ iterates to return to Z.

We give examples of systems satisfying the above conditions in the next section. These conditions let us apply the main results to obtain an SRB measure.

Theorem 3.1. *Let g be a $C^{1+\alpha}$ perturbation of an Axiom A system, and suppose that g satisfies conditions **(C1)**–**(C3)**. Then g has an SRB measure.*

The proof of Theorem 3.1 is given in Section 6 and goes as follows. Given a small admissible curve $W \in \tilde{\mathcal{A}}$, we study the sequence of escape times through Z for a trajectory starting at $x \in W$. This is a sequence of random variables with respect to m_W, and while this sequence is not independent or identically distributed, **(C2)** lets us control the average value of this sequence. This in turn gives good bounds on the sum of $\lambda(x)$ along a trajectory, and also controls the frequency with which the angle between stable and unstable cones degenerates. Ultimately, we will conclude that m_W-a.e. point $x \in W$ lies in the set $\hat{S}$ (see (2.14)), which allows us to apply Theorem C and deduce existence of an SRB measure.

3.2. **Maps on the boundary of Axiom A: neutral fixed points.** We give a specific example of a map for which the conditions of Theorem 3.1 can be verified. Let $f: U \to M$ be a $C^{1+\alpha}$ Axiom A diffeomorphism onto its image with $\overline{f(U)} \subset U$, where $\alpha \in (0,1)$. Suppose that f has one-dimensional unstable bundle.

Let p be a fixed point for f. We perturb f to obtain a new map g that has an indifferent fixed point at p. The case when M is two-dimensional and f is volume-preserving was studied by Katok [Kat79]. We allow manifolds of arbitrary dimensions and (potentially) dissipative maps. For example, one can choose f to be the Smale-Williams solenoid or its sufficiently small perturbation.

We suppose that there exists a neighborhood $Z \ni p$ with local coordinates in which f is the time-1 map of the flow generated by

$$\dot{x} = Ax \tag{3.3}$$

for some $A \in GL(d, \mathbb{R})$. Assume that the local coordinates identify the decomposition $E^u \oplus E^s$ with $\mathbb{R} \oplus \mathbb{R}^{d-1}$, so that $A = A_u \oplus A_s$, where $A_u = \gamma \,\mathrm{Id}_u$

[11] A slightly more restrictive condition would be to require that $\sum_{j=k}^{\tau(x)} \lambda(g^j x) \geq -C$ for each k. This does not automatically follow from (3.2) because the bounds there are only one-sided. The stronger condition would allow us to use Theorem B in place of Theorem C to prove Theorem 3.1. However, (3.2) is easier to verify.

and $A_s = -\beta \operatorname{Id}_s$ for some $\gamma, \beta > 0$. (This assumption of conformality in the stable direction is made primarily for technical convenience and should not be essential.) Note that in the Katok example we have $d = 2$ and $\gamma = \beta$ since the map is area-preserving. In the more general setting when $\gamma \neq \beta$, certain estimates from the original Katok example no longer hold.

From now on we use local coordinates on Z and identify p with 0. Fix $0 < r_0 < r_1$ such that $B(0, r_1) \subset Z$, and let $\psi\colon Z \to [0,1]$ be a $C^{1+\alpha}$ function such that

(1) $\psi(x) = \|x\|^\alpha$ for $\|x\| \leq r_0$;
(2) $\psi(x) = 1$ for $\|x\| \geq r_1$.

Let $\mathcal{X}\colon Z \to \mathbb{R}^d$ be the vector field given by $\mathcal{X}(x) = \psi(x)Ax$. Let $g\colon U \to M$ be given by the time-1 map of this vector field on Z and by f on $U \setminus Z$. Note that g is $C^{1+\alpha}$ because $\mathcal{X}$ is $C^{1+\alpha}$.

Theorem 3.2. *The map g satisfies conditions **(C1)**–**(C3)**, hence g has an SRB measure by Theorem 3.1.*

Remark. Note that g does not have a dominated splitting because of the indifferent fixed point, and hence this example is not covered by [ABV00].

We also observe that if ψ is taken to be C^∞ away from 0, then g is also C^∞ away from the point p. The condition $\psi(x) = \|x\|^\alpha$ near 0 for $\alpha < 1$ takes the place of the condition in [Kat79] that $1/|\psi|$ be integrable ensuring, in the case of area preserving maps, existence of a finite absolutely continuous invariant measure for the map g.

The proof of Theorem 3.2 is given in Section 7.

4. Proof of Theorem D

Our first tool for the proof of Theorem D is a generalisation of the Lebesgue decomposition theorem, following the proof given in [Bro71].

Let (X, Ω) be a measurable space, and let $\mathcal{M}$ denote the collection of all finite measures on X. We say that a collection of subsets $\mathcal{N} \subset \Omega$ is a *candidate collection of null sets* if it is closed under passing to subsets and countable unions:

(1) if $E \in \mathcal{N}$ and $F \in \Omega$, $F \subset E$, then $F \in \mathcal{N}$;
(2) if $\{E_n\} \subset \mathcal{N}$ is a countable collection, then $\bigcup_n E_n \in \mathcal{N}$.

Given a candidate collection of null sets, let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ be the subspace of $\mathcal{M}$ defined by

$$\mathcal{S} = \{\mu \in \mathcal{M} \mid \mu(E) = 0 \text{ for all } E \in \mathcal{N}\}. \tag{4.1}$$

Given a subspace $\mathcal{S} \subset \mathcal{M}$, define the space of singular measures by

$$\mathcal{S}^\perp = \{\nu \in \mathcal{M} \mid \nu \perp \mu \text{ for all } \mu \in \mathcal{S}\}.$$

If $\mathcal{S}$ is given by (4.1), then the subspaces $\mathcal{S}$ and $\mathcal{S}^\perp$ give a decomposition of $\mathcal{M}$.

Lemma 4.1. *Let $\mathcal{N}$ be a candidate collection of null sets, and let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ be given by* (4.1)*. Then $\mathcal{M} = \mathcal{S} \oplus \mathcal{S}^\perp$.*

Proof. Fix $\nu \in \mathcal{M}$; we need to show that there is a unique decomposition $\nu = \nu_1 + \nu_2$, where $\nu_1 \in \mathcal{S}$ and $\nu_2 \in \mathcal{S}^\perp$. To this end, consider the following collection of subsets:

$$\mathcal{N}' = \{E \in \mathcal{N} \mid \nu(E) > 0\}.$$

Let $\theta = \sup\{\nu(E) \mid E \in \mathcal{N}'\}$, and let $E_n \in \mathcal{N}'$ be a countable collection such that $\nu(E_n) \to \theta$. Consider the union $A = \bigcup_n E_n$, and observe that $\nu(A) = \theta$ and $A \in \mathcal{N}'$.

We claim that $\nu_1 = \nu|_{X \setminus A}$ and $\nu_2 = \nu|_A$ gives the desired decomposition. Indeed, $\nu_2 \perp \mu$ for all $\mu \in \mathcal{S}$ since $\mu(A) = 0$ and $\nu_2(X \setminus A) = 0$, so $\nu_2 \in \mathcal{S}^\perp$. Furthermore, given $E \in \mathcal{N}$, we may write $E' = E \setminus A$ and $F = E \cap A$; then $\nu_1(F) = 0$ by definition, and if $\nu(E') = \nu_1(E') > 0$, we would have $\nu(E' \cup A) = \nu(E') + \nu(A) > \theta$, contradicting the definition of θ. It follows that $\nu_1(E) = 0$, and since this holds for all $E \in \mathcal{N}$, we have $\nu_1 \in \mathcal{S}$.

Finally, uniqueness of the decomposition follows from the fact that $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$. □

Now we move to the setting of Theorem D, so M is a compact Riemannian manifold, $U \subset M$ an open set, and $f \colon U \to M$ a $C^{1+\alpha}$ local diffeomorphism such that $\overline{f(U)} \subset U$. Let $\Lambda = \bigcap_{n \geq 1} f^n(U)$ be the attractor for f. Let $\mathcal{M}^{\mathrm{ac}}$ and $\mathcal{M}^{\mathrm{h}}$ be as defined in Section 2.2. Note that the set $\mathcal{N} = \mathcal{N}(\mathcal{R})$ defined in (2.24) is a candidate collection of null sets, and (2.25) gives $\mathcal{M}^{\mathrm{ac}} = \mathcal{S}(\mathcal{N})$, so Lemma 4.1 shows that $\mathcal{M}(M) = \mathcal{M}^{\mathrm{ac}} \oplus (\mathcal{M}^{\mathrm{ac}})^\perp$.

We prove Proposition 2.1, which claims that $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$ is precisely the set of SRB measures for f.

Proof of Proposition 2.1. First we show that every SRB measure μ is in $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$. Every SRB measure is hyperbolic, so $\mu \in \mathcal{M}^{\mathrm{h}}$. To show that $\mu \in \mathcal{M}^{\mathrm{ac}}$, first observe that μ is invariant and supported on Λ. As discussed before Definition 2.1, μ can be expressed in terms of conditional measures on local unstable manifolds. More precisely, if we write $\mathcal{R}'$ for the set of all local unstable manifolds (so that in particular $\mathcal{R}' \subset \mathcal{R}$), then for each ℓ one can take a measurable partition of the regular set Y_ℓ into sets of the form $W \cap Y_\ell$, where $W \in \mathcal{R}'$, and let $\{\mu_W \mid W \in \mathcal{R}'\}$ be the conditional measures of μ relative to each element of this partition. This means that there is a measure η on $\mathcal{R}'$ such that

$$\mu(E) = \int_{\mathcal{R}'} \mu_W(E)\, d\eta(W) \tag{4.2}$$

for every measurable set E. By Definition 2.1 we have $\mu^W \ll m_W$ for η-a.e. W, and since every local unstable manifold is contained in $\mathcal{R}$, this shows that $\mu(E) = 0$ for all $E \in \mathcal{N}(\mathcal{R})$, so $\mu \in \mathcal{M}^{\mathrm{ac}}$.

Conversely, if $\mu \in \mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$, we show that μ is an SRB measure. Since μ is invariant, it is supported on Λ. Because hyperbolicity is given, this amounts to showing that the conditional measures generated by μ on local unstable manifolds are absolutely continuous with respect to the leaf volume. Once again using the decomposition of μ in (4.2), we show that $\mu_W \ll m_W$ for η-a.e. W. Indeed, if there is a positive η-measure set of W such that $\mu_W \not\ll m_W$, then we may write $\mathcal{W}$ for this set and take for each $W \in \mathcal{W}$ a set $E_W \subset W$ with $m_W(E_W) = 0$ and $\mu_W(E_W) > 0$. Taking E to be the union of these E_W yields a set with $E \subset \bigcup_{W\in\mathcal{W}} W$ and $m_W(E) = 0$ for every $W \in \mathcal{W}$, so $E \in \mathcal{N}(\mathcal{R})$, and moreover $\mu(E) > 0$ since $\mu_W(E_W) > 0$, which contradicts the assumption that $\mu \in \mathcal{M}^{\mathrm{ac}}$. □

By Proposition 2.1, the proof of Theorem D will be complete if we show that the limiting measure μ has an ergodic component in $\mathcal{M}^{\mathrm{ac}} \cap \mathcal{M}^{\mathrm{h}}$. The following result is key.

Proposition 4.2. *For every choice of* $\mathbf{K}$ *and* N*,* $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},N}$ *is weak* compact.*

Proof. Recall that $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},N} = \Phi^*(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N}))$, so it suffices to exhibit a topology on $\mathcal{R}'_{\mathbf{K},N}$ in which $\mathcal{R}'_{\mathbf{K},N}$ is compact and the map Φ^* is continuous (relative to the weak* topologies on $\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N})$ and $\mathcal{M}(M)$).

Elements of $\mathcal{R}'_{\mathbf{K},N}$ are pairs (W,ρ), where W is an admissible manifold with geometry and N-fold backwards dynamics controlled by $\mathbf{K} = (\theta,\gamma,\kappa,r,C,\overline{\lambda},\beta,L)$, and $\rho\colon W \to [1/L, L]$ is a C^α density function with $|\rho|_\alpha \leq L$ by (2.23). As specified in (2.17), the manifold W is required to be the image under the exponential map of the graph of a function ψ – that is, $W = \exp_x \operatorname{graph}\psi$, where $x \in U$, $T_xM = G \oplus F$, $\measuredangle(G,F) \geq \theta$, and $\psi\colon B_G(r) \to F$ is $C^{1+\alpha}$ with $\|D\psi\| \leq \gamma$ and $|D\psi|_\alpha \leq \kappa$.

The elements of $\mathcal{R}'_{\mathbf{K},N}$ are in one-to-one correspondence with quintuples (x,G,F,ψ,ρ) satisfying the conditions above. To define a neighborhood of such a quintuple, we first take a small neighborhood of x in U, which can be identified with $\mathbb{R}^n$. Then the second coordinate can be identified with the collection of all k-dimensional subspaces of $\mathbb{R}^n$, the third with all $(n-k)$-dimensional subspaces, the fourth with C^1 functions $B_{\mathbb{R}^k}(r) \to \mathbb{R}^{n-k}$, and the fifth with C^0 functions $B_{\mathbb{R}^k}(r) \to [1/L, L]$. This specifies a natural topology on each coordinate: the Grassmanian topology on subspaces, and the C^1 and C^0 topologies on functions. Thus we may define a topology on $\mathcal{R}'_{\mathbf{K},N}$ as the product topology over each such Euclidean neighbourhood in U.

Note that the map $\Phi\colon \mathcal{R}'_{\mathbf{K},N} \to \mathcal{M}(M)$ is continuous with respect to the topology just described, and so $\Phi^*\colon \mathcal{M}(\mathcal{R}'_{\mathbf{K},N}) \to \mathcal{M}(M)$ is continuous as well. Thus it remains only to show compactness.

First we note that each of the geometric conditions for inclusion in $\mathcal{P}_{(\theta,\gamma,\kappa,r)}$ is compact – that is, given (x_k,G_k,F_k,ψ_k) such that the above conditions are satisfied, compactness of the Grassmanian for M guarantees existence of a subsequence for which (x_k,G_k,F_k) converges to (x,G,F), the uniform bound on the angle guarantees that we still have $T_xM = G \oplus F$ in the limit, and the Arzelà–Ascoli theorem guarantees that we can get C^1 convergence of ψ_k by passing to a further subsequence; moreover, the limiting function ψ is $C^{1+\alpha}$ and satisfies the same bounds in terms of γ and κ.

A similar argument with Arzelà–Ascoli gets a subsequence for which the densities ρ_k converge (in C^0) to $\rho \in C^\alpha(W, [1/L, L])$ satisfying $|\rho|_\alpha \leq L$. Thus to verify compactness of $\mathcal{R}'_{\mathbf{K},N}$, it remains only to check the dynamical conditions controlled by $C, \overline{\lambda}, N, \beta$, which are specified in (2.18), (2.19), and (2.20).

It is straightforward that if $W_k \to W$ and $W_k \in \mathcal{Q}_{(C,\overline{\lambda},N)}$, then $W \in \mathcal{Q}_{(C,\overline{\lambda},N)}$ as well. For inclusion in $\mathcal{R}_{\mathbf{K},N}$, we observe that if $W_k \to W$ and $x_k \in H_{(C,\overline{\lambda},N)}(W_k)$ are such that $x_{k_i} \to x$, then just as before, we have $x \in H_{(C,\overline{\lambda},N)}(W)$. In other words,

$$H_{(C,\overline{\lambda},N)}(W) \supset \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{k\geq m} H_{(C,\overline{\lambda},N)}(W_k)}.$$

Writing $Y_m = \overline{\bigcup_{k\geq m} H_{(C,\overline{\lambda},N)}(W_k)}$, we observe that Y_m is a nested sequence of compact sets.

Let $\nu_W = \mu_W / \|\mu_W\|$, and similarly for ν_{W_k}. Compactness of Y_m and the fact that $\nu_{W_k} \to \nu_W$ (in the weak* topology) guarantees that $\nu_W(Y_m) \geq \overline{\lim}_{k\to\infty} \nu_{W_k}(Y_m) \geq \beta$, where we have used the fact that $Y_m \supset H_{(C,\overline{\lambda},N)}(W_k)$ for all $k \geq m$ and the condition in (2.20). Because the Y_m are nested and we have $\nu_W(Y_m) \geq \beta$ for every m, we conclude that $\nu_W(H_{(C,\overline{\lambda},N)}(W)) \geq \beta$, which completes the proof that $(W, \rho) \in \mathcal{R}'_{\mathbf{K},N}$.

As discussed above, this shows that $\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},N})$ is weak* compact, and continuity of Φ^* completes the proof that $\mathcal{M}^{\text{ac,h}}_{\mathbf{K},N}$ is weak* compact. □

Returning to the proof of Theorem D, we suppose that we are given measures $\mu_n \to \mu \in \mathcal{M}(\Lambda, f)$ and $\nu_n = \Phi^*(\eta_n) \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},q_n}$ such that $\nu_n \leq \mu_n$.

By Lemma 4.1, there is a unique decomposition $\mu = \mu_1 + \mu_2$ such that $\mu_1 \in \mathcal{M}^{\text{ac}}$ and $\mu_2 \in (\mathcal{M}^{\text{ac}})^\perp$. Note that $\mathcal{M}^{\text{ac}}$ and $(\mathcal{M}^{\text{ac}})^\perp$ are both f-invariant: thus f-invariance of μ gives $\mu = f_*\mu = f_*\mu_1 + f_*\mu_2$, and uniqueness of the decomposition gives $f_*\mu_1 = \mu_1$ and $f_*\mu_2 = \mu_2$.

Now we construct a convergent subsequence. First note that

$$\nu_n \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},N} \text{ whenever } q_n \geq N, \tag{4.3}$$

and also that $q_n \to \infty$. Because each ν_n is contained in $\mathcal{M}^{\text{ac,h}}_{\mathbf{K},1}$, which is weak* compact by Proposition 4.2, there is a convergent subsequence $\nu_{n_k} \to \nu$. Now for every N, we see from (5.24) that $\nu_{n_k} \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},N}$ for sufficiently large k, and thus $\nu \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},N}$. In particular, this gives $\nu \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K}}$.

The relation $\nu_n \leq \mu_n$ passes to the limit, and we get $\nu \leq \mu$. Because $\mu_2 \in (\mathcal{M}^{\text{ac}})^\perp$ and $\nu \in \mathcal{M}^{\text{ac,h}} \subset \mathcal{M}^{\text{ac}}$, we get $\nu \leq \mu_1$. Note that every ergodic component of μ_1 is in $\mathcal{M}^{\text{ac}}$.

Now let E be the set of Lyapunov regular points for which there is a zero Lyapunov exponent, and let E^c be its complement. Note that for every

$(W,\rho) \in \mathcal{R}'_{\mathbf{K}}$, we have $m_W(E^c) \geq \beta \|m_W\|$. Using the bounds on ρ we have

$$\int_{E^c \cap W} \rho(x)\, dm_W(x) \geq \frac{\beta}{L} \|m_W\| \geq \frac{\beta}{L^2} \int_W \rho(x)\, dm_W(x).$$

Since ν is a convex combination of the measures $\rho\, dm_W$, we conclude that $\nu(E^c) \geq (\beta L^{-2})\|\nu\|$. In particular, since $\mu_1 \geq \nu$, some ergodic component m of μ_1 gives positive weight to E^c, and hence is in $\mathcal{M}^{\mathrm{h}}$. As remarked above, $m \in \mathcal{M}^{\mathrm{ac}}$, and hence m is an SRB measure by Proposition 2.1.

5. Proof of Theorems A–C

Broadly speaking, the idea of the proof of Theorems A–C is as follows.

(1) Under the assumptions of Theorem A, we foliate a small neighbourhood with manifolds that are tangent to the unstable cone family for a positive Lebesgue measure set of points in S. By selecting an appropriate leaf of the foliation, we reduce to the problem of proving Theorem B.
(2) Given the leaf W we just found and any $x \in S \cap W$, we apply Theorem E at each of the "good" times for x and conclude that a uniformly large piece of $f^n(\xi(x))$ is admissible.
(3) Using the fact that "good" times have positive frequency for points in S, the measures $\mu_n \in \mathcal{M}(M)$ project to measures $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},q_n}$ by considering only those parts of μ_n that sit on admissible manifolds of large size in a certain foliation; moreover, these measures have uniformly positive weight. This allows us to apply Theorem D.

We observe that throughout the proof, the condition $x \in S$ is only used to guarantee that (2.8) and (2.14) hold, and so S can be replaced with $\hat{S}$ (throughout the proof we simply write S), which gives Theorem C.

Before delving into the details of the proof, we need to establish certain basic consequences of the assumption that $x \in S$. We first observe that the definition of S immediately implies that all Lyapunov exponents in $K^u(x)$ are positive, in fact, we have a stronger statement:

$$\varliminf_{n\to\infty} \frac{1}{n} \log \|Df^n v\| > 0 \text{ for all } v \in K^u(x). \tag{5.1}$$

This remains true under the assumption of Theorem C if we restrict to $n \in \Gamma^e_{\bar{\lambda}}(x) \cap \Gamma^s_{C,\bar{\lambda},q}(x)$. We will shortly see a similar result in the stable direction.

Because the dimension $d_u(x)$ of the unstable cone takes only finitely many values (and similarly for $d_s(x)$), we may without loss of generality assume that it is constant on S, since $\operatorname{Leb} S > 0$ implies that one of the level sets intersects S in a set of positive Lebesgue measure.

5.1. An invariant transverse distribution.

Our first step is to use the invariant cone family $K^s(x)$ to obtain for each $x \in S$ a subspace $E^s(x)$ that is transverse to $K^u(x)$ in the sense that for every d_u-dimensional subspace $E \subset K^u(x)$ we have $T_xM = E \oplus E^s(x)$.

Remark. The notation suggests that $E^s(x)$ is the stable subspace. Indeed, in the setting of Theorems A and B it is not hard to use (2.4) and (2.5) to show that Lyapunov exponents in $E^s(x)$ are negative for every $x \in S$. However, in the setting of Theorem C, the definition of $\hat{S}$ does not guarantee this for all $x \in \hat{S}$. Nevertheless, the definition of the collections of measures $\mathcal{M}^{\mathrm{ac,h}}_{\mathbf{K},n}$ will guarantee that the limiting measures we construct have all Lyapunov exponents non-zero.

We construct $E^s(x)$ via a standard argument: for every $x \in S$, consider the set

$$\hat{E}^s(x) = \bigcap_{n\geq 0} Df^{-n}K^s(f^n(x)) = \bigcap_{n\geq 0} \overline{Df^{-n}K^s(f^n(x))}. \tag{5.2}$$

(Equality of the two intersections is a consequence of the invariance property of $K^s(x)$.) We claim that $\hat{E}^s(x)$ contains a d_s-dimensional subspace of T_xM.

To prove this claim, let $S_xM = \{v \in T_xM \mid \|v\| = 1\}$ be the set of unit tangent vectors at x. Then $\hat{E}^s(x) \cap S_xM$ is an intersection of nested non-empty compact sets, and is hence non-empty. Indeed, for every n we can choose d_s mutually orthogonal unit vectors $v_i^n \in Df^{-n}K^s(f^n(x)) \cap S_xM$, and then choose a subsequence n_k such that $v_i^{n_k} \to v_i \in \hat{E}^s(x) \cap S_xM$ for every $1 \leq i \leq d_s$. It follows that $\hat{E}^s(x)$ contains a d_s-dimensional subspace.

Now let $S' = \{x \in S \mid x \neq f^n(y) \text{ for any } y \in S, n > 0\}$. Given $x \in S'$, let $E^s(x)$ be any d_s-dimensional subspace of $\hat{E}^s(x)$. Define $E^s(x)$ on $\bigcup_{n>0} f^nS'$ by $E^s(f^n(x)) = Df^n(x)E^s(x)$ for $x \in S'$. Decompose $S \setminus \bigcup_{n\geq 0} f^nS'$ as a disjoint union of bi-infinite orbits $\{f^nx \mid n \in \mathbb{Z}\}$, and let $S'' \subset (S \setminus \bigcup_{n\geq 0} f^nS')$ contain exactly one point from each orbit (this requires the axiom of choice). As with S', let $E^s(x)$ be any d_s-dimensional subspace of $\hat{E}^s(x)$ for each $x \in S''$, and then define $E^s(x)$ on the remainder of S by $E^s(f^nx) = Df^n(x)E^s(x)$ for all $n \in \mathbb{Z}$ and $x \in S''$. This is well-defined by our choice of S''.

The previous paragraph defines an invariant distribution $E^s(x) \subset K^s(x)$ on S. The transversality property claimed above follows from the transversality of $K^s(x)$ and $K^u(x)$.

5.2. **Admissible foliations and uniformity assumptions.** The angle between the cones $K^s(x), K^u(x)$ is given by the measurable function $\theta(x)$. Let $X_n = \{x \in A \mid \theta(x) > 1/n\}$, and observe that $S \subset \bigcup_{n\geq 1} X_n$, so there exists n such that $\mathrm{Leb}(S \cap X_n) > 0$. By restricting S to include only points in X_n, we may assume without loss of generality that $\theta(x) \geq \bar{\theta} = \frac{1}{n}$ for all $x \in S$, while still guaranteeing that $\mathrm{Leb}\, S > 0$.

By again decreasing S to a smaller set that still has positive Lebesgue measure, we will obtain a foliation of a neighbourhood of S such that every leaf $V(x)$ of the foliation is tangent to $K^u(x)$ – that is, $T_xV(x) \subset K^u(x)$ for every $x \in S$.

To this end, for every $z \in M$ we fix a neighbourhood $0 \in B_z \subset T_zM$ such that $\exp_x^{-1}$ is well-defined on $\exp_z(B_z)$ for every $x \in \exp_z(B_z)$. Let $\pi_{z,x} := \exp_x^{-1} \circ \exp_z : B_z \to T_xM$, and observe that $\pi_{z,x} = I_{z,x} + g_{z,x}$, where $I_{z,x} = D\pi_{z,x}$ is an isometry and where $g_{z,x}$ is a smooth map (as smooth as the manifold) with $Dg_{z,x}(0) = 0$.

As $d(z,x)$ goes to zero, the quantity $\|g_{z,x}\|_{C^2}$ goes to zero as well. It follows that we can choose for every $z \in U$ a neighbourhood Ω_z such that

(1) $\exp_x^{-1}$ is well-defined on Ω_z for all $x \in \Omega_z$;
(2) $\|g_{x,z}\|_{C^2} \le \bar\theta/2$ for all $x \in \Omega_z$.

Now we choose a finite set $E \subset U$ such that the neighbourhoods $\{\Omega_z \mid z \in E\}$ cover the entire trapping region U. Thus we can fix $z \in E$ such that $\mathrm{Leb}(S \cap \Omega_z) > 0$.

Let $\mathcal{G}_z^{d_u}$ denote the collection of d_u-dimensional subspaces of T_zM, endowed with the metric given by angle (or equivalently the Hausdorff metric on the intersections of subspaces with the unit sphere). Given $P \in \mathcal{G}_z^{d_u}$, denote by W_x^P the affine subspace of T_zM passing through $\exp_z^{-1}(x)$ parallel to P. Given $x \in \Omega_z \cap S$, let

$$\Delta_x = \{P \in \mathcal{G}_z^{d_u} \mid T_x \exp_z(W_x^P \cap B_z) \subset K^u(x)\};$$

that is, Δ_x comprises those d_u-dimensional subspaces $P \subset T_zM$ for which the manifold $\pi_{z,x}(W_x^P \cap B) \subset T_xM$ is tangent to the unstable cone $K^u(x)$ at $0 \in T_xM$.

Each such Δ_x contains a ball of radius $\bar\theta/2$ in $\mathcal{G}_z^{d_u}$. To see this, consider the cone $I_{z,x}^{-1}K^u(x) \subset T_zM$, which contains a ball of radius $\bar\theta$ in $\mathcal{G}_z^{d_u}$, and observe that if $P \in \mathcal{G}_z^{d_u}$ lies in this cone and is at least a distance of $\bar\theta/2$ from its boundary, then $P \in \Delta_x$ since $\|g_{z,x}\|_{C^2} \le \bar\theta/2$.

Now let $F \subset \mathcal{G}_z^{d_u}$ be a finite $\bar\theta/2$-dense set, and given $P \in F$, let $X_P = \{x \in \Omega_z \cap S \mid P \in \Delta_x\}$. Then $\bigcup_{P\in F} X_P = S \cap \Omega_z$, and by finiteness, there exists $P \in F$ such that $\mathrm{Leb}\, X_P > 0$.

It follows that by restricting S to include only points in X_P, we may assume that $T_x \exp_z(W_x^P \cap B_z) \subset K^u(x)$ for every $x \in S$. Denote by $\xi = \xi(P)$ the foliation of Ω_z whose leaves are $V(x) := \exp_z(W_x^P \cap B_z)$.

To summarise, we assume without loss of generality that

(1) $\mathrm{Leb}\, S > 0$;
(2) $\bar\theta, \delta > 0$ are such that every $x \in S$ satisfies $\theta(x) \ge \bar\theta$ and

$$\underline{\delta}\left(\Gamma^e_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)\right) \ge 3\delta \text{ for all } q \in \mathbb{N}. \tag{5.3}$$

(3) $S \subset \Omega_z \ni z$, where $\exp_z^{-1}$ is well-defined on the open set Ω_z;
(4) there is a d_u-dimensional subspace $P \subset T_zM$ so that $\exp_z(W_x^P \cap B_z)$ is $(\gamma, 1)$-admissible at x for all $x \in S$. In particular, the foliation of T_zM by d_u-dimensional affine subspaces parallel to P projects under

$\exp_z$ to a foliation ξ of Ω_z such that for every $x \in S$, the leaf $V(x)$ of ξ passing through x is $(\gamma, 1)$-admissible and is tangent to $K^u(x)$.

Now we have a smooth foliation ξ such that every leaf V of ξ is in $\mathcal{P}_{(\bar\theta,\bar\gamma,1,\bar r)}$. There are density functions $\rho_V \in L^1(V, m_V)$ for every $V \in \xi$ and a measure $\eta \in \mathcal{M}(\mathcal{P}_{(\bar\theta,\bar\gamma,1,\bar r)})$ such that

$$\mathrm{Leb}(S) = \int_{\mathcal{P}} \int_{V\cap S} \rho_V(x)\, dm_V(x)\, d\eta(V).$$

Since $\mathrm{Leb}\, S > 0$, there exists $W \in \xi \subset \mathcal{P}_{(\bar\theta,\bar\gamma,1,\bar r)}$ such that $m_W(S) > 0$, where we restrict S to include only those points x for which $T_x W \in K^u(x)$.

This also follows directly from the hypotheses of Theorem B, and so from this point forward the proofs of Theorems A and B are the same. Moreover, the only property we require of points in S is that (5.3) holds, so what follows will also prove Theorem C.

5.3. **Effective hyperbolic times.**

5.3.1. *Conditions of Theorem E.* Given $x \in S$, we now wish to apply Theorem E to the sequence of maps $T_{f^n(x)}M \to T_{f^{n+1}(x)}M$ given by writing f in local coordinates and considering the subspaces $E^u_n = T_{f^n(x)}V^n(x)$, $E^s_n = E^s(f^n(x))$. Write

$$\Gamma^{e'}_{\overline{\lambda}}(x) = \left\{ n \mid \sum_{j=k}^{n-1} \lambda^e_j \geq \overline{\lambda}(n-k) \text{ for all } 0 \leq k < n \right\} \tag{5.4}$$

for the set of times $n \in \mathbb{N}$ at which (2.31) holds and Theorem E can be applied. Note that $\lambda^u - \Delta \leq \lambda$ and so $\Gamma^{e'}_{\overline{\lambda}}(x) \subset \Gamma^{e}_{\overline{\lambda}}(x)$, but the containment may be proper since whenever $\theta(f^j x) < \bar\theta$ we have $\lambda^e_j = -L''$, which may be less than $\lambda^u_j - \Delta_j$.

Nevertheless, by choosing $\bar\theta > 0$ sufficiently small, we can guarantee that positive density is preserved upon passing to $\Gamma^{e'}_{\overline{\lambda}}(x)$. More precisely, by (5.3) and the application of [CP13, Proposition 9.3] with (2.8), there is $\bar\theta > 0$ such that

$$\underline{\delta}\left(\Gamma^{e'}_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)\right) > 2\delta \text{ for every } q. \tag{5.5}$$

In particular, Theorem E can be applied for every n in the given set of times, and every such n also satisfies a condition on transverse contraction leading up to n, which will help us obtain admissible manifolds in $\mathcal{R}_{\mathbf{K},q_n}$, as required by Theorem D.

The key consequence of (5.5) is the following lemma.

Lemma 5.1. *For every $q \in \mathbb{N}$ there is $N(q) \in \mathbb{N}$ such that for every $n \geq N(q)$, we have*

$$\int_W \frac{1}{n} \#([1,n] \cap \Gamma^{e'}_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x))\, dm_W(x) \geq \delta m_W(S). \tag{5.6}$$

Proof. Given $q, m \in \mathbb{N}$, let

$$X_m^q = \{x \mid \#([1,n] \cap \Gamma_{\overline{\lambda}}^{e'}(x) \cap \Gamma_{C,\overline{\lambda},q}^s(x)) \geq 2\delta n \text{ for every } n \geq m\}.$$

By (5.5) we have $\bigcup_m X_m^q = S$ for every q, and in particular there is $N(q)$ such that $m_W(X_{N(q)}^q) \geq \frac{1}{2} m_W(S)$. The result follows. □

5.3.2. *Consequences of Theorem E.* We write

$$S_n = \{x \in S \mid n \in \Gamma_{\overline{\lambda}}^{e'}(x) \cap \Gamma_{C,\overline{\lambda},q}^s(x)\}. \tag{5.7}$$

Applying Theorem E, we obtain constants $\bar{\gamma}, \bar{\kappa}, \bar{r}, \bar{\chi}, \bar{\theta} > 0$ such that for every $x \in S$ and $n \in \Gamma_{\overline{\lambda}}^{e'}(x)$ (in particular, for any $n \in \mathbb{N}$ and $x \in S_n$), there is an admissible manifold

$$W_n^x \subset W \cap B(x, \bar{r} e^{-\bar{\chi} n})$$

with the property that $f^n W_n^x$ is a $(\bar{\gamma}, \bar{\kappa})$-admissible manifold of size $\bar{r}$ transversal to $E^s(f^n(x))$, and

$$\measuredangle(T_{f^n(x)} V^n(x), E^s(f^n(x))) \geq \bar{\theta}.$$

Moreover, the backwards contraction condition in (2.18) holds, and we conclude that

$$f^n W_n^x \in \mathcal{P}_{(\bar{\theta},\bar{\gamma},\bar{\kappa},\bar{r})} \cap \mathcal{Q}_{(1,\bar{\lambda},n)}. \tag{5.8}$$

To obtain $f^n W_n^x \in \mathcal{R}_{\mathbf{K},q}$, we need to find "enough" $y \in W_n^x$ with $f^n(y) \in H_{C,\overline{\lambda},q}(f^n W_n^x)$. Observe that this holds for every $y \in W_n^x$ with $n \in \Gamma_{C,\overline{\lambda},q}^s(y)$ and $n \geq q$. In particular, it holds for every $y \in S_n$. In the next section we give a relationship between the measures of the sets $H_{C,\overline{\lambda},q}(f^n W_n^x)$ and $W_n^x \cap S_n$.

5.4. **Projecting to $\mathcal{M}_{\mathbf{K},q}^{\mathrm{ac,h}}$.** We need a bounded distortion result. Let $g_n \colon f^n W_n^x \to W_n^x$ be a local inverse to f^n. Given $y \in W_n^x$, let

$$\phi_n(y) = \det(Df^n)(y)|_{T_y W} \tag{5.9}$$

and define $\rho_n^x \in C(f^n W_n^x, \mathbb{R}^+)$ by

$$\rho_n^x(z) = \frac{\phi_n(x)}{\phi_n(g_n(z))}. \tag{5.10}$$

It follows immediately that

$$\frac{d(f_*^n m_{W_n^x})}{d m_{f^n W_n^x}} = \frac{\rho_n^x}{\phi_n(x)}. \tag{5.11}$$

We will show that ρ_n^x is a well-behaved function.

Lemma 5.2. *There exists $L > 0$ such that if $x \in S_n$ and W_n^x is as above, then $\rho_n^x(z) \in [1/L, L]$ and $|\rho_n^x|_\alpha \leq L$ for every $z \in W_n^x$.*

Proof. By [CP13, Theorem D], there is $\gamma > 0$ such that the manifolds $f^k W_n^x$ for $0 \le k < n$ have the property that the Grassmanian distance between $T_y(f^k W_n^x)$ and $T_z(f^k W_n^x)$ is smaller than some fixed constant γ (independent of k, n) for every $y, z \in f^k W_n^x$. In particular, because f is $C^{1+\alpha}$, there exists $K > 0$ such that

$$|\det Df(y)|_{T_y(f^k W_n^x)} - \det Df(z)|_{T_z(f^k W_n^x)}| \le K d(y,z)^\alpha$$

for every $0 \le k < n$ and $y, z \in f^k W_n^x$. Using the backwards contraction property of $f^n W_n^x$, we see that for every $z_1, z_2 \in f^n W_n^x$, we have

$$\begin{aligned}|\phi_n(g_n(z_1)) - \phi_n(g_n(z_2))| &\le \sum_{k=1}^n K d(f^{n-k}(g_n(z_1)), f^{n-k}(g_n(z_2)))^\alpha \\ &\le K \sum_{k=1}^n (e^{-\bar\chi k} d(z_1, z_2))^\alpha \\ &= K e^{-\bar\chi\alpha}(1 - e^{-\bar\chi\alpha})^{-1} d(z_1, z_2)^\alpha.\end{aligned}$$

Write $K' = K e^{-\bar\chi\alpha}(1 - e^{-\bar\chi\alpha})^{-1}$, so that

$$|\phi_n(g_n(z_1)) - \phi_n(g_n(z_2))| \le K' d(z_1, z_2)^\alpha. \tag{5.12}$$

Applying this with $z_1 = z$ and $z_2 = f^n(x)$ yields

$$\left| \frac{\phi_n(g_n(z))}{\phi_n(x)} - 1 \right| \le K' \bar r^\alpha e^{-\bar\chi d_u n}$$

for every $z \in f^n(W)$, where we use the fact that $\phi_n(x) \ge e^{\bar\chi d_u n}$. Writing $K'' = K' \bar r^\alpha + 1$, we see that

$$\frac{1}{K''} \le \frac{\phi_n(g_n(z))}{\phi_n(x)} \le K'',$$

which proves the first inequality for ρ_n^x. (Note that K'' depends only on K, $\bar r$, $\bar\chi$, and α.)

To show that the functions ρ_n^x are uniformly Hölder, we fix $z_1, z_2 \in f^n W_n^x$, write $y_i = g_n(z_i)$, and observe that

$$\begin{aligned}|\rho_n^x(z_1) - \rho_n^x(z_2)| &= \left| \frac{\phi_n(x)}{\phi_n(y_1)} - \frac{\phi_n(x)}{\phi_n(y_2)} \right| \\ &\le \frac{\phi_n(x)|\phi_n(y_1) - \phi_n(y_2)|}{\phi_n(y_1)\phi_n(y_2)} \\ &\le K' K'' d(z_1, z_2)^\alpha e^{-\bar\chi d_u n}.\end{aligned}$$

This completes the proof of the lemma. □

Now we are in a position to give conditions under which $f^n W_n^x \in \mathcal{R}_{\mathbf{K},q}$ and $f_*^n m_{W_n^x} \in \mathcal{M}_{\mathbf{K},q}^{\text{ac,h}}$. Fix $\beta' > 0$ and suppose $x \in S_n$ is such that

$$m_W(W_n^x \cap S_n) \ge \beta' m_W(W_n^x). \tag{5.13}$$

Note that $f^n(W_n^x \cap S_n) \subset H_{C,\bar\lambda,q}(f^n W_n^x)$ as long as $n \geq q$. By Lemma 5.2, we have

$$\begin{aligned} m_{f^n(W)}(f^n(W_n^x \cap S_n)) &= \int_{W_n^x \cap S_n} \frac{\phi_n(x)}{\rho_n^x(z)}\, dm_W(z) \\ &\geq \frac{\phi_n(x) m_W(W_n^x \cap S_n))}{L} \geq \phi_n(x) \frac{\beta'}{L} m_W(W_n^x), \end{aligned}$$

and similarly,

$$m_{f^n(W)}(f^n W_n^x) \leq L\phi_n(x) m_W(W_n^x),$$

so we conclude that

$$m_{f^n(W)}(H_{C,\bar\lambda,q}(f^n W_n^x)) \geq \frac{\beta'}{L^2} m_{f^n(W)}(f^n(W_x^n)).$$

In particular, taking $\beta = \beta'/L^2$ gives $f^n W_n^x \in \mathcal{R}_{\mathbf{K},q}$. Moreover, Lemma 5.2 shows that $(f^n W_n^x, \rho_n^x) \in \mathcal{R}'_{\mathbf{K},q}$, and since

$$f_*^n m_{W_n^x}(E) = \frac{1}{\phi_n(x)} \int_{f^n(E)} \rho_n^x(z)\, dm_{f^n W_n^x}(z),$$

this shows that $f_*^n m_{W_n^x} \in \mathcal{M}_{\mathbf{K},q}^{\mathrm{ac,h}}$, as desired.

Remark. The constant β' should be taken to be smaller than $1/p$, where $p \in \mathbb{N}$ is as in Lemma 5.3 below and depends only on d_u.

5.5. **A Besicovitch covering lemma.** In order to complete the proof, we need the following version of the Besicovitch covering lemma for the 'balls' $W_n^x \subset W$ with centres $x \in S_n$.

Lemma 5.3. *There is $p \in \mathbb{N}$, depending only on $d_u, \theta, \gamma, \kappa$, and r, such that for every n there are subsets $A_1, \dots, A_p \subset S_n$ such that*

- $S_n \subset \bigcup_{i=1}^p \bigcup_{x\in A_i} W_n^x$;
- *for each $1 \leq i \leq p$ and $x, y \in A_i$, we have $x = y$ or $W_n^x \cap W_n^y = \emptyset$.*

Remark. The sets W_n^x are not necessarily close to being balls in the metric d_W, but their images $f^n W_n^x$ are almost balls in the metric $d_{f^n W}$. Thus we could obtain Lemma 5.3 from the Besicovitch covering lemma in [Fe69, 2.8.14] if we could prove that $\bigcup_{x\in S_n} f^n W_n^x \subset f^n W$ is *directionally limited.* It is shown in [Fe69, 2.8.9] that C^2 Riemannian manifolds are directionally limited, but we only know that $f^n W$ is $C^{1+\alpha}$. Thus we give a direct proof of Lemma 5.3 taking advantage of the fact that the sets $f^n W_n^x$ have uniformly controlled radii. (In the general Besicovitch covering lemma the radii are allowed to vary.)

Proof of Lemma 5.3. Given $n \in \mathbb{N}$ and $x \in S_n$, we will write $\hat x = f^n x$, $\hat W_n^x = f^n W_n^x$, etc., in order to simplify notation. The uniform expansion of $f^{n-k}\colon f^k W_n^x \to \hat W_n^x$ guaranteed by Theorem E shows that $\hat W_n^x$ is the graph of a function $\psi\colon E^u(f^n x) \to E^s(f^n x)$; moreover, we have $\|D\psi\| \leq \bar\gamma$ and $|D\psi|_\alpha \leq \bar\kappa$, and the angle between $E^u(\hat x)$ and $E^s(\hat x)$ is at least $\bar\theta$. Thus the map $\Psi\colon E^u(\hat{x}) \to T_{\hat{x}}M$ given by $\Psi(v) = v + \psi(v)$ is Lipschitz with constant L that depends only on $\bar{\gamma}, \bar{\kappa}, \bar{\theta}, \bar{r}$.

We recall some terminology and notation from §5.2. Given $a \in M$, let $\mathcal{G}_a^{d_u}$ be the Grassmanian collection of d_u-dimensional subspaces of T_aM, with metric ρ given by angle. Given $z, x \in M$ nearby, we write $\pi_{z,x} = \exp_x^{-1} \circ \exp_z \colon B_z \to T_xM$, observing that $\pi_{z,x} = I_{z,x} + g_{z,x}$, where $I_{z,x}$ is an isometry and $g_{z,x}$ is smooth with $Dg_{z,x}(0) = 0$.

From the observations in the first paragraph above, there is $\varepsilon > 0$ such that for every $x \in S_n$ and $z, z' \in \exp_{\hat{x}}^{-1} \hat{W}_n^x \subset T_{\hat{x}}M$, we have

$$\rho(T_z \exp_{\hat{x}}^{-1} \hat{W}_n^x, T_{z'} \exp_{\hat{x}}^{-1} \hat{W}_n^x) < \varepsilon. \tag{5.14}$$

(We commit a slight abuse of notation by conflating $T_{\hat{x}}M$ with $T_zT_{\hat{x}}M$ for each $z \in T_{\hat{x}}M$ so that we can compare the angles.) Let $E_1(\hat{x}) = T_{\hat{x}}\hat{W}_n^x$ and $E_2(\hat{x}) = E_1(\hat{x})^\perp$, and let $P\colon T_{\hat{x}}M \to E_1(\hat{x})$ be orthogonal projection along $E_2(\hat{x})$. Then $P(\exp_{\hat{x}}^{-1} \hat{W}_n^x) \subset B_{E_1(\hat{x})}(0, r')$.

We conclude that if $d_{\hat{W}}$ denotes distance on $\hat{W}$ and $B_{\hat{W}}(y, r)$ denotes the $d_{\hat{W}}$-ball of radius r centred at y, then there are $0 < r < \bar{r} < r'$ such that

$$B_{\hat{W}}(\hat{x}, r) \subset \hat{W}_n^x \subset B_{\hat{W}}(\hat{x}, r') \text{ for all } x \in S_n. \tag{5.15}$$

Let $A \subset S_n$ be such that $\hat{A} = f^n A$ is a maximal r-separated subset of $\hat{S}_n = f^n S_n$. Then we have

$$\hat{S}_n \subset \bigcup_{x \in A} B_{\hat{W}}(\hat{x}, r) \subset \bigcup_{x \in A} \hat{W}_n^x, \tag{5.16}$$

and in particular, $S_n \subset \bigcup_{x \in A} W_n^x$. Let G be the graph whose vertex set is A, with an edge between $x, y \in A$ if and only if $W_n^x \cap W_n^y \neq \emptyset$. Write $x \leftrightarrow y$ when this occurs.

To complete the proof of Lemma 5.3 it suffices to show that there is $p \in \mathbb{N}$, depending only on $d_u, \theta, \gamma, \kappa, r$, such that the chromatic number of G is $\leq p$. It suffices to show that every vertex of G has degree $\leq p$.

Let $W_n = \bigcup_{x \in S_n} W_n^x$. Fix $x \in A$, and consider the set

$$V_n^x := \bigcup \{B_{W_n}(\hat{y}, r') \mid y \in S_n, y \leftrightarrow x\},$$

together with $\hat{V}_n^x = f^n V_n^x$. Note that $\hat{V}_n^x \subset B_{\hat{W}_n}(\hat{x}, 3r')$, and that for every $y \leftrightarrow x$ we have $B_{\hat{W}_n}(\hat{y}, \frac{r}{2}) \subset \hat{V}_n^x$. Because $\hat{A}$ is r-separated, the sets $\{B_{\hat{W}_n}(\hat{y}, \frac{r}{2}) \mid y \in A\}$ are pairwise disjoint. In particular, we have

$$\begin{aligned} m_{\hat{W}_n}(\hat{V}_n^x) &\geq \sum \{m_{\hat{W}_n}(B_{\hat{W}_n}(\hat{y}, \tfrac{r}{2})) \mid y \in A, y \leftrightarrow x\} \\ &\geq (\deg x) \inf_{y \in S_n} m_{\hat{W}_n}(B_{\hat{W}_n}(\hat{y}, \tfrac{r}{2})). \end{aligned} \tag{5.17}$$

For a lower bound on $m_{\hat{W}_n}(B_{\hat{W}_n}(\hat{y}, \frac{r}{2}))$, we write $V(r, d_u)$ for the volume of the Euclidean ball of radius r in dimension d_u, and let L be such that the exponential map $\exp_a\colon T_aM \to M$ is L-Lipschitz on the ball of radius

$3r'$ for every $a \in M$. (Here we use compactness of M.) Using (5.14), we see that for every $y \in S_n$ we have

$$m_{\hat{W}_n}(B_{\hat{W}_n}(\hat{y}, \tfrac{r}{2})) \geq L^{-d_u}(1+\varepsilon)^{-d_u} V(\tfrac{r}{2}, d_u). \tag{5.18}$$

For the upper bound on $m_{\hat{W}_n}(\hat{V}_n^x)$, we first observe that $\exp_{\hat{x}}^{-1} \hat{V}_n^x$ is not contained in $\exp_{\hat{x}}^{-1} \hat{W}_n^x$. However, for every $y \leftrightarrow x$ there is $u \in W_n^x \cap W_n^y$, and thus for any $z \in \exp_{\hat{x}}^{-1} \hat{W}_n^y$ we have

$$\begin{aligned}\rho(T_z \exp_{\hat{x}}^{-1} \hat{V}_n^x, E^1(\hat{x})) &\leq \rho(T_z \exp_{\hat{x}}^{-1} \hat{W}_n^y, T_{\exp_{\hat{x}}^{-1} \hat{u}} \exp_{\hat{x}}^{-1} \hat{W}_n^y) \\ &\qquad + \rho(T_{\exp_{\hat{x}}^{-1} \hat{u}} \exp_{\hat{x}}^{-1} \hat{W}_n^y, E^1(\hat{x})) \\ &\leq \|g_{\hat{y},\hat{x}}\|_{C^2} \rho(T_{\pi_{y,x}^{-1} z} \exp_{\hat{y}}^{-1} \hat{W}_n^y, T_{\exp_{\hat{y}}^{-1} \hat{u}} \exp_{\hat{y}}^{-1} \hat{W}_n^y) + \varepsilon.\end{aligned}$$

There is $C > 0$ such that for every $x, y \in S_n$ with $d_{\hat{W}}(\hat{x}, \hat{y}) < 2r'$ we have $\|g_{\hat{y},\hat{x}}\|_{C^2} \leq C$, and so for every $z, z' \in \exp_{\hat{x}}^{-1} \hat{V}_n^x \subset T_{\hat{x}}M$, we have

$$\rho(T_z \exp_{\hat{x}}^{-1} \hat{V}_n^x, T_{z'} \exp_{\hat{x}}^{-1} \hat{V}_n^x) < 2\varepsilon(1+C). \tag{5.19}$$

Then as in (5.18) we have

$$m_{\hat{W}_n}(\hat{V}_x^n) \leq L^{d_u}(1+2\varepsilon(1+C))^{d_u} V(d_u, 3r'). \tag{5.20}$$

Combining (5.17), (5.18), and (5.20), we see that there is $p \in \mathbb{N}$, depending only on $d_u, r, r', \varepsilon, L, C$, such that $\deg x \leq p$ for every $n \in \mathbb{N}$ and $x \in S_n$. This completes the proof of Lemma 5.3. $\square$

Let $S_n^g = \{x \in S_n \mid (5.13) \text{ holds}\}$ be the set of "good" points in S_n, and $S_n^b = S_n \setminus S_n^g$ the set of "bad" points. Using Lemma 5.3, we have

$$\begin{aligned} m_W(S_n) &\leq \sum_{i=1}^p \sum_{x \in A_i} m_W(S_n \cap W_n^x) \\ &= \left(\sum_{i=1}^p \sum_{x \in A_i \cap S_n^b} m_W(S_n \cap W_n^x) \right) + \left(\sum_{i=1}^p \sum_{x \in A_i \cap S_n^g} m_W(S_n \cap W_n^x) \right) \\ &\leq p\beta' m_W(S_n) + \sum_{i=1}^p \sum_{x \in A_i \cap S_n^g} m_W(W_n^x), \end{aligned}$$

and conclude that there exists i such that

$$\sum_{x \in A_i \cap S_n^g} m_W(W_n^x) \geq \left(\frac{1}{p} - \beta' \right) m_W(S_n). \tag{5.21}$$

Write $\beta'' = \frac{1}{p} - \beta' > 0$ and $S_n' = S_n^g \cap A_i$. Then $x \neq y \in S_n'$ implies $W_n^x \cap W_n^u = \emptyset$, and so writing $W_n = \bigcup_{x \in S_n'} W_n^x$ the arguments in Section 5.4 give

$$f_*^n m_{W_n} \in \mathcal{M}_{\mathbf{K},q}^{\mathrm{ac,h}} \tag{5.22}$$

whenever $n \geq q$. Moreover, (5.21) gives

$$m_W(W_n) \geq \beta'' m_W(S_n). \tag{5.23}$$

5.6. **Completion of the proof.** Consider the measures

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k(m_W). \tag{5.24}$$

This sequence has the property that every weak* limit is f-invariant. In order to apply Theorem D, we must find a sequence $q_n \to \infty$ such that for every n, there is $\nu_n \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},q_n} = \Phi^*(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{K},q_n}))$ for which $\mu_n \geq \nu_n$ and for which $\|\nu_n\|$ is uniformly bounded away from 0.

Let δ be as in Lemma 5.1 and fix $0 < \bar{\delta} < \delta$. Let $q_n \to \infty$ be such that $q_n < \bar{\delta} n$ and $n \geq N(q_n)$, where $N(q)$ is as in Lemma 5.1.

As shown in (5.22), for every $k \geq q_n$ we have $f_*^k m_{W_k} \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},q_n}$. Moreover, $m_W \geq m_{W_k}$, and pushing forward by f^k gives $f_*^k m_W \geq f_*^k m_{W_k}$. Averaging over k gives

$$\mu_n \geq \nu_n := \frac{1}{n}\sum_{k=q_n}^{n-1} f_*^k m_{W_k} \in \mathcal{M}^{\text{ac,h}}_{\mathbf{K},q_n}$$

and so it only remains to estimate $\|\nu_n\|$. Using (5.23), we see that

$$\begin{aligned}
\|\nu_n\| &= \frac{1}{n}\sum_{k=q_n}^{n-1} \|m_{W_k}\| = \frac{1}{n}\sum_{k=q_n}^{n-1} m_W(W_k)\\
&\geq \frac{\beta''}{n}\sum_{k=q_n}^{n-1} m_W(S_k)\\
&= \frac{\beta''}{n}\int_W \#\left([q_n,n] \cap \Gamma^{e'}_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)\right)\, dm_W(x)\\
&\geq \beta''(\delta - \bar{\delta}) m_W(S),
\end{aligned}$$

where the final inequality follows from Lemma 5.1. Applying Theorem D completes the proof.

6. Proof of Theorem 3.1

6.1. **"Good" iterates.** The first observation we need is that by uniform hyperbolicity on $U \setminus Z$, **(C2)** can be extended to all $W \in \tilde{\mathcal{A}}$, not just those that enter Z. In fact, it can be strengthened slightly. Recall that $\lambda(x) = \min(\lambda^u(x) - \Delta(x), \lambda^s(x))$ as in (2.4).

Lemma 6.1. *There are constants $L, \varepsilon, Q, \nu > 0$ and a function $p\colon \mathbb{N} \to [0,1]$ such that $\sum_{t\geq 1} tp(t) < \infty$ and every $W \in \tilde{\mathcal{A}}$ has an admissible decomposition satisfying*

(1) $m_W(W(t)) \leq p(t) m_W(W)$ for every $t \in \mathbb{N}$;
(2) $\bar{G}(W_j) \in \tilde{\mathcal{A}}$ for every j;

(3) *if* $x, y \in W_j$ *then* $\log \frac{|D\bar{G}(x)|_{T_xW}|}{|D\bar{G}(y)|_{T_yW}|} \leq Qd(\bar{G}x, \bar{G}y)^{\alpha^2}$,

(4) $\sum_{j=k}^{t(x)} \lambda(g^j x) \geq \nu$ *for all* $x \in W$ *and* $0 \leq k \leq t(x)$;

Proof. If $g(W)$ does not intersect Z then it suffices to take $t \equiv 1$ and partition W into either 1 or 2 pieces, depending on whether $\bar{G}(W) = g(W)$ has length greater or less than 2ε.

If $g(W)$ does intersect Z, then start with the decomposition $W = \bigsqcup_j W_j$ from **(C2)**. Let $\hat{\tau}\colon W \to \mathbb{N}$ be the inducing function given there, and $\hat{p}\colon \mathbb{N} \to [0, 2\varepsilon)$ be the probability envelope. Let n be the minimum time it takes for an f-orbit leaving Z to return to Z. Note that by **(C3)** we have $n > C/\log\chi$. Let $\tau = \hat{\tau} + n$ and let $p(t) = \hat{p}(t-n)$. Convergence of $\sum tp(t)$ follows from convergence of $\sum t\hat{p}(t)$.

To get the desired decomposition, note that for all j we have $\hat{G}(W_j) = g^n(g^{\hat{\tau}_j}(W_j)) \subset U \setminus Z$. By invariance of K^u we see that $\hat{G}(W_j)$ is tangent to K^u on its entire length, and thus it can be decomposed into a disjoint union $\bigsqcup_\ell \hat{G}(W_{j,\ell})$, where each $\hat{G}(W_{j,\ell})$ is in $\tilde{\mathcal{A}}$. Thus $W = \bigsqcup_{j,\ell} W_{j,\ell}$ is the desired decomposition and (2) is verified.

Condition (4) follows from **(C3)**, putting $\nu = n\log\chi - C$. So it only remains to prove the bounded distortion condition (3).

Recall that the Hölder curvature of every $W \in \mathcal{A}$ is bounded by L. Thus given $W \in \mathcal{A}$ and two nearby points $x, y \in W$, the Grassmanian distance between T_xW and T_yW is bounded above by $L'd(x,y)^\alpha$. Because f is $C^{1+\alpha}$ and M is compact, this gives

$$|Dg(x)|_{T_xW} - Dg(y)|_{T_yW}| \leq Kd(x,y)^{\alpha^2} \tag{6.1}$$

for some uniform constant K. As long as W lies outside of Z, we can use uniform expansion together with the observation that $\log a - \log b \leq (a-b)\frac{1}{b}$ whenever $a > b$ to get

$$\log \frac{|Dg(x)|_{T_xW}|}{|Dg(y)|_{T_yW}|} \leq Kd(x,y)^{\alpha^2}. \tag{6.2}$$

We also observe that there is $\hat{\chi} < 1$ such that for every $t \in \mathbb{N}$, every $x, y \in W_j \subset W(t)$, and every $k \geq 0$ such that $g^{t-k}x \in U \setminus Z$, we have

$$d(g^{t-k}x, g^{t-k}y) \leq \hat{\chi}^k d(g^t x, g^t y) = \hat{\chi}^k d(\bar{G}(x), \bar{G}(y)). \tag{6.3}$$

For convenience of notation, given $j \in \mathbb{N}$ we write

$$D_j g(x) = Dg(g^j x)|_{T_{g^j x}(g^j W)}.$$

Then we can use (6.3) together with **(C2)**(iii) and (6.2) to get

$$\begin{aligned}
\log \frac{|D\bar{G}(x)|_{T_xW}|}{|D\bar{G}(y)|_{T_yW}|} &\leq \log \frac{|Dg^{\hat{\tau}}(x)|_{T_xW}|}{|Dg^{\hat{\tau}}(y)|_{T_yW}|} + \sum_{k=1}^{n} \log \frac{|D_{\hat{\tau}+n-k}g(x)|}{|D_{\hat{\tau}+n-k}g(y)|} \\
&\leq Q\hat{\chi}^{\alpha n} d(\bar{G}x, \bar{G}y)^\alpha + \sum_{k=1}^{n} K\hat{\chi}^{k\alpha^2} d(\bar{G}x, \bar{G}y)^{\alpha^2},
\end{aligned}$$

which suffices to complete the proof of Lemma 6.1. □

Now we can iterate Lemma 6.1 and follow a procedure similar to the one in [BV00]. Given $W \in \tilde{\mathcal{A}}$, let $W = \bigsqcup_{j_1=1}^{k_1} W(j_1)$ be the partition given by Lemma 6.1. Then for every j_1, the curve $\bar{G}(W(j_1))$ is in $\tilde{\mathcal{A}}$, and so the lemma can be applied to this curve as well, giving a decomposition $W(j_1) = \bigsqcup_{j_2=1}^{k_2} W(j_1, j_2)$, where each $\bar{G}^2(W(j_1, j_2))$ is in $\tilde{\mathcal{A}}$. (Note that k_2 may depend on j_1.)

To simplify notation we write $\mathbf{j} = (j_1, \dots, j_n)$ and $|\mathbf{j}| = n$. Iterating the above procedure yields a partition $W = \bigsqcup_{\mathbf{j}} W(\mathbf{j})$ such that $\bar{G}^n(W(\mathbf{j})) \in \tilde{\mathcal{A}}$. Moreover, writing $T(\mathbf{j}) = \sum_{i=1}^{|\mathbf{j}|} t_{j_i}$, Lemma 6.1(4) gives

$$\sum_{i=0}^{T(\mathbf{j})} (\lambda^u(g^i x) - \Delta(g^i x)) \geq \nu|\mathbf{j}|, \qquad \sum_{i=0}^{T(\mathbf{j})} \lambda^s(g^i x) \leq -\nu|\mathbf{j}|. \tag{6.4}$$

Given $\mathbf{j}$ with $|\mathbf{j}| = n$ and any $x, y \in W(\mathbf{j})$, let γ be a path on $\bar{G}^n W(\mathbf{j})$ that connects $\bar{G}^n x$ to $\bar{G}^n y$. Then there is a path η on $W(\mathbf{j})$ such that $\bar{G}^n(\eta) = \gamma$, and by (6.4), the lengths of η and γ are related by

$$|\eta| \leq e^{-\nu n} |\gamma|.$$

Writing d_W for distance on W, this implies that

$$d_{W(\mathbf{j})}(x, y) \leq e^{-\nu n} d_{\bar{G}^n W(\mathbf{j})}(\bar{G}^n x, \bar{G}^n y).$$

Because $W(\mathbf{j})$ and $\bar{G}^n W(\mathbf{j})$ both have Hölder curvature bounded by L, there is a constant $L' > 0$ such that

$$d(x, y) \leq L' e^{-\nu|\mathbf{j}|} d(\bar{G}^n x, \bar{G}^n y) \tag{6.5}$$

whenever $x, y \in W(\mathbf{j})$, where d is the usual metric on M. We can use this to get the following bounded distortion control.

Lemma 6.2. *There exists $K \in \mathbb{R}$ such that given $W \in \tilde{\mathcal{A}}$ and $x, y \in W(\mathbf{j})$, we have*

$$K^{-1} \leq \frac{|Dg^{T(\mathbf{j})}(x)|_{T_x W}|}{|Dg^{T(\mathbf{j})}(y)|_{T_y W}|} \leq K. \tag{6.6}$$

Proof. Given $x, y \in W(\mathbf{j})$, and $0 \leq i < n$, we adopt the shorthand notation

$$D_i \bar{G}(x) = D\bar{G}(\bar{G}^i x)|_{T_{\bar{G}^i x}(\bar{G}^i W)},$$

so that

$$Dg^{T(\mathbf{j})}(x)|_{T_x W} = \prod_{i=0}^{n-1} D_i \bar{G}(x). \tag{6.7}$$

We see from (6.5) that

$$d(\bar{G}^i x, \bar{G}^i y) \leq L' e^{-\nu(n-i)} d(g^{T(\mathbf{j})} x, g^{T(\mathbf{j})} y). \tag{6.8}$$

Now Lemma 6.1(3), together with (6.7) and (6.8), yields

$$\begin{aligned}\log \frac{|Dg^{T(\mathbf{j})}(x)|_{T_xW}|}{|Dg^{T(\mathbf{j})}(y)|_{T_yW}|} &= \sum_{i=0}^{n-1} \log \frac{|D_i\bar{G}(x)|}{|D_i\bar{G}(y)|} \\ &\leq \sum_{i=0}^{n-1} Qd(\bar{G}^{n-i}x, \bar{G}^{n-i}y)^{\alpha^2} \leq \sum_{i=0}^{n-1} Qe^{-\nu\alpha^2 i}(L')^{\alpha^2} d(g^{T(\mathbf{j})}x, g^{T(\mathbf{j})y})^{\alpha^2}.\end{aligned}$$

This completes the proof since $\sum e^{-\nu\alpha^2 i}$ converges and the roles of x, y are symmetric. □

Now given $W \in \tilde{\mathcal{A}}$ and $x \in W$, let $j_1(x), j_2(x), \dots$ be such that $x \in W(j_1(x), \dots, j_n(x))$ for all n. Define a sequence of $\mathbb{N}$-valued random variables $t_1, t_2, \dots$ on (W, m_W) by $t_n(x) = t_{j_n(x)}$.

Proposition 6.3. *With p, ε as in Lemma 6.1 and K as in Lemma 6.2, we have $\mathbb{P}[t_n = T \mid t_1, \dots, t_{n-1}] \leq Kp(T)$.*

Proof. Observe that t_n is constant on $W(\mathbf{j})$ whenever $|\mathbf{j}| \geq n$, and so

$$\mathbb{P}[t_n = T \mid t_1, \dots, t_{n-1}] \leq \sup_{|\mathbf{j}|=n-1} \frac{m_W(W(\mathbf{j}, T))}{m_W(W(\mathbf{j}))}. \tag{6.9}$$

By Lemma 6.2, for every $\mathbf{j}$, the map $g^{T(\mathbf{j})}$ carries $W(\mathbf{j})$ to $V_{\mathbf{j}} := g^{T(\mathbf{j})}W(\mathbf{j}) \in \tilde{\mathcal{A}}$ with distortion bounded by K, and so in particular for $|\mathbf{j}| = n-1$ we have

$$\frac{m_W(W(\mathbf{j}, T))}{m_W(W(\mathbf{j}))} \leq K \frac{m_{V_{\mathbf{j}}}(V_{\mathbf{j}}(T))}{m_{V_{\mathbf{j}}}(V_{\mathbf{j}})} \leq Kp(T),$$

where the last inequality uses **(C2)**. □

6.2. **Asymptotic averages of return times.** Now we are in a position to prove that the asymptotic average of the return times t_n is bounded for m_W-a.e. initial condition. The arguments used here are well-known, but we give full details as our setting differs slightly from that in which the strong law of large numbers is usually proved. We follow [Bil79, Theorem 22.1], which gives an argument that goes back to Etemadi.

Consider the probability space (W, m_W), where $W \in \tilde{\mathcal{A}}$ is as in the previous section and we take m_W to be normalized. Let $\mathcal{F}_n$ be the increasing sequence of σ-algebras generated by the sets $W(j_1, \dots, j_n)$. By Condition (C2) and Proposition 6.3, we can choose $p \colon \mathbb{N} \to [0, 1]$ such that

$$\mathbb{P}[t_n = T \mid \mathcal{F}_{n-1}] \leq p(T), \tag{6.10}$$

$$R := \sum_{T=1}^{\infty} Tp(T) < \infty. \tag{6.11}$$

Note that the "new" $p(t)$ is obtained by multiplying the "old" one from Condition (C2) by the distortion constant K coming from Lemma 6.2.

Proposition 6.4. *If t_n is any sequence of random variables satisfying* (6.10) *and* (6.11)*, then* $\overline{\lim}_{n\to\infty} \frac{1}{n}\sum_{k=1}^n t_k \leq R$ *almost surely.*

Proof. Consider the truncated random variables $s_n = t_n \mathbf{1}_{[t_n \leq n]}$, and note that by (6.10) and (6.11) we have

$$0 \leq s_n \leq t_n \Rightarrow \mathbb{E}[s_n \mid \mathcal{F}_{n-1}] \leq R.$$

Now consider the random variables $r_n = s_n + R - \mathbb{E}[s_n \mid \mathcal{F}_{n-1}]$. Note that r_n is $\mathcal{F}_n$-measurable, and moreover

$$\mathbb{E}[r_n \mid \mathcal{F}_{n-1}] = R \text{ for all } n. \tag{6.12}$$

Let $X_N = \frac{1}{N}(r_1 + \cdots + r_N) - R$, so that $\mathbb{E}[X_N] = 0$. The idea is to use (6.12) to obtain an efficient estimate on $\mathbb{E}[X_N^2]$, which via Chebyshev's inequality gives a bound on $\mathbb{P}[X_N \geq \varepsilon]$. A careful use of Borel-Cantelli will lead to the result.

We begin with the observation that

$$\begin{aligned}\mathbb{E}[X_N^2] &= \frac{1}{N^2}\mathbb{E}\left[\left(\sum_{k=1}^N (r_k - R)\right)^2\right] \\ &= \frac{1}{N^2}\sum_{k=1}^N \mathbb{E}[(r_k - R)^2] + \frac{2}{N^2}\sum_{i<j}\mathbb{E}[(r_i - R)(r_j - R)].\end{aligned}$$

Given $i < j$, the fact that r_i is $\mathcal{F}_{n-1}$-measurable together with (6.12) gives $\mathbb{E}[(r_i - R)(r_j - R)] = 0$, hence

$$\mathbb{E}[X_N^2] = \frac{1}{N^2}\sum_{k=1}^N \mathbb{E}[(r_k - R)^2] \leq \frac{1}{N^2}\sum_{k=1}^N (\mathbb{E}[r_k^2] + 2R\mathbb{E}[r_k] + R^2). \tag{6.13}$$

Let T_0 be such that $\sum_{T \geq T_0} p(T) \leq 1$, and let Y be a random variable taking the value T with probability $p(T)$ for $T \geq T_0$. Note that by the definition of s_n and r_n, we have $r_n \leq s_n + R \leq n + R$, thus

$$\mathbb{E}[r_k^2] = \sum_{T=1}^{k+R} T^2 \mathbb{P}[r_k = T] \leq \sum_{T=1}^{k+R} T^2 p(T) \leq C + \mathbb{E}[Y^2 \mathbf{1}_{[Y \leq k+R]}]$$

for some fixed constant C. Note that the final expression is non-decreasing in k, and so together with (6.13) we have

$$\mathbb{E}[X_N^2] \leq \frac{1}{N}\left(C' + \mathbb{E}[Y^2 \mathbf{1}_{[Y \leq N+R]}]\right), \tag{6.14}$$

where again C' is a fixed constant. Fixing $\varepsilon > 0$ and using (6.14) in Chebyshev's inequality yields

$$\mathbb{P}[|X_N| \geq \varepsilon] \leq \frac{1}{\varepsilon^2 N}\left(C' + \mathbb{E}[Y^2 \mathbf{1}_{[Y \leq N+R]}]\right). \tag{6.15}$$

Fix $\alpha > 1$ and let $u_n = \lfloor \alpha^n \rfloor$. Putting $K = \frac{2\alpha}{\alpha - 1}$, choosing $y \in \mathbb{R}$, and letting $m = m(y)$ be the smallest number such that $u_m \geq y$, we have

$$\sum_{u_n \geq y} u_n^{-1} \leq 2 \sum_{n \geq m} \alpha^{-n} = K\alpha^{-m} \leq Ky^{-1},$$

and thus (6.15) yields

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}[|X_{u_n}| \geq \varepsilon] &\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} u_n^{-1} \left(C' + \mathbb{E}[Y^2 \mathbf{1}_{[Y \leq u_n + R]}] \right) \\ &\leq \frac{KC'}{\varepsilon^2} + \frac{1}{\varepsilon^2} \mathbb{E}\left[\sum_{n=1}^{\infty} Y^2 u_n^{-1} \mathbf{1}_{[Y \leq u_n + R]} \right] \\ &\leq \frac{KC'}{\varepsilon^2} + \frac{1}{\varepsilon^2} \mathbb{E}[Y^2 K (Y - R)^{-1}] < \infty, \end{aligned}$$

where the last inequality uses (6.11) and the definition of Y. Since the probabilities of the events $|X_{u_n}| \geq \varepsilon$ are summable, it follows from the first Borel–Cantelli lemma that with probability 1, only finitely many of these events occur. In particular, we have $\overline{\lim}_{n\to\infty} |X_{u_n}| \leq \varepsilon$ almost surely; in terms of the random variables r_n, this means that

$$\overline{\lim_{n\to\infty}} \left| \left(\frac{1}{u_n} \sum_{k=1}^{u_n} r_k \right) - R \right| \leq \varepsilon \text{ almost surely.}$$

Taking an intersection over all rational $\varepsilon > 0$ gives

$$\lim_{n\to\infty} \frac{1}{u_n} \sum_{k=1}^{u_n} r_k = R \text{ a.s.} \tag{6.16}$$

Let $Z_k = \sum_{i=1}^{k} r_k$. Because $r_k \geq 0$ we have $Z_{u_n} \leq Z_k \leq Z_{u_{n+1}}$ for all $u_n \leq k \leq u_{n+1}$, and in particular

$$\frac{u_n}{u_{n+1}} \frac{Z_{u_n}}{u_n} \leq \frac{Z_k}{k} \leq \frac{u_{n+1}}{u_n} \frac{Z_{u_{n+1}}}{u_{n+1}}.$$

Taking the limit and using (6.16) gives

$$\frac{1}{\alpha} R \leq \underline{\lim_{k\to\infty}} \frac{1}{k} Z_k \leq \overline{\lim_{k\to\infty}} \frac{1}{k} Z_k \leq \alpha R \text{ a.s.}$$

Taking an intersection over all rational $\alpha > 1$ gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} r_k = R \text{ a.s.} \tag{6.17}$$

Finally, we recall from the definition of s_n and r_n that $t_n \leq r_n$ whenever $t_n \leq n$. In particular, we may observe that

$$\sum_{n=1}^{\infty} \mathbb{P}[t_n > r_n] \leq \sum_{n=1}^{\infty} \mathbb{P}[t_n > n] \leq \mathbb{E}[t_n] \leq R$$

and apply Borel-Cantelli again to deduce that with probability one, $t_n > r_n$ for at most finitely many values of n. In particular, (6.17) implies that

$$\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k \le R \text{ a.s.} \tag{6.18}$$

which completes the proof of Proposition 6.4. □

6.3. **Positive rate of effective hyperbolicity.** It remains to verify the conditions of Theorem C.[12] Recall that S_W is the set of points $x \in W$ for which $\Gamma^e_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)$ has uniformly (in q) positive lower asymptotic density for some $C, \overline{\lambda} > 0$, and for which (2.8) holds. The sets $\Gamma^e_{\overline{\lambda}}(x)$ and $\Gamma^s_{C,\overline{\lambda},q}(x)$ are defined in (2.11) and (2.12), respectively, and in our setting can be controlled using the sequence t_n.

Lemma 6.5. *Let $\nu > 0$ be as in Lemma 6.1(4) and fix $0 < \overline{\lambda} < \nu' < \nu/R$. If $\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k(x) \le R$ then we have $\underline{\delta}(\Gamma^e_{\overline{\lambda}}(x) \cap \Gamma^s_{C,\overline{\lambda},q}(x)) \ge \frac{\nu'-\overline{\lambda}}{L-\overline{\lambda}}$ for every q, where L is a uniform bound for $\lambda^u - \Delta$ and λ^s.*

Proof. Let $T_n(x) = \sum_{k=1}^{n} t_k(x)$. By Lemma 6.1(4) and **(C3)**, there are sequences $a_k, b_k \in [-L, L]$ such that the following hold:

1. $(\lambda^u - \Delta)(g^k x) \ge a_k$;
2. $\lambda^s(g^k x) \le b_k$;
3. $\sum_{k=T_n(x)}^{T_{n+1}(x)-1} a_k = \nu$;
4. $\sum_{k=T_n(x)}^{T_{n+1}(x)-1} b_k = -\nu$;
5. $\sum_{k=r}^{s} a_k \ge -C$ for all $T_n(x) \le r \le s < T_{n+1}(x)$;
6. $\sum_{k=r}^{s} b_k \le -C$ for all $T_n(x) \le r \le s < T_{n+1}(x)$.

Now if $\varlimsup \frac{1}{n} T_n(x) \le R$, then the above imply $\varliminf \frac{1}{k}\sum_{j=0}^{k} a_j \ge \nu/R > \nu'$. Thus by Pliss' lemma [BP07, Lemma 11.2.6] there is $\Gamma_a \subset \mathbb{N}$ with $\underline{\delta}(\Gamma_a) \ge \frac{\nu'-\overline{\lambda}}{L-\overline{\lambda}}$ such that for all $n \in \Gamma_a$ and $0 \le k < n$, we have

$$\sum_{j=k}^{n-1} (\lambda^u - \Delta)(g^j x) \ge \sum_{j=k}^{n-1} a_j \ge \overline{\lambda}(n-k).$$

It follows that $\Gamma_a \subset \Gamma^e_{\overline{\lambda}}(x)$. Moreover, the properties listed above guarantee that every $n \in \Gamma_a$ is also in $\Gamma^s_{C,\overline{\lambda},q}(x)$ for $q \le n$. This yields the desired lower bound on the asymptotic density. □

Combining Proposition 6.4 and Lemma 6.5, we see that the only remaining problem is to estimate the angle between the stable and unstable cones.

[12] The argument that follows could also be given using the simpler Theorem B – that is, using (2.5) instead of (2.14) – but it would require a more restrictive Condition **(C3)**, as discussed in the footnote to (3.2).

6.4. **Angle between stable and unstable cones.** Because g is a diffeomorphism and the angle between K^s and K^u is uniformly positive on $U \setminus Z$, we see that for every $\bar{\theta} > 0$ there is T such that if $x \in W \in \tilde{\mathcal{A}}$ and $\tau(x) \le T$, then $\theta(g^k x) \ge \bar{\theta}$ for all $0 \le k \le \tau(x)$.

Fix $W \in \tilde{\mathcal{A}}$ and let $t_n(x)$ be as in the previous section. We conclude from the above observations that in order to bound the density of the set $\{k \in \mathbb{N} \mid \theta(g^k(x)) < \bar{\theta}\}$, it suffices to bound the quantity

$$\frac{\sum_{j=1}^m t_j(x) \mathbf{1}_{[t_j(x) \ge T]}}{\sum_{j=1}^m t_j(x)}.$$

To see this, we fix $\bar{\theta} > 0$, let $T = T(\bar{\theta})$ be as above, and write $T_n(x) = \sum_{i=1}^n t_i(x)$. For a fixed $x \in W$, let m be the largest number such that $T_m(x) \le n$, and put $d_j = 1$ if $t_j(x) \ge T$, and $d_j = 0$ otherwise. Then

$$\#\{1 \le k \le n \mid \theta(g^k x) < \bar{\theta}\} \le \left(\sum_{j=1}^m t_j(x) d_j(x) \right) + (n - T_m(x)) d_{m+1}(x).$$

If $d_{m+1} = 0$, we have

$$\frac{1}{n} \#\{1 \le k \le n \mid \theta(g^k x) < \bar{\theta}\} \le \frac{1}{T_m(x)} \sum_{j=1}^m t_j d_j,$$

and if $d_m = 1$ we have the same inequality with m replaced by $m + 1$ on the right-hand side. Writing $\rho(x) = \overline{\lim}_{n\to\infty} \frac{1}{n} \#\{1 \le k \le n \mid \theta(g^k(x)) < \bar{\theta}\}$, this implies that

$$\rho(x) \le \overline{\lim_{m\to\infty}} \frac{\sum_{j=1}^m t_j(x) \mathbf{1}_{[t_j(x) \ge T]}}{\sum_{j=1}^m t_j(x)}. \tag{6.19}$$

Given a fixed T, let $\tilde{t}_j = t_j \mathbf{1}_{[t_j \ge T]}$, and observe that with $p(t)$ as in (6.10) and (6.11), we have

$$\mathbb{P}[\tilde{t}_j = t \mid \mathcal{F}_{j-1}] \le \tilde{p}(t),$$

where $\tilde{p}(t) = p(t)$ when $t \ge T$ and $\tilde{p}(t) = 0$ otherwise. In particular, for every $\varepsilon > 0$ there is T such that $\sum t\tilde{p}(t) < \varepsilon$. Then Proposition 6.4 implies that

$$\overline{\lim} \frac{1}{n} \sum_{k=1}^n \tilde{t}_k \le \varepsilon \text{ a.s.},$$

which together with Proposition 6.4 applied to t_j yields $\rho(x) \le \varepsilon/R$. Because $\varepsilon > 0$ can be taken arbitrarily small by sending $\bar{\theta} \to 0$, we conclude that m_W-a.e. $x \in W$ satisfies (2.8). Together with Proposition 6.4 and Lemma 6.5, this shows that m_W-a.e. $x \in W$ is in $\hat{S}_W$, and thus Theorem C applies, establishing the existence of an SRB measure for g.

7. Proof of Theorem 3.2

First we outline how the conditions **(C1)**–**(C3)** will be verified. The key will be to obtain estimates on the flow generated by $\mathcal{X}$, in particular on solutions $x(t)$ of the flow itself and on tangent vectors $v(t)$ that evolve under the flow. Roughly speaking, this will go in three stages.

(1) Estimate $\theta(t)$, the angle between $x(t)$ and the unstable direction, which starts near $\pi/2$ and decays towards 0. This is done in (7.2).
(2) Estimate $\rho(t)$, the angle between $v(t)$ and the unstable direction, which we want to keep small to obtain good expansion estimates. This is done in Lemma 7.6, and a key result is Corollary 7.7.
(3) Given two trajectories starting on the same W_j, estimate the difference between the corresponding tangent vectors as t varies. This is done using Lemma 7.8 and (7.23).

The conditions **(C1)**–**(C3)** will be verified as follows. For **(C1)**, we use Lemma 7.6 that shows decay of $\rho(t)$ for large values of t. For **(C3)**, we use (7.24) and Corollary 7.7, the key being that $\int_{t_1}^{t_2} \tan\rho(t)\,dt$ is uniformly bounded.

Verification of **(C2)** is the most involved. Given $W \in \tilde{\mathcal{A}}$ that is about to enter Z, the natural partition to use is $W = \bigsqcup_j W_j$ where W_j is the set of points in W for which j is the first return time to $U \setminus Z$ under the action of g. Indeed, there is $\varepsilon > 0$ such that if we partition $f^{-1}(Z) \setminus Z$ into level sets of the return time, then any curve in $\mathcal{A}$ that crosses one of these level sets completely is mapped by G to a curve of length approximately ε. The issue is that there may be some level sets that W does not cross completely, so that $G(W_j)$ is too short. In this case we join W_j with the neighbouring W_i and increase τ by 1, so that the image under $\bar{G}$ has length between ε and 2ε.

The above procedure describes an admissible decomposition such that $\bar{G}(W_j)$ has the right length for each j. Now **(C2)**(i) will come from the expansion estimate (7.30) since we can take $p(t) = t^{-(1+\frac{1}{\alpha})}$ (up to a constant). The rest of **(C2)**(ii) will come from (7.44), which gives bounds on the Hölder curvature of $\bar{G}(W_j)$. Finally, **(C2)**(iii) will come from (7.45).

7.1. **Trajectories near the fixed point.** All of the computations and estimates described in Section 3.2 will actually be carried out on $Y := B(0, r_0) \subset Z$, where $\mathcal{X}$ has the specific form $\|x\|^\alpha Ax$. As g is uniformly hyperbolic on $Z \setminus Y$ and trajectories spend a uniformly bounded time in this region, the conditions continue to hold when we consider passages through the larger region Z.

Let φ_t be the flow on Z generated by the vector field $\mathcal{X}$. When we consider a single trajectory of the flow, we will generally write $x(t) = \varphi_t(x)$ to keep the notation more compact.

We start by verifying Condition **(C1)**. We first study how a trajectory moves through Z; in particular, how the relative distance from x to the stable and unstable manifolds of the fixed point varies.

More precisely, let $Y = B(0, r_0)$ so that $\mathcal{X}(x) = \|x\|^\alpha Ax$ on Y. Let $x\colon [0,T] \to Y$ be a trajectory of the flow determined by $\mathcal{X}$, and write $x = x_u + x_s$ for $x_u \in E^u \times \{0\}$ and $x_s \in \{0\} \times E^s$. (Recall that $E^u = \mathbb{R}$ and $E^s = \mathbb{R}^{d-1}$.) Let $\theta(t)$ be the positive angle between x and $E^u \times \{0\}$, so that $\tan\theta(t) = \|x_s(t)\|/\|x_u(t)\|$. Let T_0 be the time at which x leaves Y, so $\|x(T_0)\| = r_0$ and $x(t) \in Y$ for all $t \in [0, T_0]$. Let $\lambda = \beta + \gamma$.

Lemma 7.1. *On $t \in [0, T_0]$, we have* $(\tan\theta)' = -\lambda\|x\|^\alpha \tan\theta$.

Proof. Observe that $\dot{x}_u = \|x\|^\alpha \gamma x_u$ and $\dot{x}_s = -\|x\|^\alpha \beta x_s$, whence

$$\|x_u\|' = \left\langle \frac{x_u}{\|x_u\|}, \dot{x}_u \right\rangle = \|x\|^\alpha \gamma \|x_u\| = \|x\|^{1+\alpha}\gamma\cos\theta,$$

and similarly $\|x_s\|' = -\|x\|^{1+\alpha}\beta\sin\theta$. Thus

$$\begin{aligned}(\tan\theta)' &= \left(\frac{\|x_s\|}{\|x_u\|}\right)' = \frac{\|x_u\|\|x_s\|' - \|x_s\|\|x_u\|'}{\|x_u\|^2} \\ &= \frac{(\|x\|\cos\theta)(-\|x\|^{1+\alpha}\beta\sin\theta) - (\|x\|\sin\theta)(\|x\|^{1+\alpha}\gamma\cos\theta)}{\|x\|^2\cos^2\theta} \\ &= -(\beta+\gamma)\|x\|^\alpha\tan\theta.\end{aligned}$$

□

Defining a quantity $J(t_1, t_2)$ by

$$J(t_1, t_2) := \lambda \int_{t_1}^{t_2} \|x(\tau)\|^\alpha \, d\tau, \tag{7.1}$$

it follows from Lemma 7.1 that

$$\tan\theta(t_2) = e^{-J(t_1,t_2)} \tan\theta(t_1). \tag{7.2}$$

Given $s \in (0, \infty)$ and a trajectory $x\colon [0, T_0] \to Y$ that leaves Y at time T_0, let T_s be the time at which $\tan\theta = s$. Note that $\tan\theta$ is strictly decreasing by Lemma 7.1, so T_s is well-defined as long as

$$\tan\theta(0) > s > \tan\theta(T_0). \tag{7.3}$$

The following lemma controls how $\|x\|$ changes.

Lemma 7.2. *The norm of x varies according to the ODE*

$$\frac{d}{dt}\|x\|^{-\alpha} = \alpha(\beta\sin^2\theta - \gamma\cos^2\theta). \tag{7.4}$$

In particular, for every piece of trajectory $x\colon [0,T] \to Y$ (that need not enter/exit Y at its endpoints), we have

$$\frac{d}{dt}\|x\|^{-\alpha} \geq -\gamma\alpha, \tag{7.5}$$

$$\|x(t)\|^{-\alpha} \leq \|x(T)\|^{-\alpha} + \gamma\alpha(T - t) \tag{7.6}$$

Finally, writing $\xi(s) = \frac{\beta s^2-\gamma}{s^2+1}$*, for all* $t \le T_s$ *(meaning that* $\tan\theta \ge s$ *on* $[0,t]$*) we have*

$$\frac{d}{dt}\|x\|^{-\alpha} \ge \alpha\xi(s), \tag{7.7}$$

$$\|x(t)\|^{-\alpha} \ge \|x(0)\|^{-\alpha} + \alpha\xi(s)t. \tag{7.8}$$

Proof. For (7.4), we observe that

$$(\|x\|^{-\alpha})' = -\alpha\|x\|^{-\alpha-1}\left\langle \frac{x}{\|x\|}, \dot{x}\right\rangle = -\alpha\|x\|^{-\alpha-2}\langle x, \|x\|^{\alpha}Ax\rangle = -\alpha\langle \hat{x}, A\hat{x}\rangle,$$

where we write $\hat{x} = x/\|x\|$. Then (7.4) follows from the form of A. Now (7.5) follows directly from (7.4), and in turn implies (7.6). For (7.7), we observe that when $\tan\theta \ge s$, we have

$$\beta\sin^2\theta - \gamma\cos^2\theta = \frac{\beta\tan^2\theta - \gamma}{\tan^2\theta + 1} \ge \frac{\beta s^2 - \gamma}{s^2+1} =: \xi(s).$$

This establishes (7.7), which in turn gives (7.8). □

Proposition 7.3. *For every* $s \in (0,\infty)$ *and every trajectory satisfying* (7.3)*, we have*

$$\int_{T_s}^{T_0} \|x\|^{\alpha}\tan\theta\,dt \le \frac{s}{\gamma+\beta}. \tag{7.9}$$

Let

$$\chi = \chi(s) = \frac{\gamma}{\gamma+\beta}\left(1 + \frac{1}{s^2}\right); \tag{7.10}$$

then if the trajectory x *enters* Y *at time* 0 *and leaves it at time* T_0*, we have*

$$T_s \le \chi T_0. \tag{7.11}$$

Remark. While (7.11) holds for all $s > 0$, it gives a meaningful statement only if s is large enough to ensure that $\chi < 1$.

Proof of Proposition 7.3. For (7.9) we observe that (7.1) gives

$$\frac{d}{dt}(e^{-J(T_s,t)}) = -\lambda\|x\|^{\alpha}e^{-J(T_s,t)},$$

and so by (7.2) we have

$$\int_{T_s}^{T_0} \|x\|^{\alpha}\tan\theta\,dt = \int_{T_s}^{T_0} \|x\|^{\alpha}se^{-J(T_s,t)}\,dt = -\frac{s}{\lambda}\left[e^{-J(T_s,t)}\right]_{T_s}^{T_0}$$

which proves (7.9). To get (7.11) we combine (7.6) and (7.8) from Lemma 7.2, yielding

$$\|x(0)\|^{-\alpha} + \xi\alpha T_s \le \|x(T_0)\|^{-\alpha} + \gamma\alpha(T_0 - T_s).$$

Since $\|x(0)\| = \|x(T_0)\| = r_0$ as both are on the boundary of Y, we obtain

$$(\xi+\gamma)T_s \le \gamma T_0. \tag{7.12}$$

From Lemma 7.2 we have

$$\xi + \gamma = \frac{\beta s^2 - \gamma}{s^2 + 1} + \gamma = (\beta + \gamma)\frac{s^2}{s^2+1};$$

together with (7.12) and the definition of χ in (7.10), this gives (7.11). □

7.2. **Tangent vectors near the fixed point.** Now let $v(t)$ be a family of tangent vectors that is invariant under the flow – that is, $D\varphi_\tau(x(t))(v(t)) = v(t+\tau)$. As with x, we write $v = v_u + v_s$ and let $\rho(t)$ be the positive angle between $v(t)$ and E^u, so that $\tan\rho(t) = \|v_s(t)\|/\|v_u(t)\|$. The following standard result governs how v evolves in time; a proof (using different notation) is given in [HS74, §15.2].

Proposition 7.4. *Let φ_t be the flow for a vector field $\mathcal{X}$ on $\mathbb{R}^n$. Let $x(t)$ be a solution to $\dot{x} = \mathcal{X}(x)$, and let $v(t) \subset T_{x(t)}M$ be a $D\varphi$-invariant family of tangent vectors. Then*

$$\dot{v}(t) = (\mathcal{L}_{v(t)}\mathcal{X})(x(t)), \tag{7.13}$$

where $\mathcal{L}_v$ is the Lie derivative in the direction of v – that is, $\mathcal{L}_v\mathcal{X}(x) = (D\mathcal{X}(x))v$.

For the vector field $\mathcal{X}(x) = \|x\|^\alpha Ax$, we have

$$\begin{aligned}\mathcal{L}_v\mathcal{X}(x) &= \langle v, \nabla\|x\|^\alpha\rangle Ax + \|x\|^\alpha Av \\ &= \alpha\|x\|^{\alpha-2}\langle v, x\rangle Ax + \|x\|^\alpha Av \\ &= \|x\|^\alpha\big(\alpha\langle v, \hat{x}\rangle A\hat{x} + Av\big),\end{aligned} \tag{7.14}$$

where we write $\hat{x} = x/\|x\|$.

Lemma 7.5. *For $t \in [0, T_0]$, if $\tan\rho \leq 1$, then*

$$(\tan\rho)' \leq -\lambda\|x\|^\alpha \tan\rho + \alpha\lambda\|x\|^\alpha \frac{\tan\theta}{1+\tan^2\theta}. \tag{7.15}$$

Proof. It suffices to consider the case when $\|v\| = 1$. Observe that

$$\begin{aligned}\|v_u\|' = \left\langle \frac{v_u}{\|v_u\|}, \dot{v}_u \right\rangle &= \|v_u\|^{-1}\langle v_u, \alpha\|x\|^{\alpha-2}\langle v, x\rangle Ax_u + \|x\|^\alpha Av_u\rangle \\ &= \gamma\|x\|^\alpha\cos\rho + \gamma\alpha\|x\|^\alpha\langle v, \hat{x}\rangle\cos\theta,\end{aligned}$$

since E^u is one-dimensional. Writing ϕ_s for the angle between v_s and x_s, a similar computation gives

$$\|v_s\|' = -\beta\|x\|^\alpha\sin\rho - \beta\alpha\|x\|^\alpha\langle v, \hat{x}\rangle\cos\phi_s\sin\theta.$$

Let $q_s = \cos\phi_s \in [-1,1]$. Now

$$\begin{aligned}(\tan\rho)' &= \left(\frac{\|v_s\|}{\|v_u\|}\right)' = \frac{1}{\cos^2\rho}(\cos\rho\|v_s\|' - \sin\rho\|v_u\|') \\ &= -(\beta+\gamma)\|x\|^\alpha \tan\rho - \frac{\cos\theta}{\cos\rho}\alpha\|x\|^\alpha\langle v,\hat{x}\rangle(\beta q_s\tan\theta + \gamma\tan\rho) \\ &= -\lambda\|x\|^\alpha\tan\rho - \frac{\alpha\|x\|^\alpha(1+q_s\tan\rho\tan\theta)(\beta q_s\tan\theta+\gamma\tan\rho)}{1+\tan^2\theta}.\end{aligned}$$

The numerator in the final term is equal to

$$\alpha\|x\|^\alpha\left(\tan\rho(\gamma + q_s^2\beta\tan^2\theta) + q_s(\gamma\tan^2\rho+\beta)\tan\theta\right),$$

and since $\tan\theta > 0$, $\tan\rho\in[0,1]$, and $q_s \geq -1$, this is bounded below by $-\alpha\lambda\|x\|^\alpha\tan\theta$. The result follows. □

Lemma 7.5 establishes the existence of an invariant cone family K^u by observing that since $r/(1+r^2) \leq 1/2$ for all $r\in\mathbb{R}$, the cone defined by $\tan\rho \leq \alpha/2$ is invariant. Applying the same argument with time reversed establishes **(C1)**.

In fact, although we will occasionally use the crude estimate

$$\tan\rho \leq \frac{\alpha}{2} < \frac{1}{2}, \tag{7.16}$$

we will need more careful estimates of $\tan\rho$ in order to establish the expansion properties in **(C2)** and **(C3)**. As in the previous section, let $T_1\in[0,T_0]$ be such that $\tan\theta(x(T_1)) = 1$; our estimate will show that $\rho(t)$ is smaller the further away t is from 0 and T_1.

Lemma 7.6. *With $x(t)$ and $v(t)$ as above, we have the following bounds:*

$$\tan\rho(t) \leq \begin{cases}(\tan\rho_0)e^{-J(0,t)} + \frac{\alpha}{2}e^{-J(t,T_1)} & t\in[0,T_1],\\ (\tan\rho(T_1) + \alpha J(T_1,t))e^{-J(T_1,t)} & t\in[T_1,T_0].\end{cases}$$

Proof. Observe that if only the first half of the right hand side of (7.15) was present, then $e^{J(t_0,t)}\tan\rho$ would be non-increasing for any t_0. Thus to estimate $\tan\rho$, we differentiate this quantity and see how much it may increase.

Since $\tan\theta > 0$ we have $\frac{\tan\theta}{1+\tan^2\theta} \leq \min(\frac{1}{\tan\theta},\tan\theta)$ – we will use the first bound on $[0,T_1]$ and the second on $[T_1,T_0]$. On the first interval, we obtain

$$\begin{aligned}(e^{J(0,t)}\tan\rho)' &\leq \alpha\lambda e^{J(0,t)}\|x\|^\alpha(\tan\theta)^{-1} \\ &= \alpha\lambda e^{J(0,T_1)}\|x\|^\alpha e^{-2J(t,T_1)} = \frac{\alpha}{2}e^{J(0,T_1)}\left(e^{-2J(t,T_1)}\right)',\end{aligned}$$

and hence

$$e^{J(0,t)}\tan\rho(t) \leq \tan\rho_0 + \frac{\alpha}{2}e^{J(0,T_1)}\left(e^{-2J(t,T_1)} - e^{-2J(0,T_1)}\right).$$

This yields the estimate

$$\begin{aligned}\tan\rho(t) &\leq (\tan\rho_0)e^{-J(0,t)} + \frac{\alpha}{2}e^{J(t,T_1)}\left(e^{-2J(t,T_1)} - e^{-2J(0,T_1)}\right) \\ &= (\tan\rho_0)e^{-J(0,t)} + \frac{\alpha}{2}\left(e^{-J(t,T_1)} - e^{-(J(0,T_1)+J(0,t))}\right),\end{aligned} \tag{7.17}$$

which proves the first half of the lemma.

On $[T_1, T_0]$, we use the bound $\frac{\tan\theta}{1+\tan^2\theta} \leq \tan\theta$, and so

$$(e^{J(T_1,t)}\tan\rho)' \leq \alpha\lambda e^{J(T_1,t)}\|x\|^\alpha \tan\theta = \alpha\lambda\|x\|^\alpha,$$

which gives

$$e^{J(T_1,t)}\tan\rho(t) \leq \tan\rho(T_1) + \alpha J(T_1,t)$$

and completes the proof of the lemma. □

The following consequence of Lemma 7.6 is crucial to many of our later estimates. Here and in the remainder of the proof we used a number of constants denoted Q_i, which will not always be explicitly introduced. The first appearance of such a constant should be understood to mean that there is some value of this constant, independent of the choice of trajectory or of the time t, for which the next statement is true.

Corollary 7.7. *There is $Q_0 \in \mathbb{R}$ such that $\int_{t_1}^{t_2} \|x\|^\alpha \tan\rho\, dt \leq Q_0$ for every choice of x, v, and $t_1, t_2 \in [0, T_0]$.*

Proof. Using Lemma 7.6, observe that on $[0, T_1]$ we have

$$\begin{aligned}\lambda\|x\|^\alpha\tan\rho &\leq \lambda\tan\rho_0\|x\|^\alpha e^{-J(0,t)} + \frac{\alpha}{2}\lambda\|x\|^\alpha e^{-J(t,T_1)} \\ &= -\tan\rho_0(e^{-J(0,t)})' + \frac{\alpha}{2}(e^{-J(t,T_1)})',\end{aligned}$$

whence

$$\int_0^{T_1} \|x\|^\alpha \tan\rho\, dt \leq \frac{1}{\lambda}\left(\tan\rho_0 + \frac{\alpha}{2}\right)(1 - e^{-J(0,T_1)}). \tag{7.18}$$

Similarly, on $[T_1, T_0]$ we see that since $\tan\rho(T_1) \leq \alpha/2$, we have

$$\begin{aligned}\lambda\|x\|^\alpha\tan\rho &\leq \alpha\lambda\|x\|^\alpha\left(\frac{1}{2} + J(T_1,t)\right)e^{-J(T_1,t)} \\ &= \alpha\left(-\left(\frac{3}{2} + J(T_1,t)\right)e^{-J(T_1,t)}\right)',\end{aligned}$$

and so

$$\int_{T_1}^{T_0} \|x\|^\alpha \tan\rho\, dt \leq \frac{3\alpha}{2\lambda}. \tag{7.19}$$

The result follows since the integrand is non-negative. □

7.3. **Expansion near the fixed point.** To estimate the expansion in the unstable cone along a trajectory, we observe that by Proposition 7.4, we have

$$\log\left(\frac{\|D\varphi_t(x)(v)\|}{\|v\|}\right) = \int_0^t \langle \hat{v}, \mathcal{L}_{\hat{v}}\mathcal{X}\rangle\, d\tau, \tag{7.20}$$

where $\hat{v}(\tau) = v(\tau)/\|v(\tau)\|$. To make the calculations simpler we drop the hat and just assume $\|v\| = 1$. Then recalling (7.14),

$$\langle v, \mathcal{L}_v\mathcal{X}\rangle = \|x\|^\alpha\big(\alpha\langle v, \hat{x}\rangle\langle v, A\hat{x}\rangle + \langle v, Av\rangle\big). \tag{7.21}$$

We estimate the first term by observing that

$$\begin{aligned}\langle v, \hat{x}\rangle\langle v, A\hat{x}\rangle &= (\cos\rho\cos\theta + q_s\sin\rho\sin\theta)(\gamma\cos\rho\cos\theta - \beta q_s\sin\rho\sin\theta)\\ &= \gamma\cos^2\rho\cos^2\theta - \ell(x,v),\end{aligned}$$

where the final term is

$$\ell(x,v) = q_s(\beta-\gamma)\sin\rho\sin\theta\cos\rho\cos\theta + q_s^2\beta\sin^2\rho\sin^2\theta. \tag{7.22}$$

Remark. In the case $\gamma = \beta$, the above yields

$$\begin{aligned}\langle v, \mathcal{L}_v\mathcal{X}\rangle &= \|x\|^\alpha\big(\alpha\gamma(\cos^2\rho\cos^2\theta - q_s^2\sin^2\rho\sin^2\theta) + \gamma(\cos^2\rho - \sin^2\rho)\big)\\ &\geq \gamma\|x\|^\alpha(1-(2+\alpha)\sin^2\rho) \geq 0,\end{aligned}$$

using the fact that $\tan\rho \leq \frac{1}{2}$ and hence $\sin^2\rho \leq \frac{1}{5}$. Thus the unstable cone $K^u(x)$ never contains any contracting vectors. This is the case for the original Katok map. In our case contraction may nevertheless occur when $\beta \neq \gamma$; however, we argue below that the total contraction along a trajectory making a single trip through Y is uniformly bounded.

We see from (7.22) that there exists $Q_1 > 0$ such that $\ell(x,\rho) \leq Q_1\tan\rho$. Using this in (7.21) together with the observation that

$$\langle v, Av\rangle = \gamma\cos^2\rho - \beta\sin^2\rho = \gamma - \lambda\sin^2\rho,$$

we obtain

$$\langle v, \mathcal{L}_v\mathcal{X}\rangle \geq \|x\|^\alpha\left(\gamma(1+\alpha\cos^2\rho\cos^2\theta) - Q_2\tan\rho\right) \tag{7.23}$$

for some constant $Q_2 > 0$. By Corollary 7.7, the contribution of the final term is uniformly bounded over all trajectories. Together with (7.20), this shows that there is Q_3 such that

$$\log\left(\frac{\|D\varphi_t(x)(v)\|}{\|v\|}\right) \geq -Q_3 + \gamma\int_0^t \|x\|^\alpha(1+\alpha\cos^2\rho\cos^2\theta)\, d\tau, \tag{7.24}$$

We note that this establishes **(C3)** because the first half of the right-hand side of (7.23) is non-negative.

We see from (7.24) that it is important to control $\int \gamma\|x\|^\alpha\, d\tau$. This can be done using Lemma 7.2. Given a trajectory that escapes Y at time T_0, we have

$$\begin{aligned}\int_{t_1}^{t_2} \gamma\|x(t)\|^\alpha\,dt &\geq \int_{t_1}^{t_2} \gamma(r_0^{-\alpha} + \gamma\alpha(T_0 - t))^{-1}\,dt \\ &= -\frac{1}{\alpha}\log(r_0^{-\alpha} + \gamma\alpha(T_0 - t))\Big|_{t_1}^{t_2} = \frac{1}{\alpha}\log\left(\frac{1 + r_0^\alpha\gamma\alpha(T_0 - t_1)}{1 + r_0^\alpha\gamma\alpha(T_0 - t_2)}\right)\end{aligned} \tag{7.25}$$

In the next section we will frequently use this in the form

$$\int_{t_1}^{t_2} \langle v, \mathcal{L}_v\mathcal{X}\rangle\,dt \geq \int_{t_1}^{t_2} \gamma\|x\|^\alpha\,dt \geq -Q_4 + \frac{1}{\alpha}\log\left(\frac{T - t_1}{T - t_2}\right), \tag{7.26}$$

where $T = T_0 + 1$ and $0 \leq t_1 < t_2 \leq T - 1$, and Q_4 is a constant independent of the trajectory. For the time being, we establish a stronger expansion bound that can be used when $x(t_2)$ has escaped Y.

Let s be large enough such that $\chi = \chi(s)$ from (7.10) in Proposition 7.3 is less than 1. Given $x \in Y$ whose trajectory escapes Y at time T_0, we put $t_0 = \max(0, T_s)$ and observe that by Corollary 7.7 and Proposition 7.3, we have

$$\int_{t_0}^{T_0} \|x\|^\alpha \tan\rho\,dt \leq Q_0, \qquad \int_{t_0}^{T_0} \|x\|^\alpha \tan\theta\,dt \leq \frac{s}{\lambda}. \tag{7.27}$$

Moreover, Proposition 7.3 gives

$$T_0 - t_0 \geq (1 - \chi)T_0. \tag{7.28}$$

Now we can use (7.24) to write

$$\begin{aligned}\log\left(\frac{\|D\varphi_{T_0}(x)(v)\|}{\|v\|}\right) &\geq -Q_3 + \gamma(1 + \alpha)\int_{t_0}^{T_0} \|x\|^\alpha\,d\tau \\ &\quad - \gamma\int_{t_0}^{T_0} \|x\|^\alpha(\sin^2\theta + \sin^2\rho + \sin^2\theta\sin^2\rho)\,d\tau\end{aligned} \tag{7.29}$$

for every $v \in K^u(x)$. By (7.16) and the fact that $t \geq T_s$, the integrand on the second line is bounded above by $\|x\|^\alpha(\tan\theta + 2\tan\rho)$. Thus by (7.27) there is a constant Q_5 such that

$$\log\left(\frac{\|D\varphi_{T_0}(x)(v)\|}{\|v\|}\right) \geq -Q_5 + \gamma(1 + \alpha)\int_{t_0}^{T_0} \|x\|^\alpha\,d\tau. \tag{7.30}$$

Using (7.25) gives

$$\log\left(\frac{\|D\varphi_{T_0}(x)(v)\|}{\|v\|}\right) \geq -Q_5 + \left(1 + \frac{1}{\alpha}\right)\log(1 + r_0^\alpha\gamma\alpha(T_0 - t_0)) \tag{7.31}$$

Together with (7.28) and the fact that $1 - \chi > 0$, this shows that $m_W(W(t))$ is bounded above by a constant multiple of $t^{-(1+\frac{1}{\alpha})}$, establishing **(C2)**(i).

7.4. **Bounded distortion near the fixed point.** To show **(C2)**(ii) and (iii), we need to study two nearby trajectories on the same admissible manifold. Fix $W \in \tilde{\mathcal{A}}$ such that $g(W) \cap Y \neq 0$, and let $x, y \in W_j$ for some j, where we fix an admissible decomposition as before. Let $T_0 = \tau_j$ and let $T = T_0 + 1$. Write $\bar{x} = \bar{G}(x)$ and $\bar{y} = \bar{G}(y)$, and note that by (7.30) we have

$$d(x(t), y(t)) \leq Q_6 (T-t)^{-(1+\frac{1}{\alpha})} d(\bar{x}, \bar{y}). \tag{7.32}$$

Let $v(0)$ and $w(0)$ be unit tangent vectors to W at $x(0)$ and $y(0)$, respectively. Set $v(t) = D\varphi_t(x(0))(v(0))$, and similarly for $w(t)$. We will write $\Delta v = v - w$, and similarly for other quantities. Recall that we write $\hat{v} = v/\|v\|$, and similarly for w, x, y, etc.

Let $\eta(t) = \|\Delta\hat{v}(t)\| = \|\hat{v}(t) - \hat{w}(t)\|$. Most of our work in this section will be to estimate this quantity.

Lemma 7.8. *Writing $z = \widehat{\Delta v} = \Delta v/\|\Delta v\|$, the quantity η satisfies*

$$\begin{aligned} \eta' &= -(\langle v, \mathcal{L}_v \mathcal{X}\rangle + \langle w, \mathcal{L}_w \mathcal{X}\rangle)\eta + \langle z, \Delta\mathcal{L}_v\mathcal{X}\rangle \\ &\leq -a(t)\eta(t) + c(t), \end{aligned} \tag{7.33}$$

where

$$\begin{aligned} a(t) &= \gamma\|x\|^\alpha - Q_7 \tan\rho, \\ c(t) &= Q_8 (T-t)^{-2} d(\bar{x}, \bar{y}). \end{aligned} \tag{7.34}$$

Proof. Note that

$$\eta^2 = \langle \hat{v} - \hat{w}, \hat{v} - \hat{w}\rangle = 2(1 - \langle \hat{v}, \hat{w}\rangle),$$

so writing $\zeta = 1 - \langle \hat{v}, \hat{w}\rangle$, we have

$$\eta\eta' = \zeta' = -\langle (\hat{v})', \hat{w}\rangle - \langle \hat{v}, (\hat{w})'\rangle. \tag{7.35}$$

Differentiating the unit tangent vector $\hat{v}$ gives

$$(\hat{v})' = \left(\frac{v}{\|v\|}\right)' = \frac{\|v\|v' - v\langle v', \hat{v}\rangle}{\|v\|^2} = \frac{v' - \hat{v}\langle v', \hat{v}\rangle}{\|v\|} = \mathcal{L}_{\hat{v}}\mathcal{X} - \langle \mathcal{L}_{\hat{v}}\mathcal{X}, \hat{v}\rangle\hat{v},$$

where the last equality uses Proposition 7.4 together with linearity of the Lie derivative in v. Thus (7.35) gives

$$\zeta' = -\big\langle \mathcal{L}_{\hat{v}}\mathcal{X} - \langle \mathcal{L}_{\hat{v}}\mathcal{X}, \hat{v}\rangle\hat{v}, \hat{w}\big\rangle - \big\langle \hat{v}, \mathcal{L}_{\hat{w}}\mathcal{X} - \langle \mathcal{L}_{\hat{w}}\mathcal{X}, \hat{w}\rangle\hat{w}\big\rangle.$$

Since the right-hand side involves no derivatives we may safely simplify the notation by considering a fixed time t and assuming that $v(t), w(t)$ are normalised so that $\hat{v} = v$ and $\hat{w} = w$. Then we have

$$\begin{aligned} \zeta' &= \langle \mathcal{L}_v\mathcal{X}, v\rangle\langle v, w\rangle - \langle \mathcal{L}_v\mathcal{X}, w\rangle + \langle \mathcal{L}_w\mathcal{X}, w\rangle\langle v, w\rangle - \langle v, \mathcal{L}_w\mathcal{X}\rangle \\ &= (\langle v, \mathcal{L}_v\mathcal{X}\rangle + \langle w, \mathcal{L}_w\mathcal{X}\rangle)(1-\zeta) - \langle w, \mathcal{L}_v\mathcal{X}\rangle - \langle v, \mathcal{L}_v\mathcal{X}\rangle \\ &= -(\langle v, \mathcal{L}_v\mathcal{X}\rangle + \langle w, \mathcal{L}_w\mathcal{X}\rangle)\zeta + \langle \Delta v, \Delta\mathcal{L}_v\mathcal{X}\rangle. \end{aligned} \tag{7.36}$$

Dividing by η yields the first half of (7.33). Now we observe that (7.14) gives

(7.37)
$$\begin{aligned}\Delta\mathcal{L}_v\mathcal{X} &= \Delta\Big(\|x\|^\alpha\big(\alpha\langle v,\hat{x}\rangle A\hat{x} + Av\big)\Big)\\ &= \Delta(\|x\|^\alpha)\big(\alpha\langle v,\hat{x}\rangle A\hat{x} + Av\big)\\ &\quad + \|y\|^\alpha\Big(\alpha\langle \Delta v,\hat{x}\rangle A\hat{x} + \alpha\langle w,\Delta\hat{x}\rangle A\hat{x} + \alpha\langle w,\hat{y}\rangle A(\Delta\hat{x}) + A\Delta v\Big).\end{aligned}$$

We start by bounding the first and the last terms in the last line. Let $\rho(t)$ be as in the previous section, so that v, w are both within ρ of E^u. It follows that $\measuredangle(\Delta v, E^s) \le \rho$. Write $z = \widehat{\Delta v} = z_u + z_s$, where $z_{u,s} \in E^{u,s}$. Decomposing $\hat{x}$ as $\hat{x} = x_u + x_s$ and using $A = -\beta\,\mathrm{Id}_u + \gamma\,\mathrm{Id}_s$, we have

$$\begin{aligned}\langle z,\hat{x}\rangle\langle\Delta v, A\hat{x}\rangle &= \eta(z_u x_u + \langle z_s, x_s\rangle)(\gamma z_u x_u - \beta\langle z_s, x_s\rangle)\\ &\le Q_9\eta|z_u| \le Q_9\eta\tan\rho,\end{aligned}$$

and similarly,

$$\langle z, A\Delta v\rangle = \eta(\gamma z_u^2 - \beta\|z_s\|^2) \le \gamma\eta\tan\rho,$$

so that using the first half of (7.33) together with (7.37) and the estimate (7.23) on $\langle v, \mathcal{L}_v\mathcal{X}\rangle$, we have

$$\eta' \le a(t)\eta + b(t), \tag{7.38}$$

where

$$\begin{aligned}b(t) &= \Delta(\|x\|^\alpha)\big(\alpha\langle v,\hat{x}\rangle\langle z, A\hat{x}\rangle + \langle z, Av\rangle\big)\\ &\quad + \alpha\|y\|^\alpha\big(\langle w,\Delta\hat{x}\rangle\langle z, A\hat{x}\rangle + \langle w,\hat{y}\rangle\langle z, A(\Delta\hat{x})\rangle\big).\end{aligned} \tag{7.39}$$

Lemma 7.9. *There are constants Q_{10}, Q_8 such that*

$$\begin{aligned}\|\Delta\hat{x}\| &\le Q_{10}(T-t)^{-1}d(\bar{x},\bar{y}),\\ \Delta(\|x\|^\alpha) &\le Q_8(T-t)^{-2}d(\bar{x},\bar{y}).\end{aligned}$$

Proof. Using (7.6) gives

$$\|x(t)\| \ge r_0(1 + r_0^\alpha\gamma\alpha(T_0 - t))^{-\frac{1}{\alpha}}, \tag{7.40}$$

and (7.30) gives

$$\|\Delta x\| \le Q_{12}\big(1 + r_0^\alpha\gamma\alpha\chi(T_0-t)\big)^{-(1+\frac{1}{\alpha})}d(\bar{x},\bar{y}), \tag{7.41}$$

Thus

$$\begin{aligned}\|\Delta\hat{x}\| \le \frac{\|\Delta x\|}{\|x\|} &\le Q_{12}\big(1 + r_0^\alpha\gamma\alpha\chi(T_0-t)\big)^{-(1+\frac{1}{\alpha})}\big(1 + r_0^\alpha\gamma\alpha(T_0-t)\big)^{\frac{1}{\alpha}}d(\bar{x},\bar{y})\\ &\le Q_{10}\big(1 + (T_0-t)\big)^{-(1+\frac{1}{\alpha})}\big(1 + (T_0-t)\big)^{\frac{1}{\alpha}}d(\bar{x},\bar{y}),\end{aligned}$$

which proves the first half. Similarly,

$$\Delta(\|x\|^\alpha) \le \alpha\|\Delta x\|\min(\|x\|,\|y\|)^{\alpha-1}$$

establishes the second half and completes the proof of Lemma 7.9. ∎

We can now complete the proof of Lemma 7.8. It suffices to apply Lemma 7.9 to (7.39) and use this estimate in the first half of (7.33), which yields the estimate in the second half of (7.33) and (7.34). □

To estimate the value of $\eta(t)$ using Lemma 7.8, we let $I(t_1, t_2) = \int_{t_1}^{t_2} a(t)\,dt$, where $a(t)$ is as in (7.34). Then Lemma 7.8 gives

$$(e^{I(0,t)}\eta)' = e^{I(0,t)}(a(t)\eta + \eta') \leq e^{I(0,t)}c(t),$$

and integrating gives

$$e^{I(0,t)}\eta(t) \leq \eta(0) + \int_0^t c(s)e^{I(0,s)}\,ds.$$

Solving for $\eta(t)$, we have

$$\eta(t) \leq \eta(0)e^{-I(0,t)} + \int_0^t c(s)e^{-I(s,t)}\,ds. \tag{7.42}$$

Now we use (7.26) and Corollary 7.7 to get

$$\eta(t) \leq \eta(0)e^{-Q_4}\left(\frac{T-t}{T}\right)^{\frac{1}{\alpha}} + \int_0^t Q_8(T-s)^{-2}d(\bar{x},\bar{y})e^{-Q_4}\left(\frac{T-t}{T-s}\right)^{\frac{1}{\alpha}}\,ds$$

and note that the integral is bounded above by

$$Q_{13}(T-t)^{\frac{1}{\alpha}}d(\bar{x},\bar{y})(T-s)^{-(1+\frac{1}{\alpha})}\Big|_0^t \leq Q_{13}(T-t)^{-1}d(\bar{x},\bar{y})$$

Thus we have

$$\eta(t) \leq Q_{13}\left(\eta(0)\left(1-\frac{t}{T}\right)^{\frac{1}{\alpha}} + (T-t)^{-1}d(\bar{x},\bar{y})\right). \tag{7.43}$$

Now since $x, y \in W$ for some $W \in \mathcal{A}$, the Hölder property of TW guarantees that

$$\eta(0) = \|\Delta v\| \leq Ld(x,y)^\alpha \leq Le^{\alpha Q_3}d(\bar{x},\bar{y})^\alpha,$$

and we conclude that

$$\|\Delta v(t)\| \leq Q_{14}\left(\left(1-\frac{t}{T}\right)^{\frac{1}{\alpha}} + (T-t)^{-1}\right)d(\bar{x},\bar{y})^\alpha. \tag{7.44}$$

In particular, by choosing r_1/r_0 sufficiently large, we guarantee that $\bar{G}(W_j)$ has Hölder curvature bounded by L, which establishes **(C2)**(ii)

It only remains to get the bounded distortion estimates. For this we observe that

$$\begin{aligned}
|\Delta\langle v, \mathcal{L}_v\mathcal{X}\rangle| &\leq |\langle \Delta v, \mathcal{L}_v\mathcal{X}\rangle| + |\langle v, \Delta\mathcal{L}_v\mathcal{X}\rangle| \\
&\leq Q_{15}\|\Delta v\|\|x\|^\alpha + \Delta(\|x\|^\alpha)\big(\alpha\langle v,\hat{x}\rangle\langle v, A\hat{x}\rangle + \langle v, Av\rangle\big) \\
&\qquad + \alpha\|y\|^\alpha\big(\langle w,\Delta\hat{x}\rangle\langle v, A\hat{x}\rangle + \langle w,\hat{y}\rangle\langle v, A(\Delta\hat{x})\rangle\big) \\
&\leq Q_{15}\|\Delta v\|\|x\|^\alpha + Q_{16}\big(\Delta(\|x\|^\alpha) + \|y\|^\alpha\|\Delta\hat{x}\|\big),
\end{aligned}$$

where the second inequality uses (7.37). Together with (7.6), (7.44), and Lemma 7.9, this gives

$$\frac{|\Delta\langle v, \mathcal{L}_v \mathcal{X}\rangle|}{d(\bar{x},\bar{y})^\alpha} \le Q_{17}\left(\frac{(T-t)^{\frac{1}{\alpha}-1}}{T^{\frac{1}{\alpha}}} + (T-t)^{-2}\right).$$

Integrating from 0 to $T-1$ gives

(7.45)

$$\frac{1}{d(\bar{x},\bar{y})^\alpha}\left|\Delta\int_0^{T-1}\langle v, \mathcal{L}_v\mathcal{X}\rangle\,dt\right| \le Q_{18}\Big[-T^{-\frac{1}{\alpha}}(T-t)^{\frac{1}{\alpha}} + (T-t)^{-1}\Big]_0^{T-1}$$
$$= Q_{18}(1 - T^{-\frac{1}{\alpha}} + 1 - T^{-1}) \le 2Q_{18}.$$

Together with (7.20), this yields the desired bounded distortion estimate.

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