q-ENTROPY FOR SYMBOLIC DYNAMICAL SYSTEMS

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ABSTRACT. For symbolic dynamical systems we use the Carathéodory construction as described in [4] to introduce the notions of q-topological and q-metric entropies. We describe some basic properties of these entropies and in particular, discuss relations between q-metric entropy and local metric entropy. Both q-topological and q-metric entropies are new invariants respectively under homeomorphisms and metric isomorphisms of dynamical systems.

1. INTRODUCTION.

Since the introduction by Kolmogorov and Sinai, entropy has become one of the most important invariants of dynamics. It comes in two incarnations: as metric entropy and as topological entropy depending on whether ergodic properties of invariant measure or topological properties of the system on invariant sets are to be studied. The metric entropy measures the maximal loss of information of the iteration of finite partitions in a measurable dynamical system and the topological entropy characterizes the exponential growth rate of the number of periodic points.

For a dynamical system f acting on a measure space X with an invariant measure μ, given a finite partition ξ of X, the classical Shannon-McMillan-Breiman theorem claims that for almost every $x \in X$ the limit

$$\lim_{n\to\infty} -\frac{\log \mu(C_n(x))}{n} = h_\mu(f, x, \xi) \tag{1.1}$$

exists. Here $C_n(x)$ is the element of the n-th shifted partition $\xi_n = \xi \vee f^{-1}\xi \vee \ldots \vee f^{-(n-1)}\xi$ that contains x and $h_\mu(f, x, \xi)$ is the *local entropy* of f with respect to the partition ξ at the point x. If μ is ergodic then $h_\mu(f, x, \xi)$ is constant almost everywhere and the common value is the entropy $h_\mu(f, \xi)$

Date: August 5, 2015.

This work is partially supported by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD). The first author is partially supported by CSC and NSFC 11371271. The second author is partially supported by NSF grants 1101165 and 1400027.

of f with respect to the partition ξ. The latter coincides with the entropy $h_\mu(f)$ of f provided the partition ξ is generating.

Utilizing the Shannon-McMillan-Breiman theorem, one can consider the *multifractal decomposition* of the set X associated to local entropies

$$X = \tilde{X} \bigcup \Big(\bigcup_{\alpha \geq 0} K_\alpha \Big), \tag{1.2}$$

where $K_\alpha = \{x \in X \colon h_\mu(f, x, \xi) = \alpha\}$ and $\tilde{X}$ is the set of points for which the limit in (1.1) does not exists – the so-called *irregular part* of the multifractal decomposition. [1]The functions $g_1(\alpha) = \dim_H K_\alpha$ and $g_2(\alpha) = h(f, K_\alpha)$ are called, respectively, the *dimension* and *entropy* spectra associated to the multifractal decomposition (1.2). Here $\dim_H Y$ is the Hausdorff dimension of the set Y and $h(f, Y)$ is the topological entropy of f on the set Y.[2] These two spectra are examples of *multifractal spectra*, whose general concept was introduced in [1]. Many multifractal spectra can be obtained by utilizing a generalized Carathéodory construction described in [4], which provides a unified approach for producing various dimension-like characteristics for dynamical systems.

Another example of a multifractal spectrum that can be obtained in this way is the well-known *Hentschel-Procaccia spectrum for dimensions*, which has numerous applications in science. It is a one-parameter family of dimension-like characteristics associated with a Borel measure in a given metric space and, a priorí, it does not require any dynamics to be present. [3] To obtain this spectrum one starts with a reference measure μ, which gives positive weight to any non-empty open set. One can then use it to build a particular weight function in the Carathéodory construction to generate the *q-dimension* of sets and measures (see [4, Section 8] for detailed exposition).

In this paper, using the Carathéodory construction approach, we introduce, in the setting of symbolic dynamical systems[4], the notion of *q-entropy* in its both topological and metric incarnations, where $q \geq 0$ is a parameter. This actually generates what we call the *Hentschel-Procaccia entropy*

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[1]While the irregular part has zero measure with respect to *any* invariant measure (and hence, is completely negligible from the point of view of measure theory), its Hausdorff dimension can be positive and indeed, as big as the Hausdorff dimension of the whole space.

[2]Note that the set Y does not have to be invariant nor compact and that the topological entropy of f on Y should be treated in the sense of Bowen, see [4].

[3]Of course, the most interesting applications of the Hentschel-Procaccia spectrum for dimensions appear when the measure is invariant with respect to some dynamical system with rich stochastic properties.

[4]We emphasize that in this setting the systems always possesses a generating partition.

spectrum. We show that this spectrum is closely related to the q-topological entropy on the support of the measure μ. Furthermore, we define the *modified Hentschel-Procaccia entropy spectrum*, which is related to the q-metric entropy and local metric entropy of the measure μ.

We stress that both q-topological and q-metric entropies are new invariants of dynamics. The q-topological entropy is an invariant under a class of homeomorphisms that *respect* measure μ that is they satisfy some special requirement with respect to the measure μ. The q-metric entropy is invariant under metric isomorphisms.

Using Markov partitions one can extend the notions of q-topological and q-metric entropies as well as of the Hentschel-Procaccia entropy spectrum to uniformly hyperbolic systems (that is axiom A diffeomorphisms) and in particular, to Anosov maps. Note that the q-topological and q-metric entropies defined in this way may depend on the choice of Markov partitions. One can show that these quantities converge to limits as the diameter of the Markov partitions go to zero, thus producing q-topological entropies for the corresponding smooth system.

2. q-TOPOLOGICAL ENTROPY

Throughout this paper, let Σ_p^+ be the space of one-sided sequences on p symbols and $\sigma(\omega_1\omega_2\omega_3\dots) = (\omega_2\omega_3\dots)$ the shift map on Σ_p^+. We assume that Σ_p^+ is equipped with the distance

$$d(\omega, \omega') = 2^{-k},$$

where $\omega = (\omega_1\omega_2\dots)$ and $\omega' = (\omega_1'\omega_2'\dots)$ and $k = \min\{n : \omega_n \neq \omega_n'\}$.

Denote by $\mathcal{M}_\sigma$ and $\mathcal{E}_\sigma$ the set of all σ-invariant respectively, ergodic σ-invariant Borel probability measures on Σ_p^+. Let μ be a Borel probability measure on Σ_p^+ such that $\mu(U) > 0$ for any non-empty open subset $U \subset \Sigma_p^+$.

2.1. **Definition of q-topological entropy.**

Given $n \in \mathbb{N}$ and $\omega \in \Sigma_p^+$, denote by $C_n(\omega) = \{\omega' \in \Sigma_p^+ : \omega_i = \omega_i', \, 1 \le i \le n\}$ the cylinder of length n that contains ω.

Let $Z \subset \Sigma_p^+$ be a subset of Σ_p^+, which does not have to be compact or σ-invariant. Given a collection of cylinders $\Gamma = \{C_{n_i}(\omega^i) \colon \omega^i \in \Sigma_p^+, n_i \in \mathbb{N}\}$, we say that Γ covers Z if $Z \subset \bigcup_i C_{n_i}(\omega^i)$, and we set $n(\Gamma) = \min_i\{n_i\}$ to be the smallest length of the cylinders in Γ.

For each subset $Z \subset \Sigma_p^+$, each $\alpha \in \mathbb{R}$, $N > 0$ and each $q \ge 0$, set

$$M_q(Z, \alpha, N) = \inf\Big\{\sum_i \mu(C_{n_i}(\omega^i))^q \exp\big(-\alpha n_i\big)\Big\},$$

where the infimum is taken over all countable covers $\Gamma = \{C_{n_i}(\omega^i)\}$ of Z with $n(\Gamma) \geq N$. Since $M_q(Z, \alpha, N)$ is monotonically increasing with N, there is the limit

$$m_q(Z, \alpha) := \lim_{N \to \infty} M_q(Z, \alpha, N).$$

It is easy to see that $m_q(Z, \alpha)$ as a function of α (for a fix set Z) has a critical "jump-up" point so that

$$h_q(\sigma, Z) := \inf\{\alpha : m_q(Z, \alpha) = 0\} = \sup\{\alpha : m_q(Z, \alpha) = \infty\}.$$

Definition 2.1. *The quantity $h_q(\sigma, Z)$ is called the* q-topological entropy *of σ on the set Z.*

Given $\alpha \in \mathbb{R}$, $N > 0$, $q \geq 0$ and $Z \subset \Sigma_p^+$, define

$$R_q(Z, \alpha, N) = \inf \Big\{ \sum_i \mu(C_N(\omega^i))^q \exp\big(- \alpha N\big) \Big\}, \tag{2.3}$$

where the infimum is taken over all covers $\Gamma = \{C_N(\omega^i)\}$ of Z. We set

$$\underline{r}_q(Z, \alpha) = \liminf_{N \to \infty} R_q(Z, \alpha, N),$$
$$\overline{r}_q(Z, \alpha) = \limsup_{N \to \infty} R_q(Z, \alpha, N)$$

and define the "jump-up" points of $\underline{r}_q(Z, \alpha)$ and $\overline{r}_q(Z, \alpha)$ as

$$\underline{h}_q(\sigma, Z) = \inf\{\alpha : \underline{r}_q(Z, \alpha) = 0\} = \sup\{\alpha : \underline{r}_q(Z, \alpha) = +\infty\},$$
$$\overline{h}_q(\sigma, Z) = \inf\{\alpha : \overline{r}_q(Z, \alpha) = 0\} = \sup\{\alpha : \overline{r}_q(Z, \alpha) = +\infty\}$$

respectively.

Definition 2.2. *The quantities $\underline{h}_q(\sigma, Z)$ and $\overline{h}_q(\sigma, Z)$ are called the* lower *and* upper q-topological entropies *of σ on the set Z.*

The above definitions of the q-topological entropy and the lower and upper q-topological entropies follow the generalized Carathéodory construction described in [4]. According to this construction the space Σ_p^+ is endowed with a special Carathéodory structure given as follows: 1) the collection $\mathcal{F}$ of *admissible* sets (that are used to cover subsets in Σ_p^+) consists of all cylinders $C_n(\omega)$; 2) given a number $q \geq 0$, the functions $\xi, \eta, \psi : \mathcal{F} \to \mathbb{R}^+$ are defined by

$$\xi(C_n(\omega)) = \mu(C_n(\omega))^q, \;\; \eta(C_n(\omega)) = e^{-n}, \;\; \psi(C_n(\omega)) = n^{-1}.$$

Here ψ measures the *size* of the cylinder, η is the *potential function* and ξ the *weight function*. Then for every subset $Z \subset \Sigma_p^+$ the quantities $h_q(\sigma, Z)$, $\underline{h}_q(\sigma, Z)$ and $\overline{h}_q(\sigma, Z)$ are respectively the Carathéodory dimension and lower and upper Carathéodory capacities of the set Z (see [4] for details).

Remark 2.1. *We stress that the quantities $h_q(\sigma, Z)$, $\underline{h}_q(\sigma, Z)$ and $\overline{h}_q(\sigma, Z)$ depend on the reference measure μ which is used as a weight function in the Carathéodory construction described in* [4]*. For $q = 0$, the weight function is trivial and these quantities are respectively the standard* topological entropy *and* lower and upper topological entropies *of σ on the set Z (see* [2] *or* [4] *for details). This is why we continue to call them "q-topological entropy". However, to avoid any confusions in the definition of metric entropies, we do not emphasize the dependence on μ in the notations.*

2.2. **Properties of q-topological entropy.** We describe some properties of the q-topological entropy and the lower and upper q-topological entropies. They follow immediately from the definitions and Theorems 1.1, 2.1 and 2.4 in [4].

Proposition 2.1. *For any $q \geq 0$, the following statements hold:*

(1) *if $Z_1 \subset Z_2$, then $\mathcal{H}_q(\sigma, Z_1) \leq \mathcal{H}_q(\sigma, Z_2)$, where $\mathcal{H}_q$ denotes either h_q or $\underline{h}_q$ or $\overline{h}_q$;*

(2) *if $Z_i \subset \Sigma_p^+$, $i \geq 1$ and $Z = \bigcup_{i \geq 1} Z_i$, then $h_q(\sigma, Z) = \sup_{i \geq 1} h_q(\sigma, Z_i)$, $\underline{h}_q(\sigma, Z) \geq \sup_{i \geq 1} \underline{h}_q(\sigma, Z_i)$ and $\overline{h}_q(\sigma, Z) \geq \sup_{i \geq 1} \overline{h}_q(\sigma, Z_i)$;*

(3) *$h_q(\sigma, Z) \leq \underline{h}_q(\sigma, Z) \leq \overline{h}_q(\sigma, Z)$ for any $Z \subset \Sigma_p^+$.*

For any subset $Z \subset \Sigma_p^+$ and any $q \geq 0$, set

$$\Lambda_q(Z, n) = \inf \Big\{ \sum_i \mu(C_n(\omega^i))^q \Big\}, \tag{2.4}$$

where the infimum is taken over all covers $\Gamma = \{C_n(\omega^i)\}$ of Z. The following equivalent description of the lower and upper q-topological entropy follows immediately from definitions and Theorem 2.2 in [4].

Proposition 2.2. *For each subset $Z \subset \Sigma_p^+$ and each $q \geq 0$ we have*

$$\underline{h}_q(\sigma, Z) = \liminf_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n),$$
$$\overline{h}_q(\sigma, Z) = \limsup_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n).$$

In the following we shall compute the q-topological entropy of σ on the space Σ_2^+ of one-sided sequences on 2 symbols.

Example 2.1. *Consider a Bernoulli measure λ on the space $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$ given by the probability vector (κ_0, κ_1) where $\kappa_0, \kappa_1 > 0$ and $\kappa_0 + \kappa_1 = 1$.*

Clearly, for each $q \geq 0$

$$\Lambda_q(\Sigma_2^+, n) = \sum_{\Gamma} \lambda(C_n(\omega^i))^q = \sum_{j=0}^{n} C_n^j (\kappa_0^j \kappa_1^{n-j})^q = (\kappa_0^q + \kappa_1^q)^n,$$

where $\Gamma = \{C_n(\omega^i)\}$ *is a disjoint cover of* Σ_2^+*. By Proposition 2.2, we obtain that*

$$\underline{h}_q(\sigma, \Sigma_2^+) = \overline{h}_q(\sigma, \Sigma_2^+) = \log(\kappa_0^q + \kappa_1^q).$$

In particular, if $\kappa_0 = \kappa_1 = \frac{1}{2}$ *then*

$$\underline{h}_q(\sigma, \Sigma_2^+) = \overline{h}_q(\sigma, \Sigma_2^+) = \log 2 - q \log 2.$$

Note that $\log 2$ *is the standard topological entropy of the system* (σ, Σ_2^+).

We shall now show that the q-topological entropy as well as lower and upper q-topological entropies are invariant under a certain class of homeomorphisms of Σ_p^+ associated with the measure μ. To this end consider a continuous map $\pi : \Sigma_p^+ \to \Sigma_p^+$. It is easy to see that there exists an integer $k_0 \geq 0$ such that for every $\omega, \omega' \in \Sigma_p^+$,

$$d(\omega, \omega') < 2^{-k_0} \Rightarrow d(\pi(\omega), \pi(\omega')) < \frac{1}{2}.$$

In the case when π commutes with the shift, i.e., $\pi \circ \sigma = \sigma \circ \pi$, this implies that for any $n > 0$ and any $\omega \in \Sigma_p^+$ we have

$$\pi(C_{n+k_0}(\omega)) \subset C_n(\pi(\omega)).$$

If in addition, π is a homeomorphism we without loss of generality may assume that the number k_0 is chosen such that $\pi^{-1}(C_{n+k_0}(\omega)) \subset C_n(\pi^{-1}(\omega))$.

Proposition 2.3. *Let* $\pi : \Sigma_p^+ \to \Sigma_p^+$ *be a homeomorphism such that* $\pi \circ \sigma = \sigma \circ \pi$*. Assume that there is* $K > 0$ *and* $N > 0$ *such that for any* $n \geq N$ *and any* $\omega \in \Sigma_p$,

$$\begin{aligned} &\frac{1}{K}\mu\big(C_{n+k_0}(\omega)\big) \leq \mu\big(C_n(\pi(\omega))\big) \leq K\mu\big(C_{n+k_0}(\omega)\big), \\ &\frac{1}{K}\mu\big(C_{n+k_0}(\omega)\big) \leq \mu\big(C_n(\pi^{-1}(\omega))\big) \leq K\mu\big(C_{n+k_0}(\omega)\big). \end{aligned} \tag{2.5}$$

Then for every $Z \subset \Sigma_p^+$,

$$\mathcal{H}_q(\sigma, Z) = \mathcal{H}_q(\sigma, \pi(Z)),$$

where as before $\mathcal{H}_q$ *denotes either* h_q *or* $\underline{h}_q$ *or* $\overline{h}_q$.

Proof. Given a cover $\Gamma = \{C_n(\omega^i)\}$ of Z with $n \geq N$, the collection of cylinders $\Gamma' = \{C_n(\pi(\omega^i))\}$ covers $\pi(Z)$. By (2.5), we have

$$\sum_{\Gamma} \mu(C_{n+k_0}(\omega^i))^q \geq \frac{1}{K^q} \sum_{\Gamma'} \mu\big(C_n(\pi(\omega^i))\big)^q$$
$$\geq \frac{1}{K^q} \Lambda_q(\pi(Z), n).$$

By Proposition 2.2, one has that

$$\underline{h}_q(\sigma, Z) \geq \underline{h}_q(\sigma, \pi(Z)) \quad \text{and} \quad \overline{h}_q(\sigma, Z) \geq \overline{h}_q(\sigma, \pi(Z)).$$

Furthermore, note that if the collection $\Gamma = \{C_{n_i+k_0}(\omega^i)\}$ covers Z with $n_i \geq N$ for all i, then the collection $\Gamma' = \{C_{n_i}(\pi(\omega^i))\}$ covers $\pi(Z)$ with $n(\Gamma') \geq N$. It follows that

$$M_q(Z, \alpha, N + k_0) \geq e^{-\alpha k_0} M_q(\pi(Z), \alpha, N)$$

yielding that

$$h_q(\sigma, Z) \geq h_q(\sigma, \pi(Z)).$$

Finally, since π is a homeomorphism, applying the above arguments to π^{-1}, we obtain the desired result. □

Example 2.2. *Let λ be the Bernoulli measure on Σ_p^+ given by the probability vector $(\kappa_0, \cdots, \kappa_{p-1})$ with $\kappa_i > 0$ for each $i = 1, \ldots, p-1$ and $\kappa_0 + \cdots + \kappa_{p-1} = 1$. Fix an integer $N > 0$ and consider a homeomorphism $\pi : \Sigma_p^+ \to \Sigma_p^+$ defined as follows: $\pi(\omega) = \omega'$ where $\omega = (\omega_i)$ and $\omega' = (\omega'_i)$ are such that $\omega'_i = \omega_i$ for all $i = N+1, N+2, \ldots$ and $(\omega'_1, \ldots, \omega'_N)$ is a given permutation of $(\omega_1, \ldots, \omega_N)$. Although the homeomorphism π does not satisfy $\pi \circ \sigma = \sigma \circ \pi$, it is easy to see that $\pi(C_n(\omega)) = C_n(\pi(\omega))$ for any ω and any $n \geq N$. This means that in our case $k_0 = 1$. Furthermore, for any $n \geq N$ it is easy to check that*

$$\min_{1 \leq i \neq j < p} \Big\{ \Big(\frac{\kappa_i}{\kappa_j}\Big)^N \Big\} \leq \frac{\lambda(C_n(\omega))}{\lambda(C_n(\pi(\omega)))} \leq \max_{1 \leq i \neq j < p} \Big\{ \Big(\frac{\kappa_i}{\kappa_j}\Big)^N \Big\}.$$

By the same arguments as in the proof of Proposition 2.3, for any $Z \subset \Sigma_p^+$ we have $\mathcal{H}_q(\sigma, Z) = \mathcal{H}_q(\sigma, \pi(Z))$.

We now obtain a formula that allows one to compute the lower and upper q-topological entropies for any $q \geq 1$. Note that the function $\omega \mapsto \mu(C_n(\omega))^{q-1}$ is measurable. Since it is bounded, it is integrable. For any measurable set $Z \subset \Sigma_p^+$ and $q \geq 1$, set

$$\varphi_q(Z, n) = \int_Z \mu(C_n(\omega))^{q-1} \, \mathrm{d}\mu(\omega).$$

Theorem 2.4. *The following statements hold:*

(1) For any $q \geq 1$ and any measurable set $Z \subset \Sigma_p^+$,

$$\underline{h}_q(\sigma, Z) \geq \liminf_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n), \quad \overline{h}_q(\sigma, Z) \geq \limsup_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n),$$

and

$$\underline{h}_q(\sigma, Z) \leq \liminf_{n\to\infty} \frac{1}{n} \log \varphi_q((Z)_n, n), \quad \overline{h}_q(\sigma, Z) \leq \limsup_{n\to\infty} \frac{1}{n} \log \varphi_q((Z)_n, n),$$

where $(Z)_n = \bigcup_{\omega\in Z} C_n(\omega)$;

(2) For any set Z of full measure and any $q \geq 1$,

$$\underline{h}_q(\sigma, Z) = \liminf_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n), \quad \overline{h}_q(\sigma, Z) = \limsup_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n).$$

Proof. For any $\beta > 0$ and $n \in \mathbb{N}$ there is a cover $\Gamma = \{C_n(\omega^i)\}_{i\geq 1}$ of Z such that

$$\sum_i \mu(C_n(\omega^i))^q \leq \Lambda_q(Z, n) + \beta.$$

Using the fact that $C_n(\omega) = C_n(\omega^i)$ for each $\omega \in C_n(\omega^i)$, we have that

$$\begin{aligned}\sum_i \mu(C_n(\omega^i))^q &= \sum_i \int_{C_n(\omega^i)} \mu(C_n(\omega^i))^{q-1}\, \mathrm{d}\mu(\omega) \\ &= \sum_i \int_{C_n(\omega^i)} \mu(C_n(\omega))^{q-1}\, \mathrm{d}\mu(\omega) \\ &\geq \varphi_q(Z, n).\end{aligned}$$

Since β can be chosen arbitrarily small it follows that

$$\Lambda_q(Z, n) \geq \varphi_q(Z, n).$$

By Proposition 2.2, this implies that

$$\underline{h}_q(\sigma, Z) \geq \liminf_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n), \text{ and } \overline{h}_q(\sigma, Z) \geq \limsup_{n\to\infty} \frac{1}{n} \log \varphi_q(Z, n).$$

To prove the reverse inequality given $n \in \mathbb{N}$, choose a cover $\Gamma = \{C_n(\omega^i)\}_i$ of Z whose elements are pairwise disjoint. Since $C_n(\omega) = C_n(\omega^i)$ for each

$\omega \in C_n(\omega^i)$ we have

$$\begin{aligned}\Lambda_q(Z,n) &\le \sum_i \mu(C_n(\omega^i))^q \\ &= \sum_i \int_{C_n(\omega^i)} \mu(C_n(\omega^i))^{q-1}\,\mathrm{d}\mu(\omega) \\ &= \sum_i \int_{C_n(\omega^i)} \mu(C_n(\omega))^{q-1}\,\mathrm{d}\mu(\omega) \\ &\le \int_{(Z)_n} \mu(C_n(\omega))^{q-1}\,\mathrm{d}\mu(\omega) \\ &= \varphi_q((Z)_n, n).\end{aligned}$$

This yields the desired results. The second statement now is a direct consequence of the first one. □

If one considers q-topological entropy and the lower and upper q-topological entropies as functions over $q \ge 0$, then we have the following proposition.

Proposition 2.5. *The following statements hold:*

(1) $h_{q_1}(\sigma,Z) \ge h_{q_2}(\sigma,Z)$, $\underline{h}_{q_1}(\sigma,Z) \ge \underline{h}_{q_2}(\sigma,Z)$, *and* $\overline{h}_{q_1}(\sigma,Z) \ge \overline{h}_{q_2}(\sigma,Z)$ *for any* $Z \subset \Sigma_p^+$ *and any* $0 \le q_1 \le q_2$;

(2) *if* $\mu(Z) = 0$ *then* $h_1(\sigma,Z) \le \underline{h}_1(\sigma,Z) \le \overline{h}_1(\sigma,Z) \le 0$ *and hence,* $h_q(\sigma,Z) \le \underline{h}_q(\sigma,Z) \le \overline{h}_q(\sigma,Z) \le 0$ *for any* $q \ge 1$;

(3) *if* $\mu(Z) > 0$ *then* $h_1(\sigma,Z) = \underline{h}_1(\sigma,Z) = \overline{h}_1(\sigma,Z) = 0$ *and hence,*

$$0 \le h_q(\sigma,Z) \le \underline{h}_q(\sigma,Z) \le \overline{h}_q(\sigma,Z) \quad \text{if } 0 \le q \le 1$$

and

$$h_q(\sigma,Z) \le \underline{h}_q(\sigma,Z) \le \overline{h}_q(\sigma,Z) \le 0 \quad \text{if } q \ge 1.$$

Proof. The first statement is obvious and the second one follows from Proposition 2.1 and Theorem 2.4. By (2.4), for $q = 1$ and every $n \in \mathbb{N}$ we obtain that

$$\mu(Z) \le \Lambda_1(Z,n) \le 1.$$

By Propositions 2.1 and 2.2, this yields that $h_1(\sigma,Z) \le \underline{h}_1(\sigma,Z) = \overline{h}_1(\sigma,Z) = 0$. On the other hand, for any $\alpha < 0$ by a direct computation we have

$$m_1(Z,\alpha) = +\infty.$$

Since α can be chosen arbitrarily, this implies that $h_1(\sigma,Z) \ge 0$. Hence,

$$h_1(\sigma,Z) = 0.$$

The last statement follows from Proposition 2.1 and the first statement. □

2.3. **The Hentschel-Procaccia entropy spectrum.** We follow the approach of defining the Hentschel-Procaccia dimension spectrum in [4] and introduce the notion of the *Hentschel-Procaccia entropy spectrum.*

Given the measure μ and $q > 1$, define

$$\underline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1} \liminf_{n\to\infty} \frac{1}{n} \log \int_X \mu(C_n(\omega))^{q-1}\, \mathrm{d}\mu(\omega),$$
$$\overline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1} \limsup_{n\to\infty} \frac{1}{n} \log \int_X \mu(C_n(\omega))^{q-1}\, \mathrm{d}\mu(\omega),$$

where X is the support of μ.

Definition 2.3. *We call the one-parameter family of pairs of quantities* $(\underline{\mathcal{HP}}_q(\mu), \overline{\mathcal{HP}}_q(\mu))$ *the* HP-spectrum for entropies*.*

It follows from Theorem 2.4 that

$$\underline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1}\underline{h}_q(\sigma, X), \quad \overline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1}\overline{h}_q(\sigma, X) \tag{2.6}$$

for any $q > 1$. Therefore, by Proposition 2.2, we can rewrite the definition of HP-spectrum for entropies in the following way:

$$\underline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1} \liminf_{n\to\infty} \frac{1}{n} \log \Lambda_q(X, n),$$
$$\overline{\mathcal{HP}}_q(\mu) = \frac{1}{q-1} \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_q(X, n).$$

It is a direct consequence of Proposition 2.5 and (2.6) that $(q-1)\underline{\mathcal{HP}}_q(\mu)$ and $(q-1)\overline{\mathcal{HP}}_q(\mu)$ are non-increasing functions over $q > 1$.

2.4. **q-Topological entropy with Gibbsian reference measures.** We consider the particular case of the reference measure μ to be a Gibbs measure, and we obtain a formula that connects q-topological entropy and the classical topological pressure. Consequently, we obtain some relations between q-topological and lower and upper q-topological entropies.

Now we recall the definition of Gibbs measures. Given a continuous functions $\varphi : \Sigma_p^+ \to \mathbb{R}$, we say that a Borel probability measure μ_φ (not necessary invariant) is a *Gibbs measure* with respect to φ on Σ_p^+, if there exists a constant $K > 0$ such that for every $n \geq 1$ and every $\omega \in \Sigma_p^+$

$$K^{-1} \leq \frac{\mu_\varphi(C_n(\omega))}{e^{-nP_{\mathrm{top}}(\varphi)+S_n\varphi(\omega)}} \leq K \tag{2.7}$$

where $S_n\varphi := \sum_{i=0}^{n-1} \varphi \circ \sigma^i$ and $P_{\mathrm{top}}(\varphi)$ is the classical topological pressure of φ (see [6] for the detailed definition and properties).

Consider the reference measure μ given by its value on cylinders $C_n(\omega)$ by the formula $\mu(C_n(\omega)) = e^{S_n\varphi(\omega)}$. One can show that $h_1(\sigma, Z)$, $\underline{h}_1(\sigma, Z)$

and $\overline{h}_1(\sigma, Z)$ are respectively the *topological pressure* and *lower and upper topological pressure* of σ on the set Z (see [4] for details). We denote them by $P_Z(\varphi)$, $\underline{CP}_Z(\varphi)$ and $\overline{CP}_Z(\varphi)$ respectively.

Theorem 2.6. *Assume that μ_φ is a Gibbs measure with respect to a continuous function $\varphi : \Sigma_p^+ \to \mathbb{R}$, then for any subset $Z \subset \Sigma_p^+$ and any $q \geq 0$ we have*

$$\mathcal{H}_q(f, Z) = \mathcal{P}_Z(q\varphi) - qP_{\text{top}}(\varphi)$$

where $\mathcal{H}_q$ denotes either h_q or $\underline{h}_q$ or $\overline{h}_q$, and $\mathcal{P}_Z$ denotes either P_Z or $\underline{CP}_Z$ or $\overline{CP}_Z$ respectively.

Proof. Fix $N > 0$ and consider a cover $\Gamma = \{C_{n_i}(\omega^i)\}_{i \geq 1}$ of Z with $n(\Gamma) > N$. By (2.7), we have that

$$\sum_i \mu(C_{n_i}(\omega^i))^q \exp(-\alpha n_i) \leq K^q \sum_i \exp\bigl(-n_i(qP_{\text{top}}(\varphi) + \alpha) + qS_{n_i}\varphi(\omega^i)\bigr).$$

It follows that

$$M_q(Z, \alpha, N) \leq K^q \inf_\Gamma \sum_i \exp\bigl(-n_i(qP_{\text{top}}(\varphi) + \alpha) + qS_{n_i}\varphi(\omega^i)\bigr).$$

Hence,

$$h_q(\sigma, Z) \leq P_Z(q\varphi) - qP_{\text{top}}(\varphi).$$

Analogously, we get the following lower bound of the q-topological entropy

$$h_q(\sigma, Z) \geq P_Z(q\varphi) - qP_{\text{top}}(\varphi).$$

Hence,

$$h_q(\sigma, Z) = P_Z(q\varphi) - qP_{\text{top}}(\varphi).$$

The other two equalities for $\underline{h}_q$ and $\overline{h}_q$ can be proven in a similar fashion. □

Under the conditions of the above theorem, and using the properties of topological pressure on non-compact subsets (see [4]), we obtain the relations between q-topological and lower and upper q-topological entropies. More precisely, for any $q \geq 0$ the following properties hold:

(1) if $Z \subset \Sigma_p^+$ is σ-invariant, then $\underline{CP}_Z(q\varphi) = \overline{CP}_Z(q\varphi)$ yielding that $\underline{h}_q(\sigma, Z) = \overline{h}_q(\sigma, Z)$;

(2) if $Z \subset \Sigma_p^+$ is σ-invariant and compact, then $\underline{CP}_Z(q\varphi) = \overline{CP}_Z(q\varphi) = P_Z(\varphi)$ yielding that

$$h_q(\sigma, Z) = \underline{h}_q(\sigma, Z) = \overline{h}_q(\sigma, Z).$$

3. q-Metric Entropy

In this section we introduce different types of q-metric entropy and we study their properties and relations between them.

3.1. **Definition of the q-metric entropy.** We follow the approach in [4] and introduce the notion of q-metric entropy using the *inverse variational principle.*

Given $q \geq 0$ and a Borel probability measure ν, let

$$\begin{aligned} h_q(\sigma,\nu) &= \inf\{h_q(\sigma,Z)\colon \nu(Z)=1\} \\ &= \lim_{\delta\to 0}\inf\{h_q(\sigma,Z)\colon \nu(Z)\geq 1-\delta\}. \end{aligned}$$

We call the quantity $h_q(\sigma,\nu)$ the *q-metric entropy* of σ with respect to ν. Let further

$$\begin{aligned} \underline{h}_q(\sigma,\nu) &= \lim_{\delta\to 0}\inf\{\underline{h}_q(\sigma,Z)\colon \nu(Z)\geq 1-\delta\}, \\ \overline{h}_q(\sigma,\nu) &= \lim_{\delta\to 0}\inf\{\overline{h}_q(\sigma,Z)\colon \nu(Z)\geq 1-\delta\}. \end{aligned}$$

We call the quantities $\underline{h}_q(\sigma,\nu)$ and $\overline{h}_q(\sigma,\nu)$ respectively the *lower* and *upper q-metric entropy* of σ with respect to ν. We mention that the q-topological entropy is defined with respect to a fixed reference measure μ, which can be different from ν.

We shall now show that the q-metric entropy as well as the lower and upper q-metric entropies are invariant under a homeomorphism that *respect* the measure μ.

Proposition 3.1. *Let $\pi\colon \Sigma_p^+\to\Sigma_p^+$ be a homeomorphism that satisfies the conditions of Proposition 2.3. Then for each $\nu\in\mathcal{M}_\sigma$,*

$$\mathcal{H}_q(\sigma,\nu)=\mathcal{H}_q(\sigma,\pi_*\nu),$$

where $\pi_*\nu=\nu\circ\pi^{-1}$.

Proof. By Proposition 2.3, we obtain that

$$\begin{aligned} h_q(\sigma,\nu) &= \inf\{h_q(\sigma,Z) : \nu(Z)=1\} \\ &= \inf\{h_q(\sigma,\pi(Z)) : \nu(Z)=1\} \\ &= \inf\{h_q(\sigma,Y) : \pi_*\nu(Y)=1\} \\ &= h_q(\sigma,\pi_*\nu). \end{aligned}$$

Here the third equality follows from the fact that π is a homeomorphism. The other two equalities for $\underline{h}_q$ and $\overline{h}_q$ can be proven in a similar fashion. □

3.2. **Relations between different q-metric entropies.** In this subsection we study the relations between various versions of q-metric entropies.

Proposition 3.2. *The following statements hold:*

(1) *For the reference measure* μ *we have* $h_1(\sigma,\mu) = \underline{h}_1(\sigma,\mu) = \overline{h}_1(\sigma,\mu) = 0$*;*

(2) *For any* $q \geq 0$ *and any Borel probability measure* ν *we have*

$$h_q(\sigma,\nu) \leq \underline{h}_q(\sigma,\nu) \leq \overline{h}_q(\sigma,\nu);$$

(3) *For any* $q \geq 0$ *and any* σ*-invariant ergodic measure* ν *we have that*

$$\overline{h}_q(\sigma,\nu) \leq h_\nu(\sigma),$$

where $h_\nu(\sigma)$ *is the standard metric entropy of* σ *with respect to* ν*, see* [6] *for details;*

(4) $0 \leq h_q(\sigma,\mu) \leq \underline{h}_q(\sigma,\mu) \leq \overline{h}_q(\sigma,\mu)$ *if* $0 \leq q \leq 1$ *and,* $h_q(\sigma,\mu) \leq \underline{h}_q(\sigma,\mu) \leq \overline{h}_q(\sigma,\mu) \leq 0$ *if* $q \geq 1$*.*

Proof. The first statement is a direct consequence of statement 3 of Proposition 2.5 and the definitions.

By Statement (3) of Proposition 2.1, for any Borel probability measure ν we have

$$h_q(\sigma,\nu) \leq \underline{h}_q(\sigma,\nu) \leq \overline{h}_q(\sigma,\nu),$$

and the second statement follows. By Proposition 2.5, for any $q \geq 0$ we have

$$\overline{h}_q(\sigma,\nu) \leq \overline{h}_0(\sigma,\nu).$$

Recall that

$$\overline{h}_0(\sigma,\nu) = \lim_{\delta\to 0} \inf\{\overline{h}_0(\sigma,Z)\colon \nu(Z) \geq 1-\delta\}.$$

By Remark 2.1, $\overline{h}_0(\sigma,Z)$ is the standard Bowen's upper topological entropy on the set Z. Since ν is ergodic, we have

$$\lim_{\delta\to 0} \inf\{\overline{h}_0(\sigma,Z)\colon \nu(Z) \geq 1-\delta\} = h_\nu(\sigma),$$

see [2] or [4, Theorem 11.6] for the proof. This gives us the third statement.

The last statement follows directly from Proposition 2.5. This completes the proof of the proposition. □

Given a Borel probability measure ν and any point $\omega \in \Sigma_p^+$, set

$$\underline{h}_\nu(\omega) := \liminf_{n\to\infty} -\frac{1}{n}\log \nu(C_n(\omega)) \quad \text{and} \quad \overline{h}_\nu(\omega) := \limsup_{n\to\infty} -\frac{1}{n}\log \nu(C_n(\omega)).$$

These two quantities are called *local lower and upper metric entropy at* ω with respect to ν respectively.

Theorem 3.3. *The following statements hold:*

(1) For any $q \geq 1$,

$$(1-q)\operatorname{ess\,inf}_{\omega\in\Sigma_p^+}\overline{h}_\mu(\omega) \leq \underline{h}_q(\sigma,\mu) \leq \overline{h}_q(\sigma,\mu) \leq (1-q)\operatorname{ess\,inf}_{\omega\in\Sigma_p^+}\underline{h}_\mu(\omega);$$

(2) For any $0 \leq q < 1$,

$$(1-q)\operatorname{ess\,sup}_{\omega\in\Sigma_p^+}\underline{h}_\mu(\omega) \leq \underline{h}_q(\sigma,\mu) \leq \overline{h}_q(\sigma,\mu) \leq (1-q)\operatorname{ess\,sup}_{\omega\in\Sigma_p^+}\overline{h}_\mu(\omega).$$

Proof. We will prove the first statement; the second one can be proven in a similar fashion. Given a small number $\beta > 0$ and a positive integer N, we define the set

$$\mathcal{L}_N = \Big\{\omega \in \Sigma_p^+ : \underline{h}_\mu(\omega) - \beta \leq -\frac{1}{n}\log\mu(C_n(\omega)) \leq \overline{h}_\mu(\omega) + \beta,\ \forall n \geq N\Big\}.$$

For any $\delta > 0$ there exists N_1 such that $\mu(\mathcal{L}_N) \geq 1-\delta$ for any $N \geq N_1$. Fix such a positive integer N. It is easy to see that for any $\omega \in \mathcal{L}_N$ and $n \geq N$,

$$\exp\big(-n(\overline{h}_\mu(\omega)+\beta)\big) \leq \mu(C_n(\omega)) \leq \exp\big(-n(\underline{h}_\mu(\omega)-\beta)\big). \tag{3.8}$$

To simplify our notations let

$$\underline{h} = \operatorname{ess\,inf}_{\omega\in\Sigma_p^+}\underline{h}_\mu(\omega), \qquad \overline{h} = \operatorname{ess\,inf}_{\omega\in\Sigma_p^+}\overline{h}_\mu(\omega).$$

For μ-almost every ω we have $\underline{h}_\mu(\omega) \geq \underline{h} - \beta$. Furthermore, there exists a subset $\widetilde{\Sigma}$ of positive measure such that $\overline{h}_\mu(\omega) \leq \overline{h} + \beta$ for every $\omega \in \widetilde{\Sigma}$. Let $Z \subset \mathcal{L}_N \cap \{\omega : \underline{h}_\mu(\omega) \geq \underline{h} - \beta\}$ be a set of positive measure and Γ a cover of Z by cylinders $C_n(\omega^i)$ with $n \geq N$. Then for $q \geq 1$ we have that

$$\begin{aligned}
\sum_i \mu(C_n(\omega^i))^q &= \sum_i \mu(C_n(\omega^i))^{q-1}\mu(C_n(\omega^i))\\
&\leq \exp\big(-n(\underline{h}-2\beta)(q-1)\big)\sum_i \mu(C_n(\omega^i))\\
&\leq \exp\big(-n(\underline{h}-2\beta)(q-1)\big)\mu((Z)_n)\\
&\leq \exp\big(-n(\underline{h}-2\beta)(q-1)\big).
\end{aligned} \tag{3.9}$$

In view of Proposition 2.2 this implies that $\overline{h}_q(\sigma, Z) \leq (1-q)(\underline{h}-2\beta)$ and hence,

$$\overline{h}_q(\sigma,\mu) \leq (1-q)(\underline{h}-2\beta).$$

Since β can be chosen arbitrarily small, one has that $\overline{h}_q(\sigma,\mu) \leq (1-q)\underline{h}$.

To prove the reverse inequality we choose a set Z with $\mu(Z) > 1-\delta$ for which $\underline{h}_q(\sigma,\mu) \geq \underline{h}_q(\sigma, Z) - \beta$. If δ is sufficiently small the set

$$Y = Z \cap \widetilde{\Sigma} \cap \mathcal{L}_N$$

has positive measure. Let Γ be a cover of Y by cylinders $C_n(\omega^i)$ with $n \geq N$. We have that

$$\begin{aligned}\sum_i \mu(C_n(\omega^i))^q &= \sum_i \mu(C_n(\omega^i))^{q-1}\mu(C_n(\omega^i)) \\ &\geq \exp\big(-n(\overline{h}+2\beta)\big)^{q-1}\mu(Y).\end{aligned}$$

This implies that

$$\underline{h}_q(\sigma,\mu) \geq \underline{h}_q(\sigma,Z) - \beta \geq \underline{h}_q(\sigma,Y) - \beta \geq (1-q)(\overline{h}+2\beta) - \beta.$$

Since β can be chosen arbitrary small, this implies that $\underline{h}_q(\sigma,\mu) \geq (1-q)\overline{h}$ completing the proof of the theorem. □

As a direct consequence of Theorem 3.3 we have: if the measure μ satisfies

$$\underline{h}_\mu(\omega) = \overline{h}_\mu(\omega) := h_\mu(\omega)$$

μ-almost everywhere, e.g., μ is a σ-invariant measure, then

$$\underline{h}_q(\sigma,\mu) = \overline{h}_q(\sigma,\mu) = (1-q)\operatorname{ess\,inf}_{\omega\in\Sigma_p^+} h_\mu(\omega)$$

for $q \geq 1$, and

$$\underline{h}_q(\sigma,\mu) = \overline{h}_q(\sigma,\mu) = (1-q)\operatorname{ess\,sup}_{\omega\in\Sigma_p^+} h_\mu(\omega)$$

for $0 \leq q < 1$.

For any $q \geq 0$ and $\alpha \in \mathbb{R}$ we define the *lower and upper q-pointwise entropy* of a Borel probability measure ν at ω by

$$\begin{aligned}\underline{h}_{\nu,\alpha,q}(\omega) &= \liminf_{n\to\infty} \frac{\alpha \log \nu(C_n(\omega))}{q\log\mu(C_n(\omega)) - \alpha n}, \\ \overline{h}_{\nu,\alpha,q}(\omega) &= \limsup_{n\to\infty} \frac{\alpha \log \nu(C_n(\omega))}{q\log\mu(C_n(\omega)) - \alpha n}.\end{aligned}$$

Given a number $h \geq 0$ and the Borel probability measure μ, define

$$\mathcal{L}_h = \Big\{\omega \in \Sigma_p : \underline{h}_\mu(\omega) = \overline{h}_\mu(\omega) = h\Big\}. \tag{3.10}$$

The following result is a straightforward calculation.

Proposition 3.4. *Assume that the measure μ is such that $\mu(\mathcal{L}_h) = 1$ for some $h > 0$. Choose numbers $q \geq 0$, $q \neq 1$ and $\epsilon > 0$ such that $h - \epsilon > 0$ and the interval $I = [h(1-q)-\epsilon, h(1-q)+\epsilon]$ does not contain 0. Then for μ-almost every ω and every $\alpha \in I$,*

$$\underline{h}_{\mu,\alpha,q}(\omega) = \overline{h}_{\mu,\alpha,q}(\omega) = \alpha h(hq+\alpha)^{-1}$$

(note that $hq + \alpha \geq h - \epsilon > 0$ although α may be negative).

Theorem 3.5. *Assume that the measure μ is such that $\mu(\mathcal{L}_h) = 1$ for some $h \geq 0$. Then for any $q \geq 0$*

(1) $h_q(\sigma,\mu)=\underline{h}_q(\sigma,\mu)=\overline{h}_q(\sigma,\mu)=h(1-q)$*;*
(2) $h_q(\sigma,\mathcal{L}_h)=\underline{h}_q(\sigma,\mathcal{L}_h)=\overline{h}_q(\sigma,\mathcal{L}_h)=h(1-q)$*.*

Proof. If $q=1$, the first statement is a direct consequence of Proposition 3.2 or Theorem 3.3 and the second statement follows from Proposition 2.5.

We now consider the case $q\neq 1$. Fix a small number $\eta>0$. For μ-almost every ω there exists a number $N(\omega)>0$ such that for any $n\geq N(\omega)$

$$\exp\big(-n(h+\eta)\big)\leq\mu(C_n(\omega))\leq\exp\big(-n(h-\eta)\big). \tag{3.11}$$

Given a positive integer N, set

$$\mathcal{L}_N=\{\omega\in\mathcal{L}_h : N(\omega)\leq N\}.$$

We have that $\mathcal{L}_N\subset\mathcal{L}_{N+1}$ and $\bigcup_{n\geq 0}\mathcal{L}_N=\mathcal{L}_h$. Hence, given $\delta>0$ one can find $N_0>0$ for which $\mu(\mathcal{L}_{N_0})>1-\delta$.

Fix a number $N\geq N_0$. Let $\Gamma=\{C_n(\omega^i)\}$ be a cover of $\mathcal{L}_N$ whose elements are pairwise disjoint. By the first inequality of (3.11), the cardinality of Γ is less than or equal to $\exp[n(h+\eta)]$ for any $n\geq N$. For all sufficiently large n we have

$$\begin{aligned}\Lambda_q(\mathcal{L}_N,n)&\leq\sum_i\mu(C_n(\omega^i))^q\\&\leq\exp[-nq(h-\eta)]\exp[n(h+\eta)]\\&=\exp[n\big((1-q)h+(1+q)\eta\big)]\end{aligned}$$

It follows that

$$\overline{h}_q(\sigma,\mathcal{L}_N)\leq(1-q)h+(1+q)\eta.$$

Since $\mu(\mathcal{L}_N)>1-\delta$, we have

$$\overline{h}_q(\sigma,\mu)\leq(1-q)h+(1+q)\eta.$$

Since η can be chosen arbitrarily small, we conclude that $\overline{h}_q(\sigma,\mu)\leq(1-q)h$.

We shall now prove that $h_q(\sigma,\mu)\geq h(1-q)$. It suffices to prove that $h_q(\sigma,Z)\geq h(1-q)$ for any subset $Z\subset\Sigma_p^+$ of full measure. Choose $\eta>0$ and $\delta\in(0,1/2)$ and denote $\lambda=(h-\eta)(1-q)$ if $0\leq q<1$ or $\lambda=(h+\eta)(1-q)$ if $q>1$. Let $\mathcal{L}'=\mathcal{L}_h\cap Z$. Clearly, $\mu(\mathcal{L}')=1$. One can find a set $\mathcal{L}_1\subset\mathcal{L}'$ with $\mu(\mathcal{L}_1)>1-\delta$ and a integer $N_1>0$ such that for any $\omega\in\mathcal{L}_1$ and $n\geq N_1$

$$\exp\big(-n(h+\eta)\big)\leq\mu(C_n(\omega))\leq\exp\big(-n(h-\eta)\big).$$

We may further assume that $\mathcal{L}_1$ is compact, since otherwise we can approximate it from within by a compact subset. Given any $N>N_1$, let Γ be a cover of $\mathcal{L}_1$ by cylinders $C_{n_i}(\omega^i)$ with $n_i\geq N$ for all i. Since $\mathcal{L}_1$ is compact, we may assume that the cover is finite and consists of cylinders $C_{n_1}(\omega^1),\cdots,C_{n_l}(\omega^l)$.

Without loss of generality, we assume that $\omega^i \in \mathcal{L}_1$ for any $1 \le i \le l$. Now for $0 \le q < 1$ we have

$$\begin{aligned}\sum_{C_{n_i}(\omega^i)\in\Gamma} \mu(C_{n_i}(\omega^i))^q \exp(-\lambda n_i) &\ge \sum_{i=1}^{l} \mu(C_{n_i}(\omega^i))^q \exp(-\lambda n_i) \\ &\ge \sum_{i=1}^{l} \mu(C_{n_i}(\omega^i)) \ge 1-\delta.\end{aligned} \tag{3.12}$$

Since the inequality holds for any cover Γ of $\mathcal{L}_1$, we conclude that $M_q(\mathcal{L}_1, \lambda, N) \ge 1-\delta$. Hence, $m_q(\mathcal{L}_1, \lambda) \ge 1-\delta > 0$. This implies that

$$h_q(\sigma, \mathcal{L}_1) \ge (h-\eta)(1-q)$$

for $0 \le q < 1$. For $q > 1$, using the same arguments we obtain that

$$h_q(\sigma, \mathcal{L}_1) \ge (h+\eta)(1-q).$$

Using Proposition 2.1 and the fact that η is arbitrary, for any $q \ge 0$, $q \ne 1$ we find that

$$h_q(\sigma, Z) \ge h_q(\sigma, \mathcal{L}_1) \ge h(1-q). \tag{3.13}$$

Therefore, by definition, $h_q(\sigma, \mu) \ge h(1-q)$.

To prove the second statement, first note that the inequality $h_q(\sigma, \mathcal{L}_h) \ge h(1-q)$ is contained in (3.13), since Z is an arbitrary set of full μ-measure and in our case $\mu(\mathcal{L}_h) = 1$.

Fix now a small number $\eta > 0$. For μ-almost every ω there exists a number $N(\omega) > 0$ such that (3.11) holds for any $n \ge N(\omega)$. Given a positive integer N, set

$$\mathcal{L}_N = \{\omega \in \mathcal{L}_h : N(\omega) \le N\}.$$

We have that $\mathcal{L}_N \subset \mathcal{L}_{N+1}$ and $\bigcup_{n\ge 0} \mathcal{L}_N = \mathcal{L}_h$. Hence, given $\delta > 0$, we can find $N_0 > 0$ for which $\mu(\mathcal{L}_{N_0}) > 1-\delta$.

Fix a number $N \ge N_0$. As in the proof of the first statement we have that

$$\overline{h}_q(\sigma, \mathcal{L}_N) \le (1-q)h + (1+q)\eta.$$

Since η can be chosen arbitrarily small, we conclude that $\overline{h}_q(\sigma, \mathcal{L}_N) \le (1-q)h$. Letting $N \to \infty$, we obtain that $\overline{h}_q(\sigma, \mathcal{L}_h) \le (1-q)h$. This completes the proof of the theorem. □

3.3. **The Modified Hentschel-Procaccia entropy spectrum.** Following the approach in [5], we introduce the *modified HP-entropy spectrum*. Given a Borel measure μ and $q>1$, define

$$\underline{\mathcal{HPM}}_q(\mu)=\frac{1}{q-1}\lim_{\delta\to 0}\inf_{Z}\liminf_{n\to\infty}\frac{1}{n}\log\int_Z \mu(C_n(\omega))^{q-1}\,\mathrm{d}\mu(\omega),$$
$$\overline{\mathcal{HPM}}_q(\mu)=\frac{1}{q-1}\lim_{\delta\to 0}\inf_{Z}\limsup_{n\to\infty}\frac{1}{n}\log\int_Z \mu(C_n(\omega))^{q-1}\,\mathrm{d}\mu(\omega),$$

where the infimum is taken over all sets $Z\subset X$ with $\mu(Z)>1-\delta$.

Definition 3.1. *We call the one-parameter family of pairs of quantities* $(\underline{\mathcal{HPM}}_q(\mu),\overline{\mathcal{HPM}}_q(\mu))$ *the* modified HP-spectrum for entropies.

The following result gives the relations between modified HP-spectrum for entropies, local lower and upper metric entropy and lower and upper q-metric entropy.

Proposition 3.6. *For any $q>1$, the following statements hold:*

(1) $\underline{\mathcal{HPM}}_q(\mu)=\frac{1}{q-1}\underline{h}_q(\sigma,\mu)$ *and* $\overline{\mathcal{HPM}}_q(\mu)=\frac{1}{q-1}\overline{h}_q(\sigma,\mu)$*;*

(2) $-\operatorname{ess\,inf}_{\omega\in\Sigma_p}\overline{h}_\mu(\omega)\le\underline{\mathcal{HPM}}_q(\mu)\le\overline{\mathcal{HPM}}_q(\mu)\le-\operatorname{ess\,inf}_{\omega\in\Sigma_p}\underline{h}_\mu(\omega)$.

Proof. The first statement follows directly from definitions and Theorem 2.4. The second statement is now a direct consequence of the first result and Theorem 3.3. □

By the second statement of Proposition 3.6, if μ satisfies that $\overline{h}_\mu(\omega)=\underline{h}_\mu(\omega):=h(\omega)$ for μ-almost every ω, then

$$\underline{\mathcal{HPM}}_q(\mu)=\overline{\mathcal{HPM}}_q(\mu)=-\operatorname{ess\,inf}_{\omega\in\Sigma_p^+}h_\mu(\omega).$$

Acknowledgements. The authors would like to thank the referee for valuable comments which helped us substantially improve the paper.

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