PARTIALLY HYPERBOLIC DYNAMICAL SYSTEMS

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CONTENTS

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1. INTRODUCTION

1.1. **Motivation.**

1.1.1. *Smooth ergodic systems.* The flows and maps that arise from equations of motion in classical mechanics preserve volume on the phase space, and their study led to the development of ergodic theory.

In statistical physics, the Boltzmann–Maxwell ergodic hypothesis, designed to help describe equilibrium and nonequilibrium systems of many particles, prompted a search for ergodic mechanical systems. In geometry, the quest for ergodicity led to the study of geodesic flows on negatively curved manifolds, where Eberhard Hopf provided the first and still only argument to establish ergodicity in the case of nonconstantly negatively curved surfaces [57]. Anosov and Sinai, in their aptly entitled work "Some smooth Ergodic Systems" [10] proved ergodicity of geodesic flows on negatively curved manifolds of any dimension.

With the development of the modern theory of dynamical systems and the availability of the Birkhoff ergodic theorem the impetus to find ergodic dynamical systems and to establish their prevalence grew stronger. Birkhoff conjectured that volume-preserving homeomorphisms of a compact manifold are generically ergodic.

1.1.2. *Hyperbolicity.* The latter 1960s saw a confluence of the investigation of ergodic properties with the Smale program of studying structural stability, or, more broadly, the understanding of the orbit structure of generic diffeomorphisms. The aim of classifying (possibly generic) dynamical systems has not been realized, and there are differing views of whether it will be. Current efforts in this direction are related to the Palis conjecture (see [4]). A promising step towards understanding generic smooth systems would clearly be an understanding of structurally stable ones, and one of the high points in the theory of smooth dynamical systems is that this has been achieved: Structural stability has been found to characterize hyperbolic dynamical systems [2].

Structural stability implies that all topological properties of the orbit structure are robust. Of these, topological transitivity has a particularly natural measurable analog, namely, ergodicity. On one hand, then, robust topological transitivity of hyperbolic dynamical systems motivated the search for broader classes of dynamical systems that are robustly transitive [30]. On the other hand, this, and the fact that volume-preserving hyperbolic dynamical systems are ergodic (with respect to volume) may have led Pugh and

Shub to pose a question at the end of [56] that amounts to asking whether ergodic toral automorphisms are *stably ergodic*, i.e., whether all their volume-preserving C^1-perturbations are ergodic. They later conjectured that stable ergodicity is open and dense among volume-preserving partially hyperbolic C^2-diffeomorphisms of a compact manifold.

1.1.3. *Partial hyperbolicity.* In this chapter we aim to give an account of significant results about partially hyperbolic systems. The pervasive guiding principle in this theory is that hyperbolicity in the system provides the mechanism that produces complicated dynamics in both the topological and statistical sense, and that, with respect to ergodic properties, it does so in essence by overcoming the effects of whatever nonhyperbolic dynamics may be present in the system.

We should point out from the start that the desired dynamical qualities (such as transitivity and ergodicity) are of an indecomposability type and evidently fail, for example, for the cartesian product of an Anosov diffeomorphism with the identity. Accordingly, suitable extra hypotheses of some sort will always be present to exclude this obvious reducibility and other less obvious ones (Section 7).

The ideas and methods in the study of partially hyperbolic dynamical systems extend those in the theory of uniformly hyperbolic dynamical systems, parts of which are briefly presented in [2], and go well beyond that theory in several aspects. Outside of these Handbook volumes, accounts of uniformly hyperbolic dynamical systems from many points of view abound; [62] provides a textbook exposition that provides sufficient background. A condensed version of the material from [62] on hyperbolicity is contained in [31], which adds a useful account of absolute continuity and ergodicity in that context. Partial hyperbolicity is also surveyed in [36, 79], with different emphasis. The one source that provides the most proofs of results only stated here is [72], which we recommend for further study of the subject. The study of partial hyperbolicity developed with two objects in mind: stable ergodicity and robust transitivity. We pay more attention to the first of these, and the recent book by Bonatti, Diaz and Viana [17] covers the second one in more detail.

1.1.4. *Extensions of classical complete hyperbolicity.* While classical hyperbolicity appeared in the 1960s, partial hyperbolicity was introduced in the early 1970s by Brin and Pesin [30] motivated by the study of frame flows, and it also arose naturally from the work of Hirsch, Pugh and Shub on normal hyperbolicity [56].

Partial hyperbolicity is but one possible extension of the notion of classical (complete) hyperbolicity, or, in fact, a pair of extensions. Classical hyperbolicity can be described as requiring that the possible uniform rates of exponential relative behavior of orbits come in two collections on either side of 1 or by requiring that the Mather spectrum of the system (Subsection 2.3) consists of parts inside and outside of the unit circle. Partial hyperbolicity (in the broad sense, Definition 2.1) merely requires that the Mather spectrum consists of two parts that are separated by some circle centered at the origin (not necessarily the unit circle), and partial hyperbolicity (in the now prevalent narrower sense, Definition 2.7) requires that the Mather spectrum has 3 annular parts of which the inner one lies inside the unit circle and the outer one lies outside of the unit circle (Theorem 2.15).

There has been some shift in terminology over time, and what exactly is meant by "partial hyperbolicity" without a further attribute often has to be inferred from the context. Furthermore, there are still minor variations in naming the various "flavors" of this notion. Partial hyperbolicity in the broad sense is the concept for which one can extend the theory of invariant distributions and foliations from the context of classical complete hyperbolicity in the most direct way (the corresponding results are presented in Sections 2 and 3). These are also the results that describe the stability of trajectories and usually precede the study of topological and ergodic properties of the system. Accordingly, in those early days, "partially hyperbolic" by itself referred to the broader notion in Definition 2.1. (This notion is also known in the study of ordinary differential equations as the presence of a *dichotomy*. We will not use this synonym.) The results on central and intermediate distributions and foliations as well as accessibility are of interest principally when one considers partially hyperbolic systems in the narrower sense of Definition 2.7, where a central direction is present (as well as two more directions that have stronger contraction and expansion, respectively). This therefore used to be called "partial hyperbolicity in the narrow sense", but recent work has so much focused on this situation that "in the narrow sense" is usually dropped. In this respect we conform to the current majority choice, but retain the option of emphasizing greater generality by using the notion of partial hyperbolicity in the broad sense of Definition 2.1.

One can weaken the notion of partial hyperbolicity in the broad sense to a semiuniform one, namely to that of having a *dominated splitting* [4], the rates are separated by a uniform factor but the location of the gap is allowed to vary with the point (see page 45). Put differently, partial hyperbolicity directly constrains the Mather spectrum (see Subsection 2.3) and the presence of a dominated splitting does not. Since partial hyperbolicity implies the presence of a dominated splitting, results proved for dynamical systems with a dominated splitting apply to those as in Definition 2.1. Dominated splittings appear in work directed at the (strong) Palis conjecture (that the C^1-generic diffeomorphism is either Ω-stable or the limit of diffeomorphisms with homoclinic tangencies or heterodimensional cycles) [4], but the objectives of this chapter versus those in that by Pujals and Sambarino [4] are fairly different. As one intriguing connection one might point out that it seems reasonable to suppose that the presence of a dominated splitting may be necessary for stable ergodicity [4, 79]; indeed, this is true for an open dense set of such systems ([14], see also [79, Theorem 19.1]). On the other hand, stably ergodic systems need not be partially hyperbolic [87]. Close on the heels of the introduction of partial hyperbolicity came another fundamental extension of the theory of uniformly hyperbolic dynamical systems in a different direction. Relaxing the assumption on uniform rates by hypotheses on the Lyapunov exponents leads to the study of the much broader class of nonuniformly hyperbolic dynamical systems, which has flourished since the 1970s and is presented in [1, 13]. While this is an extension in a different direction, there are significant points of intersection with the theory of partial hyperbolicity, some of which we mention in due course (see Definition 6.3), and whose importance is likely to grow.

Indeed, it is natural to proceed further to the study of systems in which both uniformity and completeness of hyperbolicity are dropped, and this theory of nonuniformly partially hyperbolic dynamical systems is described in [1].

1.2. **Outline.** This chapter consists of three major portions, each of which is summarized below. The first of these introduces the basic definitions and those parts of the theory that are most directly analogous to corresponding ones in the theory of uniformly hyperbolic systems. The second part examines the central and intermediate distributions and foliations, which has a quite different character. The third part explores accessibility, ergodicity and stable ergodicity.

1.2.1. *Basic notions and results (Sections 2 and 3).* We first (in Section 2) present various definitions of partial hyperbolicity as well as basic examples. Conceptually the most "compact" way of thinking about uniform partial versus complete hyperbolicity is in terms of the Mather spectrum (Subsection 2.3), where partial hyperbolicity amounts to having other possibilities for the radii and number of rings.

We then proceed to a discussion of the invariant structures associated with the various spectral rings. These come in two fundamentally distinct classes. Section 3 is an unsurprising generalization of the stable manifold theory for uniformly hyperbolic dynamical systems to partially hyperbolic ones. It produces, in the presence of different rates of contraction or expansion, a hierarchy of fast stable or fast unstable manifolds that corresponds to collections of "inner" or "outer" rings of the Mather spectrum, respectively. We briefly discuss their regularity, including absolute continuity, which is important for the ergodic theory of partially hyperbolic systems. Neither the phenomena nor the methods here are particularly unexpected given any familiarity with the classical stable manifold theory.

1.2.2. *Central and intermediate foliations (Sections 4–6)*. The study of the central distribution turns out to be quite a different matter. The Hirsch–Pugh–Shub theory of normal hyperbolicity helps control both the (moderate) regularity of its leaves and provide some robustness under perturbation—once the central foliation is known to exist. Existence is a rather delicate matter and is known only under several rather stringent assumptions, while nonexistence is an open property. We present some weak forms of integrability that are more easily obtained. Here the integral manifolds for different points may intersect without coinciding, i.e., one does not obtain a foliation in the proper sense.

Considering this as the study of the invariant structures associated with the central ring of the Mather spectrum, it is natural to do the same with other rings in the Mather spectrum as well, and the associated intermediate distributions and foliations turn out to be even more delicate.

Sections 4 and 5 study primarily topological aspects of these distributions and foliations, and in Section 6 we turn to measurable aspects. We discuss, using examples and general constructions, the possible failure of the central foliation to be absolutely continuous. On one hand we present results to the effect that even when the central distribution is integrable to a foliation with smooth leaves, absolute continuity may indeed fail in the worst possible way: There is a set of full measure that intersects almost every leaf in a bounded number of points only. On the other hand, there is evidence to support the widely held surmise that singularity of the central foliation is not only possible, but indeed typical. It would not be an overstatement to say that for a partially hyperbolic system to be stably ergodic its central foliation has to fail to be absolutely continuous in most cases. See Subsection 6.2 for details.

1.2.3. *Accessibility and ergodicity*. As we will explain more carefully, the previously mentioned Hopf argument to establish ergodicity relies on the local product structure; in a uniformly hyperbolic dynamical system any two nearby points have a heteroclinic point, i.e., the local stable leaf of one point intersects the local unstable of the other. In particular, one can take a short curve in the local stable leaf of the first point to the intersection point and concatenate it with a short arc in the unstable manifold of the second point to join the points by what one then calls a us-path (Definition 7.1). In a partially hyperbolic system this certainly fails when the two foliations are jointly integrable, such as in the case of (Anosov $\times$ identity). A priori there could be a whole spectrum of intermediate possibilities:

- the foliations are jointly integrable in some places but not in others,
- the foliations are both subordinate to a common foliation with leaves of dimension larger than the sum of stable and unstable dimensions,
- nearby points might only be connectable by us-paths with long arcs,
- it might take multiple concatenations of us-paths.

Whether any of these possibilities are indeed realizable remains to be seen. But it is conjectured that generically only one possibility occurs: Any two points are accessible, i.e., can be connected by a us-path with finitely many legs.

Accordingly, the next 2 sections of this chapter are devoted to the notion of *accessibility*, which has become a central idea in the study of partially hyperbolic systems. Section 7 presents this concept, and this enables us to present next the Pugh–Shub ergodicity theory of partially hyperbolic dynamical systems in Section 8.

Finally, we discuss Sinai–Ruelle–Bowen measures (or "physical measures") in the last section.

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2. Definitions and examples

2.1. **Definition of partial hyperbolicity.** Our basic definitions require 2-sided estimates of the norms of images of linear maps, and it will be convenient to have a compact notation at our disposal. Suppose V, W are normed linear spaces, $A\colon V \to W$ a linear map and $U \subset V$. Then we define the *norm* and *conorm* of A *restricted* to U by

$$\|A \restriction U\| := \sup\{\|Av\|/\|v\| \mid v \in U \smallsetminus \{0\}\},$$
$$\|\lfloor A \restriction U \rfloor\| := \inf\{\|Av\|/\|v\| \mid v \in U \smallsetminus \{0\}\}.$$

2.1.1. *Partial hyperbolicity in the broad sense.* The first and broader definition of partial hyperbolicity is modeled on that of hyperbolicity, where the rates of exponential behavior are separated by the unit circle, by considering separation by a different circle:

Definition 2.1. Consider a manifold M, an open subset U and an embedding $f\colon U \to M$ with an invariant set Λ. Then f is said to be *partially hyperbolic* (in the broad sense) on Λ, or Λ is said to be a partially hyperbolic invariant set of f in the broad sense [30] if Λ is closed and there exist numbers $0 < \lambda < \mu$, $c > 0$, and subspaces $E_1(x)$ and $E_2(x)$ for all $x \in \Lambda$, such that

(1) $E_1(x)$ and $E_2(x)$ form an invariant splitting of the tangent space, i.e.,

$$\begin{aligned} T_xM =& E_1(x) \oplus E_2(x), \\ d_xfE_1(x) = E_1(f(x)), &\quad d_xfE_2(x) = E_2(f(x)); \end{aligned} \tag{2.1}$$

(2) if $n \in \mathbb{N}$ then $\|d_xf^n \upharpoonright E_1(x)\| \leq c\lambda^n$ and $c^{-1}\mu^n \leq \|\lfloor d_xf^n \upharpoonright E_2(x)\rfloor\|$.

If $\lambda < 1$ the subspace $E_1(x)$ is stable (in the usual sense [2]) and will be denoted by $E^s(x)$. If $\mu > 1$ the subspace $E_2(x)$ is unstable, and we use the notation $E^u(x)$.

Clearly, either $\lambda < 1$ or $\mu > 1$ (or both) and without loss of generality we assume the former.

Remark 2.2. In [55, p. 53] this is called (absolute) "pseudo-hyperbolicity", and in [92], "(λ, μ)-splitting".

A diffeomorphism f of a smooth compact Riemannian manifold is said to be *partially hyperbolic in the broad sense* if the whole manifold is a partially hyperbolic set for f in the broad sense.

2.1.2. *Lyapunov metrics.* If $0 < \lambda < \lambda' < \mu' < \mu$ define the *Lyapunov inner product* or *Lyapunov metric* $\langle \cdot, \cdot \rangle'$ by

$$\begin{aligned} \langle v, w\rangle'_x &:= \sum_{k=0}^{\infty} \langle df^k v, df^k w\rangle_{f^k(x)} {\lambda'}^{-2k} && \text{for } v, w \in E_1(x), \\ \langle v, w\rangle'_x &:= \sum_{k=0}^{\infty} \langle df^{-k} v, df^{-k} w\rangle_{f^{-k}(x)} {\mu'}^{2k} && \text{for } v, w \in E_2(x), \\ \langle v, w\rangle'_x &:= \langle v_1, w_1\rangle'_x + \langle v_2, w_2\rangle'_x \end{aligned}$$

for $v = v_1 + v_2 \in T_xM$ and $w = w_1 + w_2 \in T_xM$ with $v_1, w_1 \in E_1(x)$ and $v_2, w_2 \in E_2(x)$. The induced *Lyapunov norm* in T_xM is denoted by $\|\cdot\|'_x$. Then $\measuredangle(E_1(x), E_2(x))' = \pi/2$, $\|v\|_x/\sqrt{2} \leq \|v\|'_x \leq c\|v\|_x$, and $\|df \upharpoonright E_1(x)\|' \leq \lambda'$, $\|df^{-1} \upharpoonright E_2(x)\|' \leq (\mu')^{-1}$.

Proposition 2.3. *An embedding f is partially hyperbolic in the broad sense if and only if there are a (not necessarily smooth) Riemannian metric $\|\cdot\|$, numbers*

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 \text{ with } \mu_1 < 1, \tag{2.2}$$

and an invariant splitting

$$T_xM = E_1(x) \oplus E_2(x), \quad dfE_i(x) = E_i(f(x)) \text{ for } i = 1, 2 \tag{2.3}$$

of the tangent bundle such that $E_1(x) \perp E_2(x)$ *for every* $x \in \Lambda$ *and*

$$\lambda_1 \leq \llbracket df \restriction E_1(x) \rrbracket \leq \|df \restriction E_1(x)\| \leq \mu_1,$$
$$\lambda_2 \leq \llbracket df \restriction E_2(x) \rrbracket \leq \|df \restriction E_2(x)\| \leq \mu_2.$$

2.1.3. *Invariant distributions.* A few basic observations are quite easy to make:

Proposition 2.4. *Consider a manifold* M*, an open set* $U \subset M$ *and an embedding* $f\colon M \to U$ *with a compact partially hyperbolic invariant set* Λ*. Then, using the notations of Definition 2.1,*

(1) $E_1(x) = \{v \in T_xM \mid \exists a > 0,\ \gamma \in [\lambda, \mu)\ \forall n \in \mathbb{N} \quad \|d_x f^n v\| \leq a\gamma^n \|v\|\}$.
(2) $E_2(x) = \{v \in T_xM \mid \exists b > 0,\ \kappa \in (\lambda, \mu]\ \forall n \in \mathbb{N} \quad \|d_x f^n v\| \geq b\kappa^n \|v\|\}$.
(3) $E_1(x)$ *and* $E_2(x)$ *are continuous, so*
(4) there exists $k > 0$ *such that* $\measuredangle(E_1(x), E_2(x)) \geq k$ *for all* $x \in \Lambda$*.*
(5) There exists $\varepsilon > 0$ *such that if* $\widetilde{E} \subset TM$ *is an invariant distribution for which* $\dim \widetilde{E}_1(x) = \dim E_1(x)$ *and* $\measuredangle(\widetilde{E}_1(x), E_1(x)) \leq \varepsilon$ *for every* $x \in \Lambda$ *then* $\widetilde{E}_1(x) = E_1(x)$*, and likewise for* $E_2(x)$*.*

On the other hand, going beyond continuity is a rather more substantial achievement. This goes back to Anosov in the hyperbolic case and to Brin and Pesin [30] in the present context. The most general version is in [22]:

Theorem 2.5. $E_1(x)$ *and* $E_2(x)$ *are Hölder continuous, i.e., there exist* $C, \alpha > 0$ *such that* $\measuredangle(E_i(x), E_i(y)) \leq C\rho(x,y)^\alpha$ *for all* $x, y \in \Lambda$ *and* $i = 1, 2$*. Indeed, the Hölder exponent can be controlled through the hyperbolicity estimates: In the context of Proposition 2.3 any*

$$\alpha < \frac{\log \lambda_2 - \log \mu_1}{\log \mu_2} \tag{2.4}$$

admits a $C > 0$ *for which* $\measuredangle(E_1(x), E_1(y)) \leq C\rho(x,y)^\alpha$*, and there is an analogous estimate for the Hölder exponent of* E_2*.*

One should not expect the distribution E_1 to be smooth even in the case of Anosov diffeomorphisms. The first example of a nonsmooth stable distribution was constructed by Anosov in [9]. Hasselblatt [51] has shown that for a "typical" Anosov diffeomorphism the stable and unstable distributions are only Hölder continuous with Hölder exponent no larger than that in (2.4) (see also [52]). Moreover, high regularity has in several classes of hyperbolic systems been shown to occur only for algebraic systems [2]. Nevertheless, there are situations where these distributions are C^1 (see, e.g., [2, 51, 56]):

(1) under the *pinching condition* $\frac{\mu_1}{\lambda_2}\mu_2 < 1$[1]
(2) if the distribution E_1 is of *codimension one*.

Remark 2.6. Our definition of partial hyperbolicity (in the broad sense) corresponds to what is also known as *absolute partial hyperbolicity* (absolute pseudo-hyperbolicity in [55, p. 53]) as opposed to a weaker *relative (or pointwise) partial hyperbolicity* (relative pseudo-hyperbolicity in [55, p. 62f]). While for the former we have

$$\sup_{x\in M}\|df\restriction E_1(x)\|(\inf_{x\in M}\|(df\restriction E_2(x))^{-1}\|)^{-1} < 1,$$

the latter is defined such that

$$\sup_{x\in M}\|df\restriction E_1(x)\|(\|(df\restriction E_2(x))^{-1}\|)^{-1} < 1.$$

See [56] where other refined versions of absolute and relative hyperbolicity are introduced. It should be stressed that one can develop essentially the whole stability theory of partially hyperbolic systems assuming only relative partial hyperbolicity. However, the study of ergodic and topological properties of partially hyperbolic systems needs the stronger assumption of absolute partial hyperbolicity.[2]

2.1.4. *Partial hyperbolicity.* The study of partially hyperbolic systems with a view to ergodicity has concentrated on those with a triple splitting that includes a central direction of weakest contraction and expansion:

Definition 2.7. An embedding f is said to be *partially hyperbolic* on Λ if there exist numbers $C > 0$,

$$0 < \lambda_1 \le \mu_1 < \lambda_2 \le \mu_2 < \lambda_3 \le \mu_3 \text{ with } \mu_1 < 1 < \lambda_3 \tag{2.5}$$

and an invariant splitting into stable, central and unstable directions

$$T_xM = E^s(x)\oplus E^c(x)\oplus E^u(x),\quad d_xfE^\tau(x) = E^\tau(f(x)),\ \tau = s, c, u \tag{2.6}$$

such that if $n \in \mathbb{N}$ then

$$\begin{aligned}
C^{-1}{\lambda_1}^n &\le \|\lfloor d_xf^n\restriction E^s(x)\rfloor\| \le \|d_xf^n\restriction E^s(x)\| \le C{\mu_1}^n,\\
C^{-1}{\lambda_2}^n &\le \|\lfloor d_xf^n\restriction E^c(x)\rfloor\| \le \|d_xf^n\restriction E^c(x)\| \le C{\mu_2}^n,\\
C^{-1}{\lambda_3}^n &\le \|\lfloor d_xf^n\restriction E^u(x)\rfloor\| \le \|d_xf^n\restriction E^u(x)\| \le C{\mu_3}^n.
\end{aligned}$$

In this case we set $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$.

There is a Lyapunov metric that is fully adapted to this situation:

[1]By (2.4), E_1 is Lipschitz in this case.
[2]We would like to thank M. Viana for pointing this out to us.

Proposition 2.8. *An embedding is partially hyperbolic if and only if there exists a Riemannian metric for which there are numbers $\lambda_i, \mu_i,\ i = 1, 2, 3$ as in (2.5) and an invariant splitting (2.6) into pairwise orthogonal subspaces $E^s(x)$, $E^c(x)$ and $E^u(x)$ such that*

$$\begin{aligned} \lambda_1 \le \| \lfloor d_x f \restriction E^s(x) \rfloor \| \le \| d_x f \restriction E^s(x) \| \le \mu_1, \\ \lambda_2 \le \| \lfloor d_x f \restriction E^c(x) \rfloor \| \le \| d_x f \restriction E^c(x) \| \le \mu_2, \\ \lambda_3 \le \| \lfloor d_x f \restriction E^u(x) \rfloor \| \le \| d_x f \restriction E^u(x) \| \le \mu_3. \end{aligned} \tag{2.7}$$

2.1.5. *The cone criterion.* Verifying partial hyperbolicity appears to require finding the invariant distributions first, so it is useful to have a more obviously robust criterion that is easier to verify. For hyperbolic dynamical systems this goes back principally to Alexeyev [7, 2].

Given a point $x \in M$, a subspace $E \subset T_x M$ and a number $\alpha > 0$, define the *cone* at x centered around E of angle α by

$$C(x, E, \alpha) = \{ v \in T_x M \mid \measuredangle(v, E) < \alpha \}.$$

Proposition 2.9. *An embedding f is partially hyperbolic in the broad sense if and only if there are $\alpha > 0$ and two continuous cone families $C_1(x, \alpha) = C(x, E_1(x), \alpha)$ and $C_2(x, \alpha) = C(x, E_2(x), \alpha)$ for which*

$$d_x f^{-1}(C_1(x, \alpha)) \subset C_1(f^{-1}(x), \alpha), \quad d_x f(C_2(x, \alpha)) \subset C_2(f(x), \alpha) \tag{2.8}$$

as well as

$$\| d_x f \restriction C_1(x, \alpha) \| \le \mu_1 < \lambda_2 \le \| \lfloor d_x f \restriction C_2(x, \alpha) \rfloor \|. \tag{2.9}$$

The evident advantage of this definition is that one can verify it having only approximations of E and F in Definition 2.1, and these approximations need not be invariant in order for suitable cones around them to satisfy (2.8) and (2.9).

Proposition 2.10. *An embedding f is partially hyperbolic if and only if there are families of stable and unstable cones*

$$C^s(x, \alpha) = C(x, E^s(x), \alpha), \quad C^u(x, \alpha) = C(x, E^u(x), \alpha)$$

and of center-stable cones or center-unstable cones

$$C^{cs}(x, \alpha) = C(x, E^{cs}(x), \alpha), \quad C^{cu}(x, \alpha) = C(x, E^{cu}(x), \alpha),$$

where

$$E^{cs}(x) = E^c(x) \oplus E^s(x), \quad E^{cu}(x) = E^c(x) \oplus E^u(x),$$

such that

$$\begin{aligned} d_x f^{-1}(C^s(x, \alpha)) \subset C^s(f^{-1}(x), \alpha), \quad & d_x f(C^u(x, \alpha)) \subset C^u(f(x), \alpha), \\ d_x f^{-1}(C^{cs}(x, \alpha)) \subset C^{cs}(f^{-1}(x), \alpha), \quad & d_x f(C^{cu}(x, \alpha)) \subset C^{cu}(f(x), \alpha) \end{aligned} \tag{2.10}$$

and there are $0 < \mu_1 < \lambda_2 \le \mu_2 < \lambda_3$ *with* $\mu_1 < 1 < \lambda_3$ *such that*

$$\begin{aligned} \|d_x f \restriction C^s(x,\alpha)\| \le \mu_1, \quad \lambda_3 \le \lfloor\!\lfloor d_x f \restriction C^u(x,\alpha) \rfloor\!\rfloor, \\ \|d_x f \restriction C^{cs}(x,\alpha)\| \le \mu_2, \quad \lambda_2 \le \lfloor\!\lfloor d_x f \restriction C^{cu}(x,\alpha) \rfloor\!\rfloor. \end{aligned} \tag{2.11}$$

2.2. **Examples of partially hyperbolic systems.**

2.2.1. *The time-t map of a hyperbolic flow.* Let φ_t be a flow on a compact smooth Riemannian manifold M with a hyperbolic invariant set Λ. Given $t \in \mathbb{R}$, the map φ_t is partially hyperbolic on Λ with 1-dimensional central direction generated by the vector field.

2.2.2. *Frame flows.* Let V be a closed oriented n-dimensional manifold of negative sectional curvature and $M = SV$ the unit tangent bundle of V. Let also N be the space of positively oriented orthonormal n-frames in TV. This produces a fiber bundle $\pi\colon N \to M$ where the natural projection π takes a frame into its first vector. The associated structure group $SO(n-1)$ acts on fibers by rotating the frames, keeping the first vector fixed. Therefore, we can identify each fiber N_x with $SO(n-1)$ where g_t is the geodesic flow. The *frame flow* Φ_t acts on frames by moving their first vectors according to the geodesic flow and moving the other vectors by parallel translation along the geodesic defined by the first vector. For each t, we have that $\pi \circ \Phi_t = g_t \circ \pi$. The frame flow Φ_t preserves the measure that is locally the product of the Liouville measure with normalized Haar measure on $SO(n-1)$. The time-t map of the frame flow is a partially hyperbolic diffeomorphism (for $t \ne 0$). The center bundle has dimension $1 + \dim SO(n-1)$ and is spanned by the flow direction and the fiber direction.

2.2.3. *Direct products.* Let $f\colon U \to M$ be an embedding with a compact hyperbolic set $\Lambda \subset M$ and $E^s_f(x)$, $E^u_f(x)$ the stable and unstable subspaces at $x \in \Lambda$. Also, let $g\colon U' \to N$ be an embedding with a compact invariant set K such that

$$\max_{x\in\Lambda} \|df \restriction E^s_f(x)\| < \min_{y\in K} \lfloor\!\lfloor dg(y) \rfloor\!\rfloor \le \max_{y\in K} \|dg(y)\| < \min_{x\in\Lambda} \lfloor\!\lfloor df \restriction E^u_f(x) \rfloor\!\rfloor.$$

Then $F\colon M \times N \to M \times N$, $F(x,y) = (f(x), g(y))$ is partially hyperbolic on $\Lambda \times K$.

Particular cases are g being the identity map of N or a rotation of $N = S^1$.

2.2.4. *Skew products.* Let $f\colon U \to M$ be an embedding with a compact hyperbolic set $\Lambda \subset M$ and $E^s_f(x)$, $E^u_f(x)$ the stable and unstable subspaces at $x \in \Lambda$. Also, let $g_x\colon U_x \to N$ be a family of embeddings of $U_x \subset N$ that depend smoothly on $x \in \Lambda$ and have a common compact invariant set

K such that

(2.12)

$$\max_{x\in\Lambda} \|df \restriction E^s_f(x)\| < \min_{x\in\Lambda}\min_{y\in K} \|\lfloor dg_x(y)\rfloor\| \le \max_{x\in\Lambda}\max_{y\in K} \|dg_x(y)\| < \min_{x\in\Lambda} \|\lfloor df \restriction E^u_f(x)\rfloor\|.$$

The map $F\colon \Lambda \times \bigcap_x U_x \to M \times N$ given by $F(x, y) = (f(x), g_x(y))$ is partially hyperbolic on $\Lambda \times K$.

A particular case is obtained by taking $\Lambda = M$, $K = N = S^1$, $\alpha\colon M \to M$ smooth and $g_x = R_{\alpha(x)}$ (rotation by $\alpha(x)$). The map

$$F = F_\alpha\colon M\times S^1 \to M\times S^1, \quad F(x, y) = (f(x), R_{\alpha(x)}(y)), \quad x \in M,\ y \in S^1$$

is partially hyperbolic with 1-dimensional central direction.

2.2.5. *Group extensions.* An "algebraic" version of the previous example is a group extension over an Anosov diffeomorphism. Let G be a compact Lie group and $\varphi\colon M \to G$ a smooth function on M with values in G. Define the map $F = F_\varphi\colon M \times G \to M \times G$ by

$$F(x, y) = (f(x), \varphi(x)g), \quad x \in M,\ g \in G.$$

The map F is partially hyperbolic since left translations are isometries of G in the bi-invariant metric. If f preserves a smooth probability measure ν then F preserves the smooth probability measure $\nu \times \nu_G$ where ν_G is the (normalized) Haar measure on G.

2.2.6. *Partially hyperbolic systems on 3-dimensional manifolds.* It is an open problem to describe compact smooth Riemannian manifolds that admit partially hyperbolic diffeomorphisms. To admit the splitting into stable, unstable and center distribution the dimension of the manifold must be at least three, so it is natural to inquire first, which 3-manifolds support partially hyperbolic diffeomorphisms. The torus $\mathbb{T}^3$ does, because an automorphism given by an integer matrix with eigenvalues λ, 1, λ^{-1}, where $|\lambda| \neq 1$, is partially hyperbolic, as is any sufficiently small perturbation (Corollary 2.17). Recently, Brin, Burago and Ivanov have begun a study of partially hyperbolic dynamical systems on 3-manifolds, and interesting results have already been obtained.

Theorem 2.11 (Brin, Burago, Ivanov [27])**.** *A compact 3-dimensional manifold whose fundamental group is finite does not carry a partially hyperbolic diffeomorphism.*

This implies that there are no partially hyperbolic diffeomorphisms on the 3-dimensional sphere $\mathbb{S}^3$.

We should mention a result that can be viewed as a precursor to Theorem 2.11. L. Díaz, E. Pujals and R. Ures showed that a robustly transitive diffeomorphism of a 3-manifold M (i.e., a diffeomorphism all of whose C^1 perturbations are topologically transitive) is generically partially hyperbolic in the broad sense, and if the center-unstable bundle is integrable then the fundamental group of M is infinite [40]. This was extended to arbitrary dimension by Bonatti, Díaz and Pujals [16]: Generically the homoclinic class of any periodic saddle is either contained in the closure of an infinite set of sinks or sources (Newhouse phenomenon), or admits a dominated splitting; in particular, robust transitivity implies dominated splitting (see also page 45 and [4, Section 5]).

One may ask a question complementary to the previous one: Of what type can partially hyperbolic diffeomorphisms of 3-manifolds be? The known robustly transitive or stably ergodic ones are

- perturbations of skew-products over an Anosov diffeomorphism on $\mathbb{T}^2$,
- perturbations of the time-1-map of a transitive Anosov flow,
- some derived-from-Anosov diffeomorphisms on $\mathbb{T}^3$.

Pujals has speculated (see [21]) that this is indeed a complete list, and recent work by Bonatti and Wilkinson [21] makes it plausible that such a classification of transitive partially hyperbolic diffeomorphisms of 3-manifolds might hold: They show that the homoclinic geometry of a single periodic orbit can determine much of the global orbit structure. (Note that volume-preserving such diffeomorphisms are generically transitive by Theorem 7.9 and Theorem 7.12.)

Specifically, in the case of a skew product a periodic orbit arises from a periodic point for the base diffeomorphism and hence comes with nearby homoclinic periodic orbits that are also embedded circles. Their first result turns this observation around:

Theorem 2.12 (Bonatti–Wilkinson [21])**.** *Let f be a transitive partially hyperbolic diffeomorphism of a compact 3-manifold M with an embedded invariant circle γ such that there is some (sufficiently large) δ for which $W^s_\delta(\gamma) \cap W^u_\delta(\gamma) \smallsetminus \gamma$ has a connected component that is a circle. Then, possibly after passing to on orientable cover, M is a circle bundle over $\mathbb{T}^2$ and f is conjugate to a topological skew-product over a linear Anosov map A of $\mathbb{T}^2$, i.e., to a map of M that preserves the fibration and projects to A.*

In the case of the time-1-map of a transitive Anosov flow the homoclinic curves to an invariant circle are noncompact. The corresponding result is a little less complete than the previous one.

Theorem 2.13 (Bonatti–Wilkinson [21])**.** *Let f be a dynamically coherent (Definition 4.4) partially hyperbolic diffeomorphism of a compact 3-manifold M with a closed periodic center leaf γ such that each center leaf in $W^s_{loc}(\gamma)$ is periodic for f. Then the center foliation supports a continuous flow conjugate to a transitive expansive flow.*

It is conjectured and seems likely to be true that the expansive flows that arise here are in turn topologically conjugate to Anosov flows.

2.3. **The Mather Spectrum.** An embedding f with a compact invariant set Λ generates a continuous linear operator f_* on the Banach space $\Gamma^0(T_\Lambda M)$ of continuous vector fields $\mathbf{v}$ on Λ by the formula

$$(f_*\mathbf{v})(x) = df\mathbf{v}(f^{-1}(x)).$$

The spectrum $Q = Q_f$ of the complexification of f_* is called the *Mather spectrum* of the dynamical system f on Λ, and it provides alternative ways of expressing our various hyperbolicity conditions as well as more detailed information about separation of expansion and contraction rates:

Theorem 2.14 (Mather [65, 72])**.** *If nonperiodic orbits of f are dense in Λ then*

(1) any connected component of the spectrum Q is a ring (or annulus) $Q_i = \{z \in \mathbb{C} \mid \lambda_i \le |z| \le \mu_i\}$ around 0 with radii λ_i and μ_i, where $0 < \lambda_1 \le \mu_1 < \cdots < \lambda_t \le \mu_t$ and $t \le \dim M$;
(2) the invariant subspace $H_i \in \Gamma^0(TM)$ of f_* corresponding to the component Q_i of the spectrum is a module over the ring of continuous functions;
(3) the collection of the subspaces $E_i(x) = \{\mathbf{v}(x) \in T_x \mid \mathbf{v} \in H_i\}$ constitutes a df-invariant continuous distribution on M and

$$T_xM = \bigoplus_{i=1}^{t} E_i(x), \text{ for all } x \in M.$$

Since density of nonperiodic orbits is an easy consequence of hyperbolicity assumptions, one can characterize various classes of dynamical systems using their Mather spectra.

Theorem 2.15 (Mather [65, 72])**.**

(1) A diffeomorphism f is Anosov if and only if 1 is not contained in its Mather spectrum Q.
(2) A diffeomorphism f is partially hyperbolic on Λ in the broad sense if and only if its Mather spectrum (over Λ) is contained in a disjoint union of two nonempty rings, $Q \subset Q_1 \cup Q_2$ with Q_1 lying inside of the unit disk or Q_2 lying outside of the unit disk.
(3) A diffeomorphism f is partially hyperbolic on Λ if and only if its Mather spectrum (over Λ) is contained in a disjoint union of three nonempty rings, $Q \subset Q_1 \cup Q_2 \cup Q_3$ with Q_1 lying inside of the unit disk and Q_3 lying outside of the unit disk.

While [65, 72] state these results only for $\Lambda = M$, the proofs readily extend to invariant subsets.

It is natural to expect that the Mather spectrum is stable under small perturbations of dynamical systems, and the most straightforward approach to establishing this would be to show that the action induced on vector fields by a perturbation is close to the original such action. Unfortunately this is always false. The expected result about the Mather spectrum nevertheless turns out to be true:

Theorem 2.16 (Pesin [71, 72])**.** *Let M be a compact manifold, $f\colon M \to M$ a diffeomorphism whose nonperiodic orbits are dense. Let*

$$Q_f = \bigcup_{i=1}^{t} Q_{f,i}$$

be the decomposition of its Mather spectrum into nonempty disjoint rings $Q_{f,i}$ with radii

$$0 < \lambda_{f,1} \leq \mu_{f,1} < \cdots < \lambda_{f,t} \leq \mu_{f,t}.$$

Let also

$$TM = \bigoplus_{i=1}^{t} E_{f,i}$$

be the corresponding decomposition of the tangent bundle into df-invariant subbundles $E_{f,i}$, $i = 1, \ldots, t$. Then for any sufficiently small $\varepsilon > 0$ there exists a neighborhood η of f in $\mathrm{Diff}^1(M)$ such that for any $g \in \eta$:

(1) the Mather spectrum Q_g is a union of disjoint components $Q_{g,i}$, each being contained in a ring with radii $\lambda_{g,i} \leq \mu_{g,i}$ satisfying

$$|\lambda_{f,i} - \lambda_{g,i}| \leq \varepsilon, \quad |\mu_{f,i} - \mu_{g,i}| \leq \varepsilon.$$

(2) the distribution $E_{g,i}$ corresponding to the component $Q_{g,i}$ satisfies

$$\max_{x \in M} \measuredangle(E_{f,i}(x), E_{g,i}(x)) \leq L\delta^{\alpha} \leq \varepsilon,$$

where $\delta = d_{C^1}(f, g)$ and $L > 0$, $\alpha > 0$ are constants.

As usual, $\mathrm{Diff}^q(M)$ is the space of C^q diffeomorphisms with the C^q topology.

Corollary 2.17. Anosov systems, partially hyperbolic systems, and partially hyperbolic diffeomorphisms form open subsets in $\mathrm{Diff}^q(M)$, $q \geq 1$.

Remark 2.18. While a component $Q_{f,i}$ of the spectrum of f may be a ring, the corresponding component $Q_{g,i}$ of the spectrum of g may consist of several rings. To illustrate this situation consider an Anosov flow φ_t on a smooth manifold M and observe that the Mather spectrum of $\varphi_0 = \mathrm{Id}$ is the unit circle while the Mather spectrum of φ_t for $t \neq 0$ (which is partially hyperbolic) contains at least two more additional rings.

Remark 2.19. There are 3 general situations in which partial hyperbolicity is known to be stable. First, we just saw that this is the case when the entire manifold is a partially hyperbolic set (Corollary 2.17). Second, Theorem 2.16 extends to partially hyperbolic attractors because attractors are stable under perturbation, so partially hyperbolic attractors are also stably partially hyperbolic. Finally, when the partially hyperbolic set is a normally hyperbolic manifold then Theorem 4.3 below together with Theorem 2.16 gives persistence of partial hyperbolicity.

There are also some particular cases when partially hyperbolic sets survive under small perturbations, such as when a partially hyperbolic set Λ is the direct product of a locally maximal hyperbolic set and a compact manifold. Indeed, Λ is foliated by leaves of its center foliation and can be viewed as a normally hyperbolic lamination in the sense of [56]. Its stability follows from Theorem 4.11. Partially hyperbolic sets of this type appear in bifurcation theory (see [44, 59]).

3. Stable and unstable filtrations

3.1. **Existence and subfoliation.** For hyperbolic dynamical systems the classical Stable-Manifold Theorem [2] establishes that the stable and unstable distributions are each tangent to a unique foliation. A moderate adaptation of the Stable-Manifold Theorem yields analogous but more finely stratified information when the Mather spectrum consists of a larger number of rings (see [72] and the references therein). We should mention that the word foliation is used here in a looser sense than in differential geometry. Even in the case of Anosov diffeomorphisms these foliations are partitions into smooth manifolds that may only admit (Hölder) continuous foliation charts; for hyperbolic sets the foliation locally only fills a Cantor set times a disk (see [2, 80]).

Definition 3.1. A partition W of M is called a *foliation of M with smooth leaves* or simply *foliation* if there exist $\delta > 0$ and $\ell > 0$ such that for each $x \in M$,

1. the element $W(x)$ of the partition W containing x is a smooth ℓ-dimensional injectively immersed submanifold; it is called the *(global) leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta)$ that contains x is called the *local leaf* at x and is denoted by $V(x)$;
2. there exists a continuous map $\varphi_x \colon B(x, \delta) \to C^1(D, M)$ (where $D \subset \mathbb{R}^\ell$ is the unit ball) such that for every $y \in M \cap B(x, \delta)$ the manifold $V(y)$ is the image of the map $\varphi_x(y) \colon D \to M$.

The function $\varphi_x(y,z) = \varphi_x(y)(z)$ is called the *foliation coordinate chart*. This function is continuous and has continuous derivative $\frac{\partial}{\partial z}\varphi_x$.

A continuous k-dimensional distribution E on M is said to be

(1) *weakly integrable* if for each point $x \in M$ there is an immersed complete C^1 manifold $W(x)$ which contains x and is everywhere tangent to E, i.e., $T_yW(y) = E(y)$ for each $y \in W(x)$ [27]. We call $W(x)$ an integral manifold of E through x (note that *a priori* the integral manifolds $W(x)$ may be self-intersecting and may not form a partition of M);
(2) *integrable* if there is a foliation whose tangent bundle is E;
(3) *uniquely integrable* if there is a foliation W with k-dimensional leaves such that any C^1 curve $\sigma\colon \mathbb{R} \to M$ satisfying $\dot\sigma(t) \in E(\sigma(t))$ for all t, is contained in $W(\sigma(0))$ (in particular, $T_xW(x) = E(x)$ for all $x \in M$);
(4) *locally uniquely integrable* if for each $x \in M$ there is a k-dimensional smooth submanifold $W_{\text{loc}}(x)$ and $\alpha(x) > 0$ such that a piecewise C^1 curve $\sigma\colon [0,1] \to M$ is contained in $W_{\text{loc}}(x)$ so long as $\sigma(0) = x$, $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0,1]$ and length $\sigma < \alpha(x)$. (In this case E is integrable and the integral foliation is unique.)

Theorem 3.2 (Hirsch, Pugh Shub [55], Brin, Pesin [30], [72])**.** *Suppose f is an embedding with a compact invariant set Λ on which the tangent space admits a df-invariant splitting*

$$T_\Lambda M = \bigoplus_{i=1}^{t} E_i \tag{3.1}$$

with

$$\lambda_i < \|\lfloor df \upharpoonright E_i(x) \rfloor\| \le \|df \upharpoonright E_i(x)\| < \mu_i, \tag{3.2}$$

for all $x \in \Lambda$, where

$$0 < \lambda_1 \le \mu_1 < \cdots < \lambda_t \le \mu_t. \tag{3.3}$$

(1) If $\mu_k < 1$ then the distribution

$$F_k^s = \bigoplus_{i=1}^{k} E_i$$

is uniquely integrable and the maximal integral manifolds of this distribution generate a foliation W_k^s of M. The global leaf $W_k^s(x)$ through $x \in M$ is a C^1-immersed submanifold of M.

(2) *If $\lambda_k > 1$ then an analogous statement holds for the distribution*

$$F_k^u = \bigoplus_{i=k}^{t} E_i;$$

the corresponding foliation is W_k^u and its leaves are $W_k^u(x)$, $x \in M$.

(3) *The foliation W_k^s is f-invariant and contracting, i.e., for any $x \in M$, $y \in W_k^s(x)$ and $n \geq 0$,*

$$\rho_k^s(f^n(x), f^n(y)) \leq C(\lambda_k + \varepsilon)^n \rho_k^s(x, y),$$

where ε is such that $0 < \varepsilon < \min\{\lambda_{k+1} - \mu_k, 1 - \mu_k\}$, $C = C(\varepsilon) > 0$ is a constant independent of x, y and n, and ρ_k^s is the distance in $W_k^s(x)$ induced by the Riemannian metric.

(4) *The foliation W_k^u is f-invariant and contracting under f^{-1}.*

(5) *If f is C^q then $W_k^s(x)$ and $W_k^u(x)$ are C^q.*

Thus, in this case the two filtrations of distributions

$$F_1^s \subset F_2^s \subset \cdots \subset F_\ell^s, \qquad F_m^u \supset \cdots \supset F_t^u$$

integrate to filtrations of foliations

$$W_1^s \subset W_2^s \subset \cdots \subset W_\ell^s, \qquad W_m^u \supset \cdots \supset W_t^u,$$

where ℓ is maximal and m is minimal such that $\mu_\ell < 1$ and $\lambda_m > 1$ (note that $m = \ell + 1$ or $\ell + 2$). W_k^s is called the k-*stable foliation* and W_k^u the k-*unstable foliation* for f. If f is partially hyperbolic the foliations $W^s = W_\ell^s$ and $W^u = W_m^u$ are called the *stable* and *unstable* foliations.

Theorem 3.3 (Hirsch, Pugh, Shub [56, Theorem 6.1])**.** *Under the assumptions of Theorem 3.2 and with $1 \leq k < \ell$ (respectively, $m \leq k < t$), the foliation W_k^s* subfoliates *the foliation W_{k+1}^s (respectively, W_{k+1}^u subfoliates W_k^u). For every $x \in M$ the leaves $W_k^s(y)$ depend C^{n_k} smoothly on $y \in W_{k+1}^s(x)$, where n_k is the largest integer such that $\mu_k < \lambda_{k+1}^{n_k}$. An analogous statement holds for W_k^u.*

3.2. **Absolute continuity.** The fact that the stable and unstable foliations may not admit smooth local foliation charts prevents us from applying the classical Fubini theorem to conclude that a set that intersects each local leaf in a set of full (leaf-) measure must itself be of full measure. Anosov [9] identified this as the central technical point in the ergodic theory of hyperbolic dynamical systems (in the Hopf argument, see [57, 10, 31] and Section 7, p. 35). In this subsection and the next we explain that while, due to the absence of smooth foliation charts, it seems possible that the foliations might be singular in the measure-theoretic sense (see Section 6), the needed property of absolute continuity still holds for the stable and unstable foliations. We will later see that the central direction is much less well behaved.

The first step is absolute continuity of the holonomy maps.

Definition 3.4. Let W be a foliation of a manifold M with smooth leaves and for $x \in M$, $r > 0$ consider the family

$$\mathcal{L}(x) = \{V(w) : w \in B(x,r)\} \tag{3.4}$$

of local manifolds, where $V(w)$ is the connected component containing w of $W(w) \cap B(x,r)$ and $B(x,r)$ is the ball centered at x of radius r.

Choose two local disks D^1 and D^2 that are transverse to the family $\mathcal{L}(x)$, and define the *holonomy map* $\pi = \pi(x,W)\colon D^1 \to D^2$ (generated by the family of local manifolds) by setting

$$\pi(y) = D^2 \cap V(w) \text{ if } y = D^1 \cap V(w) \text{ and } w \in B(x,r).$$

The holonomy map π is a homeomorphism onto its image.

Let m denote the Riemannian volume. Given a submanifold D in M, let m_D be the Riemannian volume on D induced by the restriction of the Riemannian metric to D.

Theorem 3.5 (Brin, Pesin [30], Pugh, Shub [76, 77], [13, 72])**.** *Let f be a partially hyperbolic C^2 diffeomorphism of a compact smooth manifold M. Given $x \in M$ and two transverse disks D^1 and D^2 to the family $\mathcal{L}(x)$ of local stable manifolds $V(y)$, $y \in B(x,r)$, the holonomy map π is absolutely continuous (with respect to the measures m_{D^1} and m_{D^2}) and the Jacobian Jac$(\pi) := dm_{D^2}/d(\pi_* m_{D^1})$ (Radon–Nikodym derivative) is bounded from above and bounded away from zero.*

The Jacobian of the holonomy map at a point $y \in D^1$ can be computed by the following formula

$$\text{Jac}\,(\pi)(y) = \prod_{k=0}^{\infty} \frac{\text{Jac}\,(d_{f^k(\pi(y))} f^{-1} | T_{f^k(\pi(y))} f^k(D^2))}{\text{Jac}\,(d_{f^k(y)} f^{-1} | T_{f^k(y)} f^k(D^1))}.$$

In particular, the infinite product on the right hand-side converges.

The issue of absolute continuity as it affects the ergodic theory of hyperbolic and partially hyperbolic dynamical systems can be put in this form: If $E \subset B(x,q)$ is a Borel set of positive volume, can the intersection $E \cap V(y)$ have zero Lebesgue measure (with respect to the Riemannian volume on $V(y)$) for almost every $y \in E$?

Theorem 3.5 is the main step towards ruling out this pathology for the stable and unstable foliations.

Theorem 3.6. *Let f be a partially hyperbolic C^2 diffeomorphism of a compact smooth manifold M, ν a smooth f-invariant probability measure on M. Then the conditional measures on each $V(w)$ are absolutely continuous with respect to the induced Riemannian volume, and likewise for transversals.*

To state this more precisely, let $\nu_V(w)$ be the conditional measures on $V(w)$ for $w \in B(x,r)$ and consider the measurable partition ξ of

$$Q(x) := \bigcup_{w \in B(x,r)} V(w)$$

into local manifolds, identifying the factor space $Q(x)/\xi$ with an open transverse disk D. Denote by $\hat{\nu}_D$ the factor measure generated by the partition ξ (supported on D), by $m_V(w)$ the Riemannian volume on $V(w)$, and by m_D the Riemannian volume on D. Then Theorem 3.6 is meant to say that

(1) the measures $\nu_V(w)$ and $m_V(w)$ are equivalent for ν-almost every $w \in B(x,r)$;
(2) the factor measure $\hat{\nu}_D$ is equivalent to the measure m_D.

As a consequence

$$d\nu_V(w)(y) = \kappa(w,y) dm_V(w)(y)$$

for every $w \in B(x,r)$ and $y \in V(w)$, where $\kappa(w,y)$ is continuous and satisfies the homological equation

$$\kappa(f(w), f(y)) = \frac{\mathrm{Jac}(df \restriction E(y))}{\mathrm{Jac}(df \restriction E(w))} \kappa(w,y).$$

It follows that

$$\kappa(w,y) = \prod_{i=0}^{\infty} \frac{\mathrm{Jac}(df \restriction E(f^i(y)))}{\mathrm{Jac}(df \restriction E(f^i(w)))}.$$

By the Hölder continuity of the distribution E (see Theorem 2.5), the product converges.

The absolute continuity property of local stable manifold described in Theorem 3.6 follows from absolute continuity of the holonomy map (see Theorem 3.5). However, the converse does not hold.

4. Central Foliations

4.1. **Normal hyperbolicity.** The notion of normal hyperbolicity in dynamical systems was introduced by Hirsch, Pugh and Shub in [55] (see also [56]; a particular case of normal hyperbolicity was considered by R. Sacker [82] in his work on partial differential equations). The theories of normal hyperbolicity and partial hyperbolicity are closely related in their results and methods. Moreover, the former provides techniques to study integrability of the central distribution and robustness of the central foliation for partially hyperbolic systems (see Subsection 4.3 and Subsection 4.4).

Definition 4.1. Let $q \geq 1$, M a C^q compact connected Riemannian manifold (without boundary), $U \subset M$ open, $f\colon U \to M$ a C^q embedding, and $N = N_f$ a compact C^1 f-invariant submanifold of M, i.e., $f(N) = N$. The map f is said to be *normally hyperbolic to* N if f is partially hyperbolic on (or "along") N, i.e., there is an invariant splitting

$$T_xM = E^s(x) \oplus T_xN \oplus E^u(x) \text{ with } df E^s(x) = E^s(f(x)) \text{ and } df E^u(x) = E^u(f(x)) \tag{4.1}$$

for every $x \in N$, such that

$$\begin{aligned}\lambda_1 \leq &\|\lfloor df \restriction E^s(x) \rfloor\| \leq \|df \restriction E^s(x)\| \leq \mu_1,\\ \lambda_2 \leq &\|\lfloor df \restriction T_xN \rfloor\| \leq \|df \restriction T_xN\| \leq \mu_2,\\ \lambda_3 \leq &\|\lfloor df \restriction E^u(x) \rfloor\| \leq \|df \restriction E^u(x)\| \leq \mu_3,\end{aligned} \tag{4.2}$$

where $0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$ and $\mu_1 < 1 < \lambda_3$.

Similarly to Theorem 2.5, the splitting (4.1) is Hölder continuous.

By the Local-Stable-Manifold Theorem one can construct, for every $x \in N$, local stable and unstable manifolds, $V^s(x)$ and $V^u(x)$ respectively, at x, such that

(1) $x \in V^s(x)$, $x \in V^u(x)$;
(2) $T_xV^s(x) = E^s(x)$, $T_xV^u(x) = E^u(x)$;
(3) if $n \in \mathbb{N}$ then

$$\rho(f^n(x), f^n(y)) \leq C(\mu_1 + \varepsilon)^n \rho(x, y), \text{ for } y \in V^s(x),$$
$$\rho(f^{-n}(x), f^{-n}(y)) \leq C(\lambda_3 - \varepsilon)^n \rho(x, y), \text{ for } y \in V^u(x),$$

where $C > 0$ is a constant and $\varepsilon > 0$ is sufficiently small.

Set

$$V^{so}(N) = \bigcup_{x \in N} V^s(x) \text{ and } V^{uo}(N) = \bigcup_{x \in N} V^u(x). \tag{4.3}$$

These are topological manifolds called local *stable* and *unstable* manifolds of N. They are f-invariant and

$$N = V^{so}(N) \cap V^{uo}(N).$$

Theorem 4.2 (Hirsch, Pugh, Shub [56, Proposition 5.7, Theorem 3.5])**.** *$V^{uo}(N)$ and $V^{so}(N)$ are Lipschitz continuous and indeed smooth submanifolds of M.*

In [56], Hirsch, Pugh and Shub used the Hadamard method for constructing local stable and unstable manifolds through N. Their approach does not rely on the existence of local stable manifolds through individual points

$x \in N$ but instead, builds local stable and unstable manifolds through N as a whole. Of course, *a posteriori* one can derive (4.3). Hirsch, Pugh and Shub obtained more complete information on local stable and unstable manifolds through N. In particular, they showed that a normally hyperbolic manifold N survives under small perturbation of the system, thus establishing stability of normal hyperbolicity.

Theorem 4.3 (Hirsch, Pugh, Shub [56, Sections 4–6])**.** *Let f be a C^q embedding with $q \geq 1$ that is normally hyperbolic to a compact smooth manifold N,*

$$\begin{aligned} \ell_u &:= \max\{j \in \{0, \dots, q\} \mid \mu_1 < \lambda_2^j\}, \\ \ell_s &:= \max\{j \in \{0, \dots, q\} \mid \mu_2^j < \lambda_3\}, \end{aligned} \qquad \ell := \min\{\ell_s, \ell_u\}, \tag{4.4}$$

where λ_i and μ_i, $i = 1, 2, 3$, are as in Definition 4.1. Then

(1) *Existence: there exist locally f-invariant submanifolds $V^{so}(N)$ and $V^{uo}(N)$ tangent to $E^s \oplus TN$ and $E^u \oplus TN$ respectively.*
(2) *Uniqueness: if N' is an f-invariant set which lies in an ε-neighborhood $U_\varepsilon(N)$ of N, for sufficiently small ε, then $N' \subset V^{so}(N) \cup V^{uo}(N)$.*
(3) *Characterization: $V^{so}(N)$ (respectively, $V^{uo}(N)$) consists of all points y for which $\rho(f^n(y), N) \leq r$ for all $n \geq 0$ (respectively, $n \leq 0$) and some small $r > 0$; indeed, $\rho(f^n(y), N) \to 0$ exponentially as $n \to +\infty$ (respectively, as $n \to -\infty$).*
(4) *Smoothness: $V^{so}(N)$ and $V^{uo}(N)$ are submanifolds in M of class C^{ℓ_s} and C^{ℓ_u} respectively; in particular, N is a C^ℓ submanifold.*
(5) *Lamination: $V^{so}(N)$ and $V^{uo}(N)$ are fibered by $V^s(x)$ and $V^u(x)$, $x \in N$; see (4.3).*

For every $\delta > 0$ there exist $r > 0$ and $\varepsilon > 0$ such that

(6) *for every embedding g of class C^q with $d_{C^1}(f, g) \leq \varepsilon$, there exists a smooth submanifold N_g, invariant under g, to which g is normally hyperbolic; N_g lies in an r-neighborhood $U_r(N_f)$ of N_f.*
(7) *$V_g^{so}(N) \in C^{\ell_s}$, $V_g^{uo}(N) \in C^{\ell_u}$ and $N_g \in C^\ell$ (where the numbers ℓ_s, ℓ_u, and ℓ are given by (4.4)); they depend continuously on g in the C^1 topology;*
(8) *there exists a homeomorphism $H : U_r \to M$ which is δ-close to the identity map in the C^0 topology and such that $H(N_f) = N_g$.*

4.2. **Integrability of the central foliation and dynamical coherence.** Let M be a compact smooth Riemannian manifold and $f : M \to M$ a diffeomorphism that is partially hyperbolic with a df-invariant splitting of the tangent bundle (2.6) satisfying (2.7). The central distribution E^c may not, in general, be integrable as Subsection 5.1 below illustrates. Nonintegrability is an open property (Theorem 4.9).

We describe some conditions that guarantee integrability of the central distribution. In fact, these conditions guarantee the stronger property of unique integrability, which we introduce now.

Definition 4.4. A partially hyperbolic embedding is said to be *dynamically coherent* if E^{cs} and E^{cu} are integrable to foliations W^{cs} and W^{cu}, respectively.

In this case E^c is integrable to the *central foliation* W^c for which $W^c(x) = W^{cu}(x) \cap W^{cs}(x)$, each leaf of W^{cs} is foliated by leaves of W^c and W^s, and each leaf of W^{cu} is foliated by leaves of W^c and W^u.

Note that these assumptions do not imply that the integral foliations are unique, so it is not clear whether the central subbundle is uniquely integrable in this case. [18] demonstrates that Hölder continuous distributions may have many different integral foliations. On the other hand, there is no known example in which a central subbundle is integrable and not uniquely integrable.

Brin communicated the following result, whose proof is essentially contained in [27]:

Theorem 4.5. *If the central subbundle is uniquely integrable then the system is dynamically coherent, and the center-stable and center-unstable subbundles are also uniquely integrable.*

Definition 4.6. A foliation W of M is said to be *quasi-isometric* if there are $a > 0$ and $b > 0$ such that $\rho_W(x, y) \leq a \cdot \rho(x, y) + b$ for every $x \in M$ and every $y \in W(x)$, where ρ_W is the distance along the leaves of W.

A partially hyperbolic embedding f is said to be *center-isometric* if it acts isometrically in the central direction, i.e., $\|df(x)v\| = \|v\|$ for every $x \in \Lambda$ and $v \in E^c(x)$.

Denote by $\widetilde{M}$ the universal cover of M and by $\widetilde{W}^s$ and $\widetilde{W}^u$ the lifts of the stable and unstable foliations to $\widetilde{M}$.

Theorem 4.7 (Brin [25, 72])**.** *Let M be a compact smooth Riemannian manifold and $f\colon M \to M$ a diffeomorphism which is partially hyperbolic with a df-invariant splitting of the tangent bundle* (2.6) *satisfying* (2.7).

If $\widetilde{W}^s$ and $\widetilde{W}^u$ are quasi-isometric in the universal cover $\widetilde{M}$, then the distributions E^{cs}, E^{cu} and E^c are locally uniquely integrable.

If f is center-isometric then the central distribution E^c is locally uniquely integrable.

As we mentioned above the central distribution may not be integrable. However, it is often weakly integrable (Definition 3.1), and this weak integrability persists under small perturbations:

Theorem 4.8 (Brin, Burago, Ivanov [27]). *Let f be a partially hyperbolic diffeomorphism of M. Assume the distributions E_f^{cs}, E_f^{cu} and E_f^c are weakly integrable. Then there is a C^1 neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is a partially hyperbolic diffeomorphism whose distributions E_g^{cs}, E_g^{cu} and E_g^c are weakly integrable.*

So long as one stays safely within the partially hyperbolic context this weak integrability is also a closed property:

Theorem 4.9 (Brin, Burago, Ivanov [27]). *Let $\{f_n\}_{n\geq 0}$ be a sequence of partially hyperbolic diffeomorphisms of M. Assume that*

(1) $f_n \to g$ in the C^1 topology;
(2) the distributions $E_{f_n}^{cs}$, $E_{f_n}^{cu}$ and $E_{f_n}^c$ are weakly integrable for all n;
(3) all f_n have the same hyperbolicity constants (2.5).

Then g is partially hyperbolic and the distributions E_g^{cs}, E_g^{cu} and E_g^c are weakly integrable.

4.3. Smoothness of central leaves via normal hyperbolicity.

Theorem 4.3 gives a fair amount of information about normally hyperbolic submanifolds. Since the center foliation, if defined, is "essentially" normally hyperbolic in that the strong contraction and expansion act transversely to it, one would like to apply Theorem 4.3 to this situation. Hirsch, Pugh and Shub developed a construction which allows one to do this.

Let f be a partially hyperbolic diffeomorphism with a df-invariant splitting of the tangent bundle (2.6) satisfying (2.7) and with E^c integrable. For $r > 0$ let $U_r(W^c(x)) \subset M$ be the tubular neighborhood of radius r of the leaf $W^c(x)$. Consider the manifold that is the *disjoint* union

$$\mathcal{M}_r = \bigcup_{x\in M} U_r(W^c(x)). \tag{4.5}$$

For sufficiently small ε, $0 < \varepsilon < r$, the map f induces a diffeomorphism $F : \mathcal{M}_\varepsilon \to \mathcal{M}_r$ which is normally hyperbolic to the submanifold

$$\mathcal{N} = \bigcup_{x\in M} W^c(x) \subset \mathcal{M}_\varepsilon.$$

The manifolds $\mathcal{M}_\varepsilon$ and $\mathcal{N}$ are not compact but complete. Theorem 4.3 extends to this situation because the proof relies only on the existence of a tubular neighborhood of the normally hyperbolic manifold, a uniform lower bound for the radius of injectivity of the exponential map, and uniform estimates (4.2). We have all this at our disposal since the manifold M is compact and the central foliation W^c is integrable. This yields a corollary of Theorem 4.3:

Theorem 4.10. *Let $f\colon M \to M$ be a partially hyperbolic embedding with integrable central distribution. Then $W^c(x) \in C^\ell$ for every $x \in M$, where ℓ is as in* (4.4).

4.4. **Robustness of the central foliation.** A far less straightforward application of the Hirsch–Pugh–Shub construction can be used to produce robustness of the central foliation W^c under small perturbation of the system; the subtlety of the matter is evidenced by the requirement that this foliation be smooth.

Theorem 4.11 (Hirsch, Pugh, Shub [56, Theorem 7.5])**.** *Assume that the central distribution E^c for f is integrable, that the corresponding foliation W^c is smooth and that g is a C^q diffeomorphism sufficiently close to f in the C^1-topology. Then g is partially hyperbolic with integrable central distribution E^c_g.*

The direct approach suggested above is carried out in [72], but this result is usually obtained as an immediate corollary of Theorems 4.13 and 4.15 below, whose proofs exploit plaque expansivity and pseudo-orbits.

In general, we do not have a smooth central foliation. Moreover, even if the central foliation for f were smooth, the central foliation for a "typical" perturbation of f would not be. Thus the assumptions of integrability and smoothness are not jointly robust; only integrability persists if we assume both at the outset. In [56], Hirsh, Pugh and Shub introduced a property of the central foliation for f called *plaque expansivity* that is weaker than smoothness (see Theorem 4.13 below) but still guarantees integrability of the central distribution for sufficiently small perturbations of f and furthermore persists itself under small perturbations (see Theorem 4.15 below).

Definition 4.12. Let W be a foliation of a compact smooth manifold M whose leaves are C^r smooth immersed submanifolds of dimension k. Given a point $x \in M$, we call the set $P(x) \subset W(x)$ a C^r *plaque* of W at x if $P(x)$ is the image of a C^r embedding of the unit ball $D \subset \mathbb{R}^k$ into $W(x)$. A *plaquation* $\mathcal{P}$ for W is a collection of plaques such that every point $x \in M$ is contained in a plaque $P \in \mathcal{P}$.

Let $\{x_n\}_{n\in\mathbb{Z}}$ be a pseudo-orbit for f (see [3, 2]). We say that the pseudo-orbit *respects* a plaquation $\mathcal{P}$ for W if for every $n \in \mathbb{Z}$ the points $f(x_n)$ and x_{n+1} lie in a common plaque $P \in \mathcal{P}$.

Assume that the foliation W is invariant under a diffeomorphism f of M. We say that f is *plaque expansive* with respect to W if there exists $\varepsilon > 0$ with the following property: if $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ are ε-pseudo-orbits which respect W and if $\rho(x_n, y_n) \le \varepsilon$ for all $n \in \mathbb{Z}$ then x_n and y_n lie in a common plaque for all $n \in \mathbb{Z}$.

Note that plaque expansivity does not depend on the choice of either the Riemannian structure in M or the plaquation $\mathcal{P}$ for W. It is indeed weaker than smoothness:

Theorem 4.13 (Hirsch, Pugh, Shub [56, Theorem 7.2])**.** *Let f be a partially hyperbolic diffeomorphism. Assume that the central distribution E^c is integrable and the central foliation W^c is smooth. Then W^c is plaque expansive.*

Remark 4.14. If $df \upharpoonright E^c(x)$ acts as an isometry for every $x \in M$ then the central distribution E^c is integrable by Theorem 4.7, and the central foliation W^c is plaque expansive (see [56, Section 7]).

Theorem 4.15 (Hirsch, Pugh, Shub [56, Theorem 7.1])**.** *Let $f \colon M \to M$. If f is partially hyperbolic with the central distribution E^c_f for f integrable and f plaque expansive with respect to the central foliation W^c_f then the same holds for any sufficiently C^1-close diffeomorphism g (with respect to E^c_g and W^c_g).*

5. Intermediate Foliations

The central distribution we studied in the previous section corresponds to the central ring in the Mather spectrum, and it is now natural to study the structures associated with other intermediate rings (as opposed to the inner- and outermost ones, which figured in Section 3).

Consider a diffeomorphism f of class C^q of a compact Riemannian manifold M admitting a df-invariant splitting (3.1) satisfying (3.2) and (3.3). Given $1 < k < t$ with $\mu_k < 1$ we now discuss the integrability problem for the invariant distribution E_k, called the *intermediate* distribution.

5.1. Nonintegrability of intermediate distributions.

In general, E_k is not integrable as we now illustrate with an example that goes back to Smale [86] and appears in [62, Section 17.3] as well as [90, p. 1549], where it provides an example of a diffeomorphism that is normally hyperbolic with respect to a smooth, 1-dimensional foliation and not conjugate to the time-one map of any Anosov flow (and can be shown to be stably ergodic using the methods of [45]).

Consider the *Heisenberg group* of matrices

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : (x, y, z) \in \mathbb{R} \right\}$$

with the usual matrix multiplication: in (x, y, z) coordinates it is given by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

The center of H is the 1-parameter subgroup

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, H is a 3-dimensional, simply connected, nonabelian nilpotent group. Its Lie algebra is

$$\mathfrak{L}(H) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : (x, y, z) \in \mathbb{R} \right\}$$

with generators

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $[X, Y] = Z$ while all other Lie brackets of generators are zero.

Let $G = H \times H$ be the Lie group with generators $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ such that $[X_i, Y_i] = Z_i$ and all other brackets of generators are zero. Its Lie algebra is

$$\mathfrak{L}(G) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A, B \in \mathfrak{L}(H) \right\}.$$

The group H has an obvious integer lattice of matrices with entries in $\mathbb{Z}$ which generates an integer lattice in G. We need another lattice in G however.

Consider the number field $\mathbb{K} = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$. It possesses a unique nontrivial automorphism σ such that $\sigma(a + b\sqrt{5}) = a - b\sqrt{5}$.

Let Γ be the subgroup of G given by $\exp_{\mathrm{Id}} \gamma$, where $\exp_{\mathrm{Id}} \colon \mathfrak{L}(G) \to G$ is the exponential map and

$$\gamma := \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \mid A \in \mathfrak{L}(H) \text{ with entries in the algebraic integers in } \mathbb{K} \right\} \subset \mathfrak{L}(G)$$

with $\sigma(A)_{ij} = \sigma(A_{ij})$. It can be shown that Γ is a lattice ([62, Section 17.3]). Define a Lie algebra automorphism Φ on $\mathfrak{L}(G)$ by

$$\Phi(X_1) = \lambda_1 X_1, \quad \Phi(Y_1) = \lambda_1^2 Y_1, \quad \Phi(Z_1) = \lambda_1^3 Z_1,$$

$$\Phi(X_2) = \lambda_1^{-1} X_2, \quad \Phi(Y_2) = \lambda_1^{-2} Y_2, \quad \Phi(Z_2) = \lambda_1^{-3} Z_2,$$

where $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$. There exists a unique automorphism $F \colon G \to G$ with $dF_{|_{\mathrm{Id}}} = \Phi$. Since λ_1 and λ_2 are units in $\mathbb{K}$, that is integers whose inverses are also integers, and $\sigma(\lambda_1) = \lambda_2$ we have $F(\Gamma) = \Gamma$. Thus, F projects to an Anosov diffeomorphism f of $\Gamma \backslash G$.

The invariant splitting for f is $T(\Gamma\backslash G) = E^s \oplus E^u$, where E^s is the 3-dimensional distribution generated by X_2, Y_2 and Z_2 and E^u is the 3-dimensional distribution generated by X_1, Y_1 and Z_1. Observe that $E^u = P \oplus Q$ where P is the 2-dimensional distribution generated by X_1, Y_1 and Q is the 1-dimensional distribution generated by Z_1. The distribution P is intermediate and is not integrable. To see this note that the generators X_1, Y_1 and Z_1 induce three vector fields x_1, y_1 and z_1 on $g \in \Gamma\backslash G$ such that $x_1(g), y_1(g) \in P(g)$ and $z_1(g) \in Q(g)$ for any $g \in \Gamma\backslash G$. Since the distribution P is smooth, by the Frobenius theorem, its integrability would imply that the Lie bracket $[x_1, y_1]$ of vector fields x_1 and y_1 lies in P, contrary to $[X_1, Y_1] = Z_1$.

It follows from Theorem 4.9 that nonintegrability in this example is an open property.

5.2. **Invariant families of local manifolds.** Positive results are available, however. After all, if $1 < k < t$ then the intermediate distribution E_k, is the central distribution in the splitting

$$TM = \Big(\bigoplus_{j=1}^{k-1} E_j\Big) \oplus E_k \oplus \Big(\bigoplus_{j=k+1}^{t} E_j\Big),$$

so we can apply results of Subsection 4.3 and Subsection 4.4.

First, Theorem 4.10 gives the class of smoothness of the leaves of the foliation W_k when E_k is integrable:

Theorem 5.1. *With the notations of* (3.2) *and* (3.3)*, suppose η_k and m_k are the largest integers such that $\mu_{k-1} < \lambda_k^{\eta_k}$ and $\mu_k < \lambda_{k+1}^{m_k}$, respectively, and let $n_k = \min\{\eta_k, m_k\}$. If E_k is integrable then the leaves of the corresponding intermediate invariant foliation W_k are C^{n_k}.*

Note that the assumptions are closely related to those of Theorem 3.3, but the conclusion is complementary. The present result asserts smoothness of leaves, whereas Theorem 3.3 is about smooth dependence of the leaves on a base point (when those leaves are known to be as smooth as the diffeomorphism by Theorem 3.2).

As to robustness of the integral foliation W_k, we wish to apply Theorem 4.15. This requires the additional assumption that $\lambda_{k+1} > 1$, which means that f is an Anosov diffeomorphism.

Theorem 5.2. *Assume W_k is plaque expansive (e.g., smooth) and that $\lambda_{k+1} > 1$. Let g be a C^q diffeomorphism sufficiently close to f in the C^1 topology. By Theorem 2.16, g possesses an invariant distribution $(E_k)_g$ corresponding to E_k. This distribution is integrable and the corresponding foliation $(W_k)_g$ is plaque expansive.*

Since the diffeomorphism f in the last theorem is Anosov, so is g. By the structural stability theorem, f and g are topologically conjugate by a Hölder homeomorphism h which is close to the identity map. It follows that $h(W_k)$ is a g-invariant foliation whose leaves are Hölder continuous submanifolds. Theorem 5.2 shows that the leaves of this foliation are indeed smooth of class C^ℓ.[3]

Even if the distribution E_k is integrable its leaves may not be C^q. To explain this phenomenon consider the linear map $A(x,y) = (\lambda x, \mu y)$ of the plane, where $0 < \lambda < \mu < 1$. The origin is an attracting fixed point. The x-axis can be geometrically characterized as consisting of points P for which

$$\rho(0, A^n P) \leq \lambda^n \rho(0, P).$$

On the other hand any curve $\gamma_C = \{(x,y) : x = Cy^{\log\lambda/\log\mu}\}$ is invariant under A and consists of points P for which

$$\rho(0, A^n P) \leq \mu^n \rho(0, P).$$

Note that for $\log\lambda/\log\mu \notin \mathbb{N}$ (nonresonance) these curves are only finitely differentiable except for the y-axis (corresponding to $C = 0$). Therefore, there is no "obvious" choice of a local leaf and it seems unlikely that the intermediate foliation will happen to include the leaf that is infinitely differentiable.

However, if $\mu_k < 1$ and some special *nonresonance* condition holds, smooth leaves are realizable: the distribution E_k admits an invariant family of local manifolds $\{V_k(x)\}_{x\in M}$ which are as smooth as the map f is—but they may not constitute a foliation.

Theorem 5.3 (Pesin [71]). *Fix k such that $0 < \lambda_k \leq \mu_k < 1$ and assume the* nonresonance condition $N := [\log\lambda_1/\log\mu_k] + 1 \leq q$ *and if* $j = 1, \dots, N$, $1 \leq i < k$ *then* $[(\lambda_k)^j, (\mu_k)^j] \cap [\lambda_i, \mu_i] = \varnothing$.

Then for every $x \in M$ there exists a local submanifold $V_k(x)$ such that:

(1) $x \in V_k(x)$ and $T_x V_k(x) = E_k(x)$;
(2) $f(V_k(x)) \subset V_k(f(x))$;
(3) $V_k(x) \in C^q$;
(4) for any $x \in M$ the collection of local manifolds $\{V_k(x)\}_{x\in M}$ is the only collection of C^N local manifolds that satisfies $T_x V_k(x) = E_k(x)$, $f(V_k(x)) \subset V_k(f(x))$, and

$$\sup_{1\leq s\leq N}\sup_{x\in M} \|d^s V_k(x)\| \leq \textit{const}.$$

[3] $h(W_k)$ is an integral foliation because by a lemma of Hirsch–Pugh–Shub, normally hyperbolic manifolds are unique and robust in the C^0-topology.

Remark 5.4. The nonintegrable intermediate distribution P for the diffeomorphism in Subsection 5.1 does not satisfy the nonresonance condition.

5.3. **Lack of smoothness of the intermediate foliations.** The following example illustrates the possible lack of smoothness of leaves for intermediate distributions. Consider an automorphism A of the torus $\mathbb{T}^3$ with eigenvalues λ_i, $i = 1, 2, 3$, such that $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$. We have an invariant splitting

$$T\mathbb{T}^3 = \bigoplus_{i=1}^{3} E_{i,A}.$$

Assume $\log \lambda_1 / \log \lambda_2 \notin \mathbb{Z}$ (nonresonance), and let $N = [\log \lambda_1 / \log \lambda_2] + 1$.

Consider the foliation $W_{2,A}$ associated to $E_{2,A}$. By Theorem 5.1 any C^∞ diffeomorphism f sufficiently C^1-close to A possesses an invariant foliation $W_{2,f}$ tangent to $E_{2,f}$ and with C^{N-1} leaves. In general, the leaves $W_{2,f}(x)$ cannot be more than C^{N-1} smooth for a "large" set of points $x \in M$. Hence, they are different from the local submanifolds given by the preceding theorem, since these submanifolds are of class C^N (indeed, of class C^∞ in this particular case).

Theorem 5.5 (Jiang, de la Llave, Pesin [61])**.** *In any neighborhood η of A in the space $\mathrm{Diff}^1(\mathbb{T}^3)$ there exists $G \in \eta$ such that*

(1) G is a C^∞ diffeomorphism and topologically conjugate to A;
(2) G admits an invariant splitting

$$T\mathbb{T}^3 = \bigoplus_{i=1}^{3} E_{i,G}$$

with $E_{i,G}$ close to $E_{i,A}$ and integrable; the integral manifold $W_{i,G}(x)$ passing through x is of class C^{N-1} but not C^N for some $x \in \mathbb{T}^3$;
(3) the set of points $\{x \mid W_{i,G}(x)$ is not of class $C^N\}$ is a residual subset of $\mathbb{T}^3$.

6. Failure of absolute continuity

Let W be a foliation of M with smooth leaves and $V(x)$, $x \in M$, the local leaf passing through x. In our discussion of the stable and unstable foliations in Section 3 we discussed the question of absolute continuity:

> If $E \subset B(x, q)$ is a Borel set of positive volume, can the intersection $E \cap V(y)$ have zero Lebesgue measure (with respect to the Riemannian volume on $V(y)$) for almost every $y \in E$?

6.1. **An example of a foliation that is not absolutely continuous.** We describe a scheme due to Katok for producing partially hyperbolic maps whose central foliation fails to be absolutely continuous in the strongest possible way: there is a set of full measure that intersects each leaf of the foliation in at most one point. This phenomenon is known as "Fubini's nightmare" since the Fubini theorem fails with respect to this foliation in the strongest possible way. (An example of this construction on an annulus was widely circulated from 1992 [33], and in 1997 a version on the square was published [66]). We thank Keith Burns for providing the presentation rendered here.

Let A be the hyperbolic automorphism of the torus $\mathbb{T}^2$ defined by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

There is a family $\{f_t \mid t \in [0,1]\}$ of diffeomorphisms preserving the area m and satisfying the following conditions:

(1) f_t is a small perturbation of A for every $t \in [0,1]$;
(2) f_t depends smoothly on t;
(3) $l'(t) \neq 0$, where $l(t)$ is the larger eigenvalue of the derivative of f_t at its fixed point.

The diffeomorphisms f_t are all Anosov, conjugate to A, and ergodic with respect to m. For any s and t in $[0,1]$, the maps f_s and f_t are conjugate via a unique homeomorphism h_{st} close to the identity, i.e., $f_t = h_{st} \circ f_s \circ h_{st}^{-1}$. The homeomorphism h_{st} is Hölder continuous. Let m_{st} be the pushforward of m by h_{st}. Then m_{st} is an ergodic invariant measure for f_t. Using the condition on $l(t)$ and the following lemma, we see that $m \neq m_{st}$ unless $s = t$.

Lemma 6.1 (de la Llave [64])**.** *Suppose $f, g\colon \mathbb{T}^2 \to \mathbb{T}^2$ are smooth area-preserving Anosov diffeomorphisms that are conjugate via an area-preserving homeomorphism h. Let p be a periodic point for f with least period k. Then $Df^k(p)$ and $Dg^k(h(p))$ have the same eigenvalues up to sign.*

Proof. Let λ and λ' be the eigenvalues of $Df^k(p)$ and $Dg^k(h(p))$ respectively that lie inside the unit circle. Since f and g are area-preserving, the other eigenvalues of $Df^k(p)$ and $Dg^k(h(p))$ are $1/\lambda$ and $1/\lambda'$ respectively. Choose $x \in W^u_{loc}(p; f) \smallsetminus \{p\}$ and $y \in W^s_{loc}(p; f) \smallsetminus \{p\}$ that are not equal to p. Let R_n be the smallest "rectangle" bounded by (parts of) $W^s_{loc}(p; f)$, $W^s_{loc}(f^{-kn}(x); f)$, $W^u_{loc}(p; f)$, and $W^u_{loc}(f^{kn}(y); f)$. Let R'_n be the smallest "rectangle" bounded by (parts of) $W^s_{loc}(h(p); g)$, $W^s_{loc}(h(f^{-kn}(x)); g)$,

$W^u_{loc}(h(p);g)$, and $W^u_{loc}(h(f^{kn}(y));g)$. Then

$$\lim_{n\to\infty}\frac{\text{area}(R_{n+1})}{\text{area}(R_n)}=\lambda^{2k}\quad\text{and}\quad\lim_{n\to\infty}\frac{\text{area}(R'_{n+1})}{\text{area}(R_n)}={\lambda'}^{2k}.$$

On the other hand, the conjugacy h takes R_n to R'_n for any n. Since h is area-preserving, it follows that $\lambda=\pm\lambda'$. □

A point is *generic* with respect to an invariant measure if the forward and backward Birkhoff averages of any continuous function are defined at the point and are equal to integral of the function with respect to the measure. If x is generic for f_s with respect to m, then $h_{st}(x)$ is generic for f_t with respect to m_{st} and hence is not generic for f_t with respect to m, unless $s=t$. (To see this, note that the Birkhoff averages of a continuous function φ along the f_t-orbit of $h_{st}(x)$ are the same as the Birkhoff averages of $\varphi\circ h_{st}$ along the f_s orbit of x.)

Now consider the diffeomorphism $F\colon\mathbb{T}^2\times[0,1]\to\mathbb{T}^2\times[0,1]$ given by $F(x,t)=(f_t(x),t)$. We have just observed that for any $x\in\mathbb{T}^2$ the set $H(x)=\{(h_{0t}(x),t)\mid t\in[0,1]\}$ contains at most one element of the set $\mathcal{G}$ of points $(y,t)\in\mathbb{T}^2\times[0,1]$ such that y is generic for f_t with respect to m.

Now, F is a small perturbation of $A\times id_{[0,1]}$ and thus partially hyperbolic. It follows from Theorem 4.11 that F has a center foliation whose leaves are small perturbations of the intervals $\{x\}\times[0,1]$ for $x\in\mathbb{T}^2$. Since F maps the tori $\mathbb{T}^2\times\{t\}$ into themselves, it is easily seen that the leaves of W^c_F are ℓ-normally hyperbolic for any ℓ, and hence are C^∞ by Theorem 4.10. On the other hand, for each $x\in\mathbb{T}^2$, the leaf of W^c_F that passes through $(x,0)\in\mathbb{T}^2\times[0,1]$ is $H(x)$.

The set $\mathcal{G}$ of generic points for F has full measure with respect to m in each torus $\mathbb{T}^2\times\{t\}$ and hence has full Lebesgue measure in $\mathbb{T}^2\times[0,1]$, but, as observed above, it intersects each center leaf in at most one point.

To construct an analogous example on $\mathbb{T}^2\times S^1$ use two periodic points simultaneously instead of the one fixed point. The example here is constructed in such a way that $l(t)=l(s)\Rightarrow t=s$. For a continuous parametrization using $t\in S^1$ this won't work, but starting from the map A^2 instead, which has several fixed points, we use perturbations for which the largest eigenvalues $l_1(t)$ and $l_2(t)$ at two fixed points $x_1(t)$ and $x_2(t)$ satisfy $l_1(t)=l_1(s)$ and $l_2(t)=l_2(s)\Rightarrow t=s\pmod 1$. For example, make $l'_1(t)>0$ on $(0,1/2)$, $l'_1(t)<0$ on $(1/2,1)$ and $l'_2(t)=0$ on $(0,1/2)$, $l'_1(t)>0$ on $(1/2,3/4)$, $l'_2(t)<0$ on $(3/4,1)$.

6.2. **Pathological foliations.** We saw earlier that even in terms of existence, uniqueness and smoothness of leaves the central foliation is a rather more delicate entity than the members of the stable and unstable filtrations, and the preceding example shows that if there is a central foliation at all it may fail to be absolutely continuous. It turns out that this is not at all exceptional.

Let A be an area-preserving linear hyperbolic automorphism of the 2-dimensional torus $\mathbb{T}^2$. Consider the map $F = A \times \mathrm{Id}$ of the 3-dimensional torus $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$. Any sufficiently small C^1 perturbation G of F is uniformly partially hyperbolic with 1-dimensional central distribution. The latter is integrable to a continuous foliation W^c of M with compact leaves (they are diffeomorphic to S^1; this foliation can be shown to be Hölder continuous [80]). There is a perturbation G of F which preserves volume and has nonzero Lyapunov exponents in the central direction [84] (see also [41]). In this case the central foliation is not absolutely continuous: for almost every $x \in M$ the conditional measure (generated by the Riemannian volume) on the leaf $W^c(x)$ of the central foliation passing through x has finite support [81].

We describe a more general version of this result. Let (X, ν) be a probability space and $f\colon X \to X$ an invertible transformation that preserves the measure ν and is ergodic with respect to ν. Let M be an n-dimensional smooth compact Riemannian manifold and $\varphi\colon X \to \mathrm{Diff}^{1+\alpha}(M)$. Assume that the skew-product transformation

$$F\colon X \times M \to X \times M,\ F(x, y) = (f(x), \varphi_x(y))$$

is Borel measurable and possesses an invariant ergodic measure μ on $X \times M$ such that $\pi_*\mu = \nu$, where $\pi\colon X \times M \to X$ is the projection.

For $x \in X$ and $k \in \mathbb{Z}$ define $\varphi_x^{(k)}\colon M \to M$ by

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)},$$

where $\varphi_x^{(0)} = \mathrm{Id}$. Since the tangent bundle to M is measurably trivial the derivative map of φ along the M direction gives a cocycle

$$\mathcal{A}\colon X \times M \times \mathbb{Z} \to GL(n, \mathbb{R}),$$

where $\mathcal{A}(x, y, k) = d_y\varphi_x^{(k)}$. If $\log^+ \|d\varphi\| \in L^1(X \times M, \mu)$ then the Multiplicative Ergodic Theorem and ergodicity of μ imply that the Lyapunov exponents $\chi_1 < \cdots < \chi_\ell$ of this cocycle are constant for μ-almost every (x, y).

Theorem 6.2 (Ruelle, Wilkinson [81])**.** *If for some $\gamma > 0$ the function φ satisfies*

$$\log^+ \|d\varphi\|_\gamma \in L^1(X, \nu), \tag{6.1}$$

where $\|\cdot\|_\gamma$ is the γ-Hölder norm, and if $\chi_\ell < 0$ then there exists a set $S \subset X \times M$ of full measure and $k \in \mathbb{N}$ such that $\mathrm{card}(S \cap (\{x\} \times M)) = k$ for almost every $x \in X$.

This phenomenon is rather typical.

Definition 6.3. A partially hyperbolic diffeomorphism that preserves a smooth measure is said to have *negative central exponents* if the Lyapunov exponents in the central direction are negative almost everywhere.

Conjecture 6.4. *The central foliation of a "typical" partially hyperbolic diffeomorphism with negative central exponents is not absolutely continuous.*

Mañé proved (unpublished) that if the central foliation is one-dimensional and has compact leaves then this foliation is not absolutely continuous provided the Lyapunov exponent in the central direction is nonzero on a set of positive measure.[4] Hirayama and Pesin [54] showed that the central foliation is not absolutely continuous if it has compact leaves and f is "central dissipative", i.e., the sum of the central exponents is nonzero on a set of positive measure (here, negative, positive or zero exponents can be present). Note that partially hyperbolic central dissipative diffeomorphisms whose central foliation has compact leaves form an open set in the space of C^1 diffeomorphisms and that any partially hyperbolic diffeomorphism whose central foliation has compact leaves can be perturbed to become central dissipative.

This motivates the question whether one can perturb a partially hyperbolic system with all central Lyapunov exponents zero to a system with negative central exponents. This has been shown to be true in some particular cases (see [8, 42, 12, 15]) but remains unknown otherwise.

Conjecture 6.5. *Given a partially hyperbolic dynamical system f whose central Lyapunov exponents are zero, there exists a partially hyperbolic dynamical system with negative central exponents arbitrarily close to f.*

7. Accessibility and Stable Accessibility

We now begin our study of the ergodic theory of partially hyperbolic dynamical systems. The strategy for establishing ergodicity is based on suitable extensions of the Hopf argument [57], see also [62, p. 217], and we describe it here in order to explain the main object of the present section.

The Hopf argument establishes ergodicity of a uniformly hyperbolic diffeomorphism as follows. By the Birkhoff Ergodic Theorem, ergodicity means that for every L^1-function φ (by L^1-density φ is without loss of generality continuous, hence uniformly continuous by compactness) the time averages or Birkhoff averages $\varphi_n := \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^n$ converge to a constant a.e. Uniform continuity of φ and the contraction of stable leaves imply that

[4] We thank A. Wilkinson for providing us with this information

the limit function is constant on stable leaves, and likewise for "backwards" time averages (obtained analogously from f^{-1}) on unstable leaves. Since the Birkhoff Ergodic Theorem implies that the forward and backward limits exist and agree a.e., one deduces that these are constant a.e. from the fact that this holds on stable and unstable leaves separately, using absolute continuity: "Almost everywhere on almost every leaf" is the same as "almost everywhere".

For partially hyperbolic dynamical systems the same argument can be attempted, but first of all, one cannot use all three foliations because the Hopf argument relies on contraction in either forward or backward time to conclude that an invariant function is constant on leaves. The center foliation lacks this feature (and may, moreover, fail to be absolutely continuous as we have seen, which would cause problems in the later stage of the argument). On the other hand, in this case it is not clear that any two nearby points have a heteroclinic point. Put differently, in the hyperbolic situation one can join any two nearby points by a path consisting of two short segments, one each in a stable and an unstable leaf. (We call such a path a us-path.) This may not be the case in a partially hyperbolic system, as one sees, for example, in the case of cartesian products of a hyperbolic dynamical system with the identity, which are evidently not ergodic. More to the point, joint integrability of the stable and unstable foliations limits these connections to pairs of points that lie in the same joint stable-unstable leaf. This motivates interest in how joint integrability can fail. It is conceivable, for example, that there are situations in which the foliations are jointly integrable in some places but not in others, or cases in which they are not jointly integrable but nevertheless both subordinate to a common foliation the dimension of whose leaves is larger than the sum of stable and unstable dimensions. Whether these are possible is not very well understood, and the question of which of these situations may occur in examples is of interest in its own right.

In terms of salvaging the Hopf argument, say, it would be natural to make the assumption that any two nearby points can be joined by a us-path consisting of two short segments in a stable and unstable leaf, respectively ("accessibility by a us-path with short legs"). This should be relatively easy to use. Under the name of "local transitivity" it was imposed by Brin and Pesin [30], but it turned out to be too restrictive to be widely applicable. Therefore one wishes to explore weaker assumptions that are still strong enough to yield topological or measurable irreducibility. One can relax this assumption by allowing "long legs", i.e., by requiring only that two nearby points be connected by a us-path whose stable and unstable pieces may be rather long. On the other hand, one may allow the connection to be established by a path consisting of a multitude of pieces that lie alternatingly in stable and unstable leaves. If one simultaneously drops the requirement that the legs be short, one obtains the notion of accessibility that is now in use.

While it is intuitive to present this notion in terms of paths, and these are used in proving topological transitivity, they are not employed in proofs of ergodicity. The most obvious technical difficulty with these would be that the transition points between stable and unstable segments must have the same forward and backward Birkhoff averages for the function at hand in order for the Hopf argument to work. But this may be tricky to arrange. Therefore one argues directly with the algebras of sets in the proofs of ergodicity, as explained in the next section.

7.1. **The Accessibility property.**

Definition 7.1. Let f be a partially hyperbolic diffeomorphism of a compact Riemannian manifold M.

Two points $p, q \in M$ are said to be *accessible*, if there are points $z_i \in M$ with $z_0 = p$, $z_\ell = q$, such that $z_i \in V^\alpha(z_{i-1})$ for $i = 1, \dots, \ell$ and $\alpha = s$ or u. The collection of points $z_0, z_1, \dots, z_\ell$ is called the us-*path* connecting p and q and is denoted variously by $[p, q]_f = [p, q] = [z_0, z_1, \dots, z_\ell]$. (Note that there is an actual path from p to q that consists of pieces of smooth curves on local stable or unstable manifolds with the z_i as endpoints.)

Accessibility is an equivalence relation and the collection of points accessible from a given point p is called the *accessibility class* of p.

A diffeomorphism f is said to have the *accessibility property* if the accessibility class of any point is the whole manifold M, or, in other words, if any two points are accessible.

If f has the accessibility property then the distribution $E^s \oplus E^u$ is not integrable (and therefore, the stable and unstable foliations, W^s and W^u, are not jointly integrable). Otherwise, the accessibility class of any $p \in M$ would be the leaf of the corresponding foliation passing through p.

There is a weaker version of accessibility which provides a useful tool in studying topological transitivity of f.

Definition 7.2. Given $\varepsilon > 0$, we say that f is ε-*accessible* if for every open ball B of radius ε the union of accessibility classes passing through B is M.

An equivalent requirement is that the accessibility class of any point should enter every open ball of radius ε, i.e., be ε-dense. Clearly, if f is accessible then it is ε-accessible for any ε. It is not hard to check that a perturbation of an accessible dynamical system is ε-accessible:

Proposition 7.3. *If a partially hyperbolic diffeomorphism f has the accessibility property and $\varepsilon > 0$ then*

(1) *there exist $\ell > 0$ and $R > 0$ such that for any $p, q \in M$ one can find a us-path that starts at p, ends within distance $\varepsilon/2$ of q, and has at most ℓ legs, each of them with length at most R;*
(2) *there exists a neighborhood $\mathcal{U}$ of f in the space $\mathrm{Diff}^2(M)$ such that every $g \in \mathcal{U}$ is ε-accessible.*

Often, an "almost-everywhere" accessibility notion is adequate:

Definition 7.4. We say that f has the *essential accessibility property* if the partition of M by the accessibility classes is trivial in the measure-theoretical sense, i.e., any measurable set that consists of accessibility classes has measure zero or one.

7.2. **Accessibility and topological transitivity.** It is not hard to see that accessibility plus volume-preservation produce a fair amount of recurrence.

Definition 7.5 ([3])**.** Given $\varepsilon > 0$ we say that an orbit is ε*-dense* if the points of the orbit form an ε-net. Clearly, a trajectory $\{f^n(x)\}_{n\in\mathbb{Z}}$ is everywhere dense in M if and only if it is ε-dense for every $\varepsilon > 0$.

We say that a point $x \in M$ is *forward* (respectively, *backward*) *recurrent* if for any $r > 0$ there exists $n > 0$ (respectively, $n < 0$) such that $f^n(x) \in B(x, r)$. If a point x is forward (respectively, backward) recurrent then for any $r > 0$ there exists a sequence $n_k \to +\infty$ (respectively, $n_k \to -\infty$) such that $f^{n_k}(x) \in B(x, r)$.

Theorem 7.6 (Burns, Dolgopyat, Pesin [32])**.** *If a partially hyperbolic diffeomorphism f is ε-accessible and preserves a smooth measure then almost every orbit of f is ε-dense.*

Proof. Fix an open ball B of radius ε. Say that a point is *good* if it has a neighborhood of which almost every point has an iterate in B. We must show that every $p \in M$ is good.

Fix $p \in M$. Since f is ε-accessible, there is a us-path $[z_0, \dots, z_k]$ with $z_0 \in B$ and $z_k = p$. Then z_0 is good, and we show by induction on j that each z_j is good.

If z_j has a neighborhood N such that $\mathcal{O}(x) \cap B \neq \varnothing$ for almost every $x \in N$ let S be the subset of N consisting of points with this property that are also both forward and backward recurrent. By the Poincaré Recurrence Theorem [3, Theorem 3.4.1], S has full measure in N. If $x \in S$ and $y \in W^s(x) \cup W^u(x)$ then $\mathcal{O}(y) \cap B \neq \varnothing$. The absolute continuity of the foliations W^s and W^u means that $\bigcup_{x\in S}(W^s(x) \cup W^u(x))$ has full measure in the set $\bigcup_{x\in N}(W^s(x) \cup W^u(x))$, which is a neighborhood of z_{j+1}. $\square$

Corollary 7.7 (Brin [23])**.** Let f be a partially hyperbolic diffeomorphism of a compact Riemannian manifold M that preserves a smooth measure on

M and has the accessibility property. Then for almost every point $x \in M$ the trajectory $\{f^n(x)\}_{n\in\mathbb{Z}}$ is dense in M. In particular, f is topologically transitive.

Remark 7.8. In fact, Brin proved this using only that every point is nonwandering. This holds in particular when the map preserves a smooth measure as well as when periodic points are dense.

One can relax accessibility to essential accessibility:

Theorem 7.9 (Burns, Dolgopyat, Pesin [32]). *If a partially hyperbolic diffeomorphism f is essentially accessible and preserves a smooth measure then it is topologically transitive.*

The assumption that f preserves a smooth measure cannot be dropped in general.

Theorem 7.10 (Nitică, Török [67]). *Consider $F = f \times Id\colon M \times S^1 \to M \times S^1$ where f is a C^1 Anosov diffeomorphism of M. There exists a C^1 neighborhood of F whose elements are accessible but not topologically transitive.*

Proof. By Theorem 7.12 below (see also Theorem 7.13) there is a C^1-open and C^1-dense set of accessible C^1-small perturbations of F, so it suffices to construct an open set of nontransitive diffeomorphisms. Choose $h \in \mathrm{Diff}^1(S^1)$ as close to the identity as desired with h having an attracting fixed point. There are open neighborhoods $U, V \subset S^1$ of this point with $h(\bar{U}) \subset V \subset \bar{V} \subset U$. If $g := f \times h$ then $g(M \times \bar{U}) \subset M \times V$ and any map that is C^0 close to g has the same property. Note that such a transformation is not topologically transitive because each positive semiorbit has at most one element in the open set $M \times (U \smallsetminus \bar{V})$. $\square$

7.3. **Stability of accessibility.** Accessibility allows one to salvage the Hopf argument for ergodicity. Since we are also interested in stable ergodicity, it is natural to begin by looking at stable accessibility.

Definition 7.11. A diffeomorphism f is said to be *stably accessible* if there exists a neighborhood $\mathcal{U}$ of f in the space $\mathrm{Diff}^1(M)$ (or in the space $\mathrm{Diff}^1(M, \nu)$ where ν is an f-invariant Borel probability measure) such that any diffeomorphism $g \in \mathcal{U}$ has the accessibility property.

7.3.1. *General theory.* The study of stable accessibility is based on the *quadrilateral argument* first introduced by Brin [24]. Roughly speaking it goes as follows (we assume for simplicity that the central distribution E^c is integrable). Given a point $p \in M$, consider a 4-legged us-path $[z_0, z_1, z_2, z_3, z_4]$ originating at $z_0 = p$. We connect z_{i-1} with z_i by a geodesic γ_i lying in the corresponding stable or unstable manifold and we obtain the curve $\Gamma_p = \bigcup_{1\le i\le 4} \gamma_i$. We parameterize it by $t \in [0,1]$ with $\Gamma_p(0) = p$.

If the distribution $E^s \oplus E^u$ were integrable (and hence, the accessibility property for f would fail) the endpoint $z_4 = \Gamma_p(1)$ would lie on the leaf of the corresponding foliation passing through p. Therefore, one can hope to achieve accessibility if one can arrange a 4-legged us-path in such a way that $\Gamma_p(1) \in W^c(p)$ and $\Gamma_p(1) \neq p$. In this case the path Γ_p can be homotoped through 4-legged us-paths originating at p to the trivial path so that the endpoints stay in $W^c(p)$ during the homotopy and form a continuous curve. Such a situation is usually persistent under small perturbations of f and hence leads to stable accessibility.

We note that all current applications of stable accessibility are to dynamically coherent systems.

The first substantial result is that the accessibility property is C^1 generic in the space of partially hyperbolic diffeomorphisms, volume-preserving or not.

Theorem 7.12 (Dolgopyat, Wilkinson [43]). *Let $q \ge 1$, $f \in \mathrm{Diff}^q(M)$ (or $f \in \mathrm{Diff}^q(M,\nu)$, where ν is a smooth invariant measure on M) be partially hyperbolic. Then for every neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ (respectively, $\mathcal{U} \subset \mathrm{Diff}^1(M,\nu)$) of f there exists a C^q diffeomorphism $g \in \mathcal{U}$ which is stably accessible.*

An outline of the proof of this theorem in the special case when the central distribution E^c is 1-dimensional and integrable can be found in [72].

In the special case when the partially hyperbolic diffeomorphism has 1-dimensional center bundle, accessibility can be shown to be an open dense property in the space of diffeomorphisms of class C^2 (see [39]).

7.3.2. *Results in special cases.* Theorem 7.12 can be improved in some special cases. In the remainder of this subsection we consider skew products over Anosov diffeomorphisms satisfying (2.12), time-t maps of suspension flows and group extensions over Anosov diffeomorphisms. These systems are partially hyperbolic and hence so are small perturbations. Their central distribution is integrable and the corresponding central foliation has compact smooth leaves. The proofs of accessibility exploit various versions of Brin's quadrilateral argument, and outlines can be found in [72].

7.3.3. *Skew products over Anosov diffeomorphisms.* In the context of Subsubsection 2.2.4 we get

Theorem 7.13 (Nitică, Török [67]). *If M is a connected manifold then there is a neighborhood of F in $\mathrm{Diff}^q(M \times S^1)$ or $\mathrm{Diff}^q(M \times S^1, \nu \times m)$ in which stable accessibility is open and dense.*

7.3.4. *Suspension flows.*

Theorem 7.14 (Brin [26], Talitskaya [88], [72]). *Let T_t be the suspension flow (see* [3, Sections 1.3j, 2.2j, 5.2j, 6.5d]*) over a C^q Anosov diffeomorphism with roof function $H: M \to \mathbb{R}^+$. There exists an open and dense set $\mathcal{U}$ of C^q functions $H: M \to \mathbb{R}^+$ such that the suspension flow T_t is stably accessible.*

7.3.5. *Group extensions.* Let G be a compact connected Lie group, $f: M \to M$ a C^q Anosov diffeomorphism, and $\varphi: M \to G$ a C^q function. Consider the G-extension

$$F = F_\varphi : M \times G \to M \times G, \quad F_\varphi(x, y) = (f(x), \varphi(x)y)$$

of f. See Subsection 2.2.

Theorem 7.15 (Brin [24], Burns, Wilkinson [37]). *For every neighborhood $\mathcal{U} \subset C^q(M, G)$ of φ there is a $\psi \in \mathcal{U}$ such that F_ψ is stably accessible. In other words, stably accessible group extensions are dense in the space of C^q group extensions over the Anosov diffeomorphism f.*

7.3.6. *Time-t maps of an Anosov flow.* Let φ_t be an Anosov flow on a compact smooth Riemannian manifold M. It turns out that stable accessibility of the time-1 diffeomorphism depends on whether the distribution $E^s \oplus E^u$ is integrable, i.e., whether the stable and unstable foliations, W^s and W^u, of the time-1 map are jointly integrable. First, let us comment on joint integrability.

Fix $\varepsilon > 0$. Given a point $x \in M$, consider a local smooth submanifold

$$\Pi(x) = \bigcup_{y \in B^u(x,\varepsilon)} \bigcup_{-\varepsilon \le \tau \le \varepsilon} \varphi_\tau(y)$$

through x. For $x, x' \in M$ let $\pi_{x,x'}: \Pi(x) \to \Pi(x')$ be the holonomy map generated by the family of local stable manifolds. The foliations W^s and W^u are *jointly integrable* if for every $y \in \Pi(x)$ the image of the local unstable leaf $V^u(y)$ under $\pi_{x,x'}$ is the local unstable leaf $V^u(\pi_{x,x'}(y))$.

Theorem 7.16 (Burns, Pugh, Wilkinson [35]). *Assume the stable and unstable foliations of the flow are not jointly integrable. Then the time-1 map φ_1 is stably accessible.*

By verifying the hypotheses of Theorem 7.16 one can establish stable accessibility of the time-1 map for

(1) geodesic flows on negatively curved manifolds (more generally, contact flows; Katok, Kononenko, [63]);
(2) C^2 volume-preserving flows on compact 3-manifolds that are not special flows with a constant height function (Burns, Pugh, Wilkinson, [35]).

We close this section with two conjectures about accessibility.

Conjecture 7.17. *A partially hyperbolic dynamical system with the accessibility property is stably accessible.*

This conjecture fails if one replaces accessibility by essential accessibility due to an example by Brin [36].

Conjecture 7.18. *The space of stably accessible partially hyperbolic dynamical systems is open and dense in the C^r topology for any $r \geq 1$. (This is known for $r = 1$ by* [43]*.)*

8. The Pugh–Shub Ergodicity Theory

8.1. **Conditions for ergodicity.** Let f be a C^2 diffeomorphism of a smooth compact Riemannian manifold M that is partially hyperbolic and that preserves a smooth measure ν. To study ergodicity of f one uses a version of the Hopf argument [2, 62, 31, 72] adapted to the case of partially hyperbolic systems.

Let $\mathcal{B}$ be the Borel σ-algebra of M. Say that $x, y \in M$ are stably equivalent if

$$\rho(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty,$$

and unstably equivalent if

$$\rho(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty.$$

Stable and unstable equivalence classes induce two partitions of M, and we denote by $\mathcal{S}$ and $\mathcal{U}$, the Borel σ-algebras they generate. Recall that for an algebra $\mathcal{A} \subset \mathcal{B}$ its saturated algebra is the set

$$\operatorname{Sat}(\mathcal{A}) = \{B \in \mathcal{B} : \text{ there exists } A \in \mathcal{A} \text{ with } \nu(A \triangle B) = 0\}.$$

It follows from the Hopf argument that f is ergodic if

$$\operatorname{Sat}(\mathcal{S}) \cap \operatorname{Sat}(\mathcal{U}) = \mathcal{T}, \tag{8.1}$$

where $\mathcal{T}$ is the trivial algebra.

For an Anosov diffeomorphism f the stable equivalence class containing a point x is the leaf $W^s(x)$ of the stable foliation. Similarly, the unstable equivalence class containing x is the leaf $W^u(x)$ of the unstable foliation. The σ-algebra $\mathcal{S}$ consists of those Borel sets S for which $W^s(x) \subset S$ whenever $x \in S$, and the σ-algebra $\mathcal{U}$ consists of those Borel sets U for which $W^u(x) \subset U$ whenever $x \in U$. The relation (8.1) holds by absolute continuity of stable and unstable foliations, which proves ergodicity for Anosov diffeomorphisms.

If a diffeomorphism f is partially hyperbolic the stable and unstable foliations W^s and W^u of M also generate Borel σ-algebras $\mathcal{M}^s$ and $\mathcal{M}^u$, respectively, so $\mathcal{S} \subset \mathcal{M}^s$ and $\mathcal{U} \subset \mathcal{M}^u$ (note that stable and unstable sets containing a point x may be larger than $W^s(x)$ and $W^u(x)$ due to possible contractions and expansions along the central directions). It follows that

$$\mathrm{Sat}(\mathcal{S}) \cap \mathrm{Sat}(\mathcal{U}) \subset \mathrm{Sat}(\mathcal{M}^s) \cap \mathrm{Sat}(\mathcal{M}^u).$$

If f is accessible then $\mathrm{Sat}(\mathcal{M}^s \cap \mathcal{M}^u) = \mathcal{T}$. In fact, essential accessibility (Definition 7.4) is enough. If f is essentially accessible then ergodicity would follow from

$$\mathrm{Sat}(\mathcal{M}^s) \cap \mathrm{Sat}(\mathcal{M}^u) \subset \mathrm{Sat}(\mathcal{M}^s \cap \mathcal{M}^u) \tag{8.2}$$

(the opposite inclusion is obvious). We describe conditions that guarantee this.

Theorem 8.1. *A volume-preserving essentially accessible dynamically coherent partially hyperbolic diffeomorphism with absolutely continuous foliations W^{cs} and W^{cu} (see Definition 4.4 and Subsection 3.2) is ergodic.*

Proof. (8.2) follows from the conditions of the theorem. □

The assumption that the foliations W^{cs} and W^{cu} are absolutely continuous is very strong. It holds for example, when the center foliation is smooth (more generally, Lipschitz continuous). However, “typically” the center foliation is not even absolutely continuous (see Subsection 6.2). Here is a more “practical” yet still technical assumption.

Definition 8.2 ([38])**.** We say that f is *center-bunched* if $\max\{\mu_1, \lambda_3^{-1}\} < \lambda_2/\mu_2$ in (2.7).

This definition due to Burns and Wilkinson imposes a much weaker constraint than earlier versions; they show in [38] that this assumption suffices to get the following:

Theorem 8.3. *A partially hyperbolic (essentially) accessible dynamically coherent center-bunched diffeomorphism is ergodic.*

Grayson, Pugh Shub [45] proved this theorem for small perturbations of the time one map of the geodesic flow on a surface of constant negative curvature. Wilkinson in her thesis extended their result to small perturbations of the time-one map of the geodesic flow on an arbitrary surface of negative curvature. Then Pugh and Shub in [77, 78] proved the theorem assuming a stronger center bunching condition. The proof of the theorem in the form stated here (with a weaker center bunching condition) was obtained by Burns and Wilkinson in [38].

The way to establish (8.2) without absolute continuity of the center-stable and center-unstable foliation is through the use of a collection of special sets at every point $x \in M$ called *juliennes*, $J_n(x)$ (they resemble slivered vegetables). We shall describe a construction of these sets which assures that

(J1) $J_n(x)$ form a basis of the topology.

(J2) $J_n(x)$ form a basis of the Borel σ-algebra. More precisely, let Z be a Borel set; a point $x \in Z$ is said to be *julienne dense* if

$$\lim_{n\to+\infty} \frac{\nu(J_n(x)\cap Z)}{\nu(J_n(x))} = 1.$$

Let $D(Z)$ be the set of all julienne dense points of Z. Then

$$D(Z) = Z \pmod 0.$$

(J3) If $Z \in Sat(\mathcal{M}^s) \cap Sat(\mathcal{M}^u)$, then $\mathcal{D}(Z) \in Sat(\mathcal{M}^s \cap \mathcal{M}^u)$.

Properties (J1)–(J3) imply (8.2).

Note that the collection of balls $B(x, 1/n)$ satisfies requirements (J1) and (J2) but not (J3). Juliennes can be viewed as balls "distorted" by the dynamics in the following sense. Fix an integer $n \geq 0$, a point $x \in M$ and numbers τ, σ such that $0 < \tau < \sigma < 1$. Denote by

$$B_n^s(x,\tau) = \{y \in W^s(x) \mid \rho(f^{-k}(x), f^{-k}(y)) \leq \tau^k\},$$
$$B_n^u(x,\tau) = \{y \in W^u(x) \mid \rho(f^{k}(x), f^{k}(y)) \leq \tau^k\}.$$

and define the julienne

$$J_n(x) := [J_n^{cs}(x) \times B_n^u(x,\tau)] \cap [B_n^s(x,\tau) \times J_n^{cu}(x)],$$

where the local foliation products

$$J_n^{cs}(x) = B_n^s(x,\tau) \times B^c(x,\sigma^n), \quad J_n^{cu}(x) = B_n^u(x,\tau) \times B^c(x,\sigma^n)$$

are the *center stable* and *center unstable juliennes*, and $B^c(x,\sigma^n)$ is the ball in $W^c(x)$ centered at x of radius σ^n. One may think of $J_n(x)$ as a substitute for $B_n^s(x,\tau) \times B^c(x,\sigma^n) \times B_n^u(x,\tau)$, which is only well-defined if the stable and unstable foliations are jointly integrable.

The proof of (J1)–(J3) is based on the the following properties of juliennes:

(1) *scaling*: If $k \geq 0$ then $\nu(J_n(x))/\nu(J_{n+k}(x))$ is bounded, uniformly in $n \in \mathbb{N}$;

(2) *engulfing*: there is $\ell \geq 0$ such that, for any $x, y \in M$, if $J_{n+\ell}(x) \cap J_{n+\ell}(y) \neq \varnothing$ then $J_{n+\ell}(x) \cup J_{n+\ell}(y) \subset J_n(x)$;

(3) *quasi-conformality*: there is $k \geq 0$ such that if $x, y \in M$ are connected by an arc on an unstable manifold that has length ≤ 1 then the holonomy map $\pi : V^{cs}(x) \to V^{cs}(y)$ generated by the family of local unstable manifolds (see Subsection 3.2) satisfies $J_{n+k}^{cs}(y) \subset \pi(J_n^{cs}(x)) \subset J_{n-k}^{cs}(y)$.

The properties (1) and (2) are possessed by the family of balls in Euclidean space and they underlie the proof of the Lebesgue Density Theorem. One can use these properties to show that juliennes are density bases. The center-unstable juliennes are a density basis on $W^{cu}(x)$ with respect to the smooth conditional measure $\nu_{W^{cu}}$ on $W^{cu}(x)$, the center-stable juliennes are a density basis on $W^{cs}(x)$ with respect to the smooth conditional measure $\nu_{W^{cs}}$ on $W^{cs}(x)$, and the juliennes are a density basis on M with respect to the smooth measure ν.

Juliennes, $J_n(x)$, are small but highly eccentric sets in the sense that the ratio of their diameter to their inner diameter increases with n (the inner diameter of a set is the diameter of the largest ball it contains). In general, sets of such shape may not form density bases, but juliennes do because their elongation and eccentricity are controlled by the dynamics; in particular, they nest in a way similar to balls.

Quasi-conformality is what is needed to prove Property (J3). Roughly speaking it means that the holonomy map (almost) preserves the shape of juliennes.

Conjecture 8.4. *A partially hyperbolic dynamical system preserving a smooth measure and with the accessibility property is ergodic.*

8.2. **The Pugh–Shub Stable-Ergodicity Theorem.**

Definition 8.5. Let $q \geq 1$. A C^q diffeomorphism f of a compact C^q Riemannian manifold M preserving a smooth measure ν is said to be *stably ergodic* if any C^1-small perturbation of f preserving ν is ergodic.

Stable ergodicity imposes some conditions on the map. Bochi, Fayad and Pujals [15] showed that there is an open and dense set of $C^{1+\alpha}$ stably ergodic (with respect to a smooth measure) diffeomorphisms with nonzero Lyapunov exponents (this answers a problem posed in [32]).

A stably ergodic diffeomorphism f need not be partially hyperbolic, Tahzibi, [87]. However, it possesses a *dominated splitting*, i.e., the tangent space splits into two invariant subspaces E and F such that for $n \in \mathbb{N}$,

$$\|df^n \restriction E(x)\| \|df^{-n} \restriction F(f^n(x))\| \leq C\lambda^n$$

with uniform $C > 0$ and $0 < \lambda < 1$. This was proved by Arbieto and Matheus assuming that f is $C^{1+\epsilon}$ and volume preserving ([11]). On the other hand if f is a symplectic stably ergodic diffeomorphism then f must be partially hyperbolic [58].

To establish stable ergodicity of a partially hyperbolic diffeomorphism f one can check whether the hypotheses of Theorem 8.3 are stable under small perturbations.

(1) If f is dynamically coherent and the central foliation W^c is of class C^1 then every diffeomorphism g which is sufficiently close to f

in the C^1 topology is dynamically coherent (see Theorem 4.11 and Theorem 4.15).

(2) If f is center-bunched then every diffeomorphism g which is sufficiently close to f in the C^1 topology is center-bunched.

Thus, for dynamically coherent center-bunched partially hyperbolic diffeomorphisms, stable ergodicity follows from stable accessibility:

Theorem 8.6 ([78, 79])**.** *A dynamically coherent center-bunched stably (essentially) accessible partially hyperbolic diffeomorphism that preserves a smooth measure ν and has a smooth or plaque-expansive center foliation is stably ergodic (and stably K* [62, Sections 3.6k, 3.7j], [37, Corollary 1.2]*).*

Conjecture 8.7. *A partially hyperbolic dynamical system preserving a smooth measure and with the accessibility property is stably ergodic. (This would follow from Conjecture 7.17 and Conjecture 8.4.)*

Conjecture 8.8. *Stably ergodic diffeomorphisms are open and dense in the space of C^r partially hyperbolic dynamical systems for $r \geq 1$. (This would follow from Conjecture 7.18 and Conjecture 8.4.)*

Combining Theorem 8.6 with the results in Subsection 7.3 we obtain several classes of stably ergodic systems:

8.2.1. *Skew product maps over Anosov diffeomorphisms.* If $F = f \times \mathrm{Id} : M \times S^1 \to M \times S^1$ then there is a neighborhood $\mathcal{U}$ of F in $\mathrm{Diff}^2(M \times S^1)$ or $\mathrm{Diff}^2(M \times S^1, \nu \times m)$ such that stable ergodicity is open and dense in $\mathcal{U}$ (here m is the length).

8.2.2. *Suspension flows over Anosov diffeomorphisms.* There exists an open and dense set of C^q functions $H\colon M \to \mathbb{R}^+$ such that the suspension flow T_t with the roof function H is stably ergodic. Field, Melbourne and Török [47] strengthened this result.

Theorem 8.9. *For $r > 0$, there exists a C^r open and dense subset $\mathcal{A}$ in the space of strictly positive C^r (roof) functions such that for every $H \in \mathcal{A}$ the suspension flow T_t with the roof function H is stably mixing. If $r \geq 2$ then $\mathcal{A}$ is open in the C^2 topology and C^∞ roof functions are $C^{[r]}$ dense in $\mathcal{A}$.*

8.2.3. *Group extensions over Anosov diffeomorphisms.* if $F_\varphi\colon M \times G \to M \times G$ is a group extension then for every neighborhood $\mathcal{U} \subset C^q(M, G)$ of the function φ there exists a function $\psi \in \mathcal{U}$ such that the diffeomorphism F_ψ is stably ergodic.

Burns and Wilkinson obtained a complete characterization of stable ergodicity for group extensions over volume-preserving Anosov diffeomorphisms. Namely, we say that the map $h : M \times Y \to M \times Y$ of class $C^{q+\alpha}$ is an *algebraic factor* of the $C^{q+\alpha}$ group extension F_φ if $Y = H \backslash G$, where H

is a closed subgroup of G, and there exists a $C^{q+\alpha}$ function $\Phi\colon M \to G/H$ for which the following diagram is commutative

$$\begin{array}{ccc} M \times G & \xrightarrow{F_\varphi} & M \times G \\ {\scriptstyle \pi_\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\Phi} \\ M \times H\backslash G & \xrightarrow{h} & M \times H\backslash G \end{array}$$

where $\pi_\Phi(x,g) = (x, \Phi(x)^{-1}g)$ (and $\Phi(x)^{-1} = \{g^{-1} | g \in \Phi(x)\}$ is an element of $H\backslash G$).

Theorem 8.10. *Let $f\colon M \to M$ be a $C^{q+\alpha}$ volume-preserving Anosov diffeomorphism of an infranilmanifold, G a compact, connected Lie group and $\varphi\colon M \to G$ a $C^{q+\alpha}$ map. If the group extension F_φ is not stably ergodic then it has an algebraic factor $h\colon M \times H\backslash G \to M \times H\backslash G$, where one of the following holds:*

(1) $H \neq G$, and h is the product of f with $\mathrm{Id}_{H\backslash G}$;
(2) h is normal, $H\backslash G$ is a circle, and h is the product of f with a rotation;
(3) h is normal, $H\backslash G$ is a d-torus, and $h = f_\psi$ where ψ is homotopic to a constant and maps M into a coset of a lower dimensional Lie subgroup of the d-torus.

If F_φ has an algebraic factor of type (1), it is not ergodic; if F_φ has an algebraic factor of type (2) but none of type (1) then it is ergodic, but not weakly mixing; otherwise F_φ is Bernoulli. In addition, F_φ is stably ergodic if and only if it is stably ergodic within skew products.

Applying this result to the case when the group G is semisimple, one can show that F_φ is stably ergodic if and only if it is ergodic.

Field, Melbourne and Török [47] studied stable ergodicity of group extensions over hyperbolic sets and generalized earlier results in [6, 48, 69, 89]. Let f be a C^2 diffeomorphism of a compact smooth manifold M possessing a locally maximal hyperbolic set Λ which is not a periodic orbit. Let μ be the unique equilibrium measure on Λ corresponding to a Hölder continuous potential (so $f|\Lambda$ is ergodic with respect to μ). Consider a compact connected Lie group G with the Haar measure m.

Theorem 8.11. *For $r > 0$ there exists a C^r open and dense subset $\mathcal{A} \subset C^r(M, G)$ such that for every $\varphi \in \mathcal{A}$ the group extension F_φ is ergodic with respect to the measure $\nu = \mu \times m$. If $f|\Lambda$ is topologically mixing then F_φ is mixing with respect to ν.*

In other words, if $f|\Lambda$ is topologically transitive (respectively, topologically mixing) then the stably ergodic (respectively stably mixing) group extensions form an open and dense set in the space of C^r group extensions for any $r > 0$.

8.2.4. *Time-t maps of Anosov flows.* If the stable and unstable foliations of an Anosov flow are not jointly integrable then the time-t map for $t \neq 0$ is stably ergodic.

Theorem 7.16 provides a strong dichotomy between joint integrability of the strong foliations and (stable) accessibility. The paper in which this theorem is proved also produces a clean dichotomy between joint integrability and (stable) ergodicity:

Theorem 8.12 (Burns, Pugh, Wilkinson [35]). *The time-one map of a volume-preserving Anosov flow is stably ergodic unless the strong stable and strong unstable foliations for the flow are jointly integrable.*

The proof follows the line of argument in [45] and uses the crucial fact that the holonomy maps are not just continuous but indeed Hölder continuous with the Hölder exponent close to 1 (see [80]) so that these maps do not distort juliennes too much.

In the special case of flows on 3-manifolds one can strengthen this result and show that *the time-one map is stably ergodic if and only if the flow is not a suspension flow over an Anosov diffeomorphism with a constant roof function*. In particular, *the time-one map of a volume-preserving topologically mixing C^2 Anosov flow is stably ergodic*. As a corollary one has that the time-one map of geodesic flows on a closed negatively curved Riemannian surface is stably ergodic (this result was earlier obtained by Wilkinson [90]).

8.2.5. *Frame flows.* There are also several cases in which the frame flow and its time-t maps are known to be ergodic [34]:

Theorem 8.13. *Let Φ_t be the frame flow on an n-dimensional compact smooth Riemannian manifold with sectional curvatures between $-\Lambda^2$ and $-\lambda^2$. Then in each of the following cases the flow is ergodic, K (*[62, Sections 3.6k, 3.7j], [5, Section 4.3]*), and even Bernoulli (*[5, Sections 6–7]*), and the time-one map of the frame flow is stably ergodic and stably K:*

(1) if the curvature is constant (Brin, [23]*);*
(2) for a set of metrics of negative curvature which is open and dense in the C^3 topology (Brin, [23]*);*
(3) if n is odd and $n \neq 7$ (Brin and Gromov, [28]*);*
(4) if n is even, $n \neq 8$, and $\lambda/\Lambda > 0.93$ (Brin and Karcher, [29]*)*
(5) if $n = 7$ and $\lambda/\Lambda > 0.99023\dots$ (Burns and Pollicott, [34]*);*
(6) if $n = 8$ and $\lambda/\Lambda > 0.99023\dots$ (Burns and Pollicott, [34]*).*

Ergodicity of the frame flow was proved by the authors cited in each case; [34] pointed out the K and Bernoulli property and used [37, Corollary 1.2] (which relies on [30]) to deduce those of the time-1-maps across all cases.

8.3. Ergodicity and stable ergodicity for toral automorphisms. Theorem 8.13 has as a particular consequence that the time-one map of the frame flow of a manifold with negative curvature is stably ergodic in all cases where it is known to be ergodic. At the end of [56] Hirsch, Pugh and Shub posed a question that might be interpreted as asking whether every ergodic automorphism of the n-torus is stably ergodic. In this context ergodicity is easily characterized by the property that the automorphism has no eigenvalue that is a root of unity [50].

The dissertation of Rodriguez Hertz [53] answers the question in the affirmative for dimension up to 5:

Theorem 8.14 (Rodriguez Hertz [53]). *Every ergodic linear automorphism of a torus of dimension up to 5 is stably ergodic. (But for dimension 4 only with respect to C^{22}-perturbations.)*

This result arises, in fact, as a consequence of rather more general ones.

Definition 8.15. An automorphism of $\mathbb{T}^n$ none of whose eigenvalues is a root of the unity and whose characteristic polynomial is irreducible over the integers and not a polynomial in t^i for any $i \geq 2$ is said to be a pseudo-Anosov automorphism.

Theorem 8.16 (Rodriguez Hertz [53]). *If $n \geq 6$ then any pseudo-Anosov automorphism of $\mathbb{T}^n$ with $\dim E^c = 2$ is stably ergodic with respect to the C^5-topology, and if $n = 4$ then any pseudo-Anosov automorphism of $\mathbb{T}^n$ is stably ergodic with respect to the C^{22}-topology.*

Rodriguez Hertz derives Theorem 8.14 from Theorem 8.16 by showing that ergodic automorphisms of $\mathbb{T}^4$ are either Anosov or pseudo-Anosov and ergodic automorphisms of $\mathbb{T}^5$ are Anosov (and hence clearly stably ergodic). In fact, the odd-dimensional case is much simplified by his observation that if n is odd and $A \in SL(n, \mathbb{Z})$ has irreducible characteristic polynomial then A is Anosov. The remaining substance of the work therefore lies in the cases $n = 4$ and $n \geq 6$, in each of which Rodriguez Hertz studies a dichotomy concerning accessibility. He considers the accessibility classes (lifted to the universal cover) for such a perturbation and shows that these are either all trivial (they intersect each stable leaf in a point) or else must all be equal to $\mathbb{R}^n$ [53, Theorem 5.1]. The latter implies accessibility, and in the former case an application of KAM-theory (or, for $n = 4$, a separate theorem of Moser) then establishes smooth conjugacy of the foliations to those of the linear system, which yields essential accessibility. Rodriguez Hertz can then apply Theorem 8.3 in either case.

His result prompted Pugh and Shub to make their earlier question more explicit in the following form:

Problem ([79])**.** Is every ergodic toral automorphism stably ergodic in the C^r-topology for some r?

9. Partially Hyperbolic Attractors

A partially hyperbolic set Λ for a diffeomorphism f of a compact manifold M is called a *partially hyperbolic attractor* if there is a neighborhood U of Λ such that $\overline{f(U)} \subset U$ and

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

An important property of a partially hyperbolic attractor is as follows.

Theorem 9.1. $W^u(x) \subset \Lambda$ *for every* $x \in \Lambda$.

Any diffeomorphism g sufficiently close to f possesses a partially hyperbolic attractor which lies in a small neighborhood of Λ.

An invariant Borel probability measure μ on Λ is said to be a u*-measure* if the conditional measures $\mu^u(x)$ generated by μ on local unstable leaves $V^u(x)$ are absolutely continuous with respect to the Riemannian volume on $V^u(x)$.

Consider a smooth measure ν on U with the density function ψ with respect to the Riemannian volume m, i.e.,

$$\text{supp}\ \psi \subset U, \qquad \int_U \psi\, dm = 1.$$

The sequence of measures

$$\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i \nu$$

is the evolution of the measure ν under the system f. Even if the sequence ν_n does not converge, any limit measure μ is supported on Λ.

Theorem 9.2 (Pesin, Sinai, [73])**.** *Any limit measure of the sequence of measures* ν_n *is an* f*-invariant* u*-measure on* Λ.

We describe another approach for constructing u-measures on Λ. For $x \in \Lambda$ and $y \in V^u(x)$ consider the function

$$\kappa(x,y) = \prod_{i=0}^{n-1} \frac{J(df \restriction E^u(f^i(y)))}{J(df \restriction E^u(f^i(x)))}.$$

Define the probability measure $\widetilde{m}_n$ on $V_n(x) = f^n(V^u(x))$ by

$$d\widetilde{m}_n(y) = c_n\kappa(f^n(x), y)dm_{V_n(x)}, \quad \text{for } y \in V_n(x),$$

where c_n is normalizing factor and $m_{V_n(x)}$ is the Riemannian volume on $V_n(x)$ induced by the Riemannian metric. We define the Borel measure m_n on Λ by

$$m_n(A) = \widetilde{m}_n(A \cap V_n(x)),$$

when $A \subset \Lambda$ is a Borel set. One can show that $m_n(A) = m_0(f^{-n}(A))$.

Theorem 9.3 (Pesin, Sinai, [73])**.** *Any limit measure of the sequence of measures m_n is an f-invariant u-measure on Λ.*

While Theorem 9.2 describes u-measures as a result of the evolution of an absolutely continuous measure in a neighborhood of the attractor, Theorem 9.3 determines u-measures as limit measures for the evolution of an absolutely continuous measure supported on a local unstable manifold. One can deduce Theorem 9.2 from Theorem 9.3 and the proof of the latter, presented in [73], exploits a method which allows one to avoid the use of Markov partitions – the classical tool to prove existence of SRB-measures for hyperbolic attractor (in general, a partially hyperbolic attractor does not have any Markov partition).

Assume now that the unstable distribution E^u splits into the sum of two invariant subdistributions $E^u = E_1 \oplus E_2$ with E_1 expanding more rapidly than E_2. One can view f as a partially hyperbolic diffeomorphism with E_1 as the new unstable distribution (and $E_2 \oplus E^c$ as the new center distribution) and construct u-measures, associated with this distribution, according to Theorem 9.3.

Theorem 9.4. *Any u-measure associated with the distribution E^u is a u-measure associated with the distribution E_1.*

The proof of this theorem can be easily obtained from the fact that the leaves of the $W_1(y)$ depend smoothly on $y \in W^u(x)$, see Theorem 3.3.

If Λ is a hyperbolic attractor and $f \restriction \Lambda$ is topologically transitive then there exists a unique u-measure. This may not be true for a general partially hyperbolic attractor and some additional strong conditions are necessary to guarantee uniqueness.

Let us denote by $\chi(x, v)$ the Lyapunov exponent at the point $x \in \Lambda$ and the vector $v \in T_xM$.

Let ν be an invariant Borel probability measure on Λ. We say that Λ has *negative central exponents* with respect to ν if there exists a set $A \subset \Lambda$ of positive measure such that $\chi(x, v) < 0$ for every $x \in A$ and $v \in E^c(x)$.

Theorem 9.5 (Burns, Dolgopyat and Pesin, [32]). *If ν is a u-measure with negative central exponents then every ergodic component of $f \restriction A$ of positive measure is open* $\pmod 0$.

Using this result one can provide conditions which guarantee uniqueness or at most finite number of u-measures.

Theorem 9.6 (Burns, Dolgopyat and Pesin, [32]). *Assume that there exists a u-measure ν with negative central exponents. Assume, in addition, that for almost every $x \in \Lambda$ the trajectory $\{f^n(x)\}$ is everywhere dense in Λ. Then $f \restriction \Lambda$ is ergodic with respect to ν.*

One can show that if for every $x \in \Lambda$ the global strongly unstable manifold $W^u(x)$ is dense then almost every orbit is dense. Moreover, under this assumption there is a unique u-measure which is also an SRB-measure for f.

Theorem 9.7 (Bonatti and Viana [20]). *Let f be a C^2 diffeomorphism possessing a partially hyperbolic attractor Λ. Assume that for every $x \in \Lambda$ and every disk $D^u(x) \subset W^u(x)$ centered at x, we have that $\chi(y, v) < 0$ for a positive Lebesgue measure subset of points $y \in D^u$ and every vector $v \in E^c(y)$. Then f has at most finitely many u-measures.*

Theorem 9.8 (Alves, Bonatti and Viana [8]). *Assume that f is nonuniformly expanding along the center-unstable direction, i.e.,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|df^{-1} \restriction E^{cu}_{f^j(x)}\| < 0 \tag{9.1}$$

for all x in a positive Lebesgue measure set $A \subset M$. Then f has an ergodic SRB-measure supported in $\bigcap_{j=0}^{\infty} f^j(M)$. Moreover, if the limit in (9.1) is bounded away from zero then A is contained $\pmod 0$ *in the union of the basins of finitely many SRB-measures.*

Let $\Lambda = \Lambda_f$ be a partially hyperbolic attractor for f. It is well-known that any C^1 diffeomorphism g which is sufficiently close to f in the C^1 topology possesses a partially hyperbolic attractor Λ_g which lies in a small neighborhood of Λ_f. The following statement shows that u-measures depend continuously on the perturbation.

Theorem 9.9 (Dolgopyat, [41]). *Let f_n be a sequence of C^2 diffeomorphisms converging to a diffeomorphism f in the C^2 topology. Let also ν_n be a u-measure for f_n. Assume that the sequence of measures ν_n converges in the weak topology to a measure ν. Then ν is a u-measure for f.*

The following statement describes a version of stable ergodicity for partially hyperbolic attractors.

Theorem 9.10. *Assume that there exist a u-measure $\nu = \nu_f$ for f with negative central exponents. Assume also that for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f. Then any C^2 diffeomorphism g which is sufficiently close to f also has negative central exponents on a set that has positive measure with respect to a u-measure ν_g; this measure is the only SRB measure for g and $g \restriction \Lambda_g$ is ergodic with respect to ν_g.*

The following statement provides conditions which guarantee uniqueness of u-measures.

Theorem 9.11 (Bonatti and Viana [20])**.** *Let f be a C^2 diffeomorphism possessing a partially hyperbolic attractor Λ. Assume that*

(1) there exist $x \in \Lambda$ and a disk $D^u(x) \subset W^u(x)$ centered at x for which $\chi(y, v) < 0$ for a positive Lebesgue measure subset of points $y \in D^u$ and every vector $v \in E^c(y)$;

(2) every leaf of the foliation W^u is dense in Λ.

Then f has a unique u-measure and it is ergodic. The support of this measure coincides with Λ.

The measure ν in this theorem is a Sinai-Ruelle-Bowen (SRB) measure (for the definition and some relevant results on SRB-measures see [1, Section 14]).

REFERENCES

[**Surveys in this and the companion volume]**

[1] L. Barreira, Y. Pesin *Smooth ergodic theory and nonuniformly hyperbolic dynamics*

[2] Boris Hasselblatt, *Hyperbolic dynamical systems*

[3] Boris Hasselblatt, Anatole Katok, *Principal Structures*

[4] E. Pujals, M. Sambarino,

[5] J.-P. Thouvenot, *Entropy, isomorphism and equivalence in ergodic theory*

[**Other references]**

[6] R. Adler, B. Kitchens and M. Shub, *Stably ergodic skew products*, Discrete Contin. Dynam. Systems **2** (1996), no. 3, 349–350.

[7] V. Alekseev, *Quasirandom dynamical systems. I. Quasirandom diffeomorphisms*, Mat. Sb. (N.S.) **76 (118)** (1968), 72–134; *Invariant Markov subsets of diffeomorphisms*, Uspehi Mat. Nauk **23** (1968), no. 2 (140), 209–210.

[8] J. F. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math. **140** (2000), no. 2, 351–398

[9] D. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. ,**90** (1969), 1–235.

[10] D. Anosov, Y. Sinai, *Certain smooth ergodic systems*, Russian Math. Surveys, **22** (1967), 103–167.

[11] A. Arbieto, C. Matheus, *A pasting lemma I: the case of vector fields*. Preprint IMPA 2003.

[12] A. Baraviera and C. Bonatti, *Removing zero Lyapunov exponents*, Ergodic Theory Dynam. Systems **23** (2003), no. 6, 1655–1670.

[13] L. Barreira, Y. Pesin *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series, 23, AMS, Providence, RI, 2001.

[14] J. Bochi, B. Fayad and E. Pujals, *Stable ergodicity implies Bernoulli*, preprint

[15] J. Bochi, B. Fayad and E. Pujals, *A remark on conservative diffeomorphisms*, preprint, 1993

[16] C. Bonatti, L. J. Díaz and E. R. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, Ann. of Math. (2) **158** (2003), no. 2, 355–418

[17] C. Bonatti and L. J. Díaz and M. Viana", *Dynamics beyond uniform hyperbolicity: A global geometric and probabilistic perspective*, Encyclopedia Math. Sciences 102, Springer Verlag, 2004

[18] C. Bonatti, J. Franks, *A Hölder continuous vector field tangent to many foliations*. In *Modern Dynamical Systems and Applications*, B. Hasselblatt, M. Brin, Y. Pesin, eds., Cambridge University Press, New York, 2004, pp. 299-306.

[19] C. Bonatti, C. Matheus, M. Viana, A. Wilkinson, *Abundance of stable ergodicity*, Commentarii Mathematici Helvetici, to appear

[20] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math. **115** (2000), 157–193.

[21] C. Bonatti and A. Wilkinson, *Transitive partially hyperbolic diffeomorphisms on 3-manifolds*, preprint 2003

[22] M. Brin, *Hölder continuity of invariant distributions*, in *Smooth ergodic theory and its applications*, A. Katok, R. de la Llave, Y. Pesin and H. Weiss eds., Proc. Symp. Pure Math., Amer. Math. Soc., 2001.

[23] M. Brin, *Topological transitivity of one class of dynamical systems and flows of frames on manifolds of negative curvature*, Func. Anal. Appl., **9** (1975), 9–19.

[24] M. Brin, *The topology of group extensions of C systems*, Mat Zametki, **18** (1975), 453–465.

[25] M. Brin, *On dynamical coherence*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 395–401.

[26] M. Brin, , PhD thesis, Moscow, 1975.

[27] M. Brin, D. Burago, S. Ivanov, *On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group*. In *Modern Dynamical Systems and Applications*, B. Hasselblatt, M. Brin, Y. Pesin, eds., Cambridge University Press, New York, 2004, pp. 307–312.

[28] M. Brin, M. Gromov, *On the ergodicity of frame flows*, Invent. Math., **60** (1980), 1–7.

[29] M. Brin, H. Karcher, *Frame flows on manifolds of negative curvature*, Compos. Math., **52** (1984), 275–297.

[30] M. Brin, Y. Pesin, *Partially hyperbolic dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 170–212.

[31] M. Brin, G. Stuck, *Dynamical Systems*, Cambridge University Press, 2002

[32] K. Burns, D. Dolgopyat, Y. Pesin, *Partial hyperbolicity, Lyapunov exponents and stable ergodicity*, J. Stat. Phys., **109** (2002), 927–942.

[33] K. Burns, L. Flaminio, *Fubini's nightmare*, unpublished note, 1992

[34] K. Burns, M. Pollicott, *Stable ergodicity and frame flows*, Geom. Dedicata 98 (2003), 189–210.

[35] K. Burns, C. Pugh, A. Wilkinson, *Stable ergodicity and Anosov flows*, Topology, **39** (2000), 149–159.

[36] K. Burns, C. Pugh, M. Shub, A. Wilkinson, *Recent results about stable ergodicity*, in *Smooth ergodic theory and its applications*, A. Katok, R. de la Llave, Y. Pesin and H. Weiss eds., Proc. Symp. Pure Math., Amer. Math. Soc., 2001.

[37] K. Burns, A. Wilkinson, *Stable ergodicity of skew products*, Ann. Sci. École Norm. Sup., **32** (1999), 859–889.

[38] K. Burns, A. Wilkinson, *Better center bunching*, preprint, 2004.

[39] Ph. Didier, *Stability of accessibility*, Ergodic Theory Dynam. Systems **23** (2003), no. 6, 1717–1731.

[40] L. J. Díaz, E. R. Pujals and R. Ures, *Partial hyperbolicity and robust transitivity*, Acta Math. **183** (1999), no. 1, 1–43

[41] D. Dolgopyat, *On dynamics of partially hyperbolic systems on three manifolds*, Preprint (1999).

[42] D. Dolgopyat, Y. Pesin *Every Compact Manifold Carries a Completely Hyperbolic Diffeomorphism*, Ergod. Theory and Dyn. Syst., **22** (2002), 1–27.

[43] D. Dolgopyat, A. Wilkinson, *Stable accessibility is C^1 dense*, Geometric methods in dynamics. II. Astérisque No. 287 (2003), xvii, 33–60.

[44] A. S. Gorodetskiĭ and Yu. S. Ilyashenko, *Some new robust properties of invariant sets and attractors of dynamical systems*, Funktsional. Anal. i Prilozhen. **33** (1999), no. 2, 16–30, 95; translation in Funct. Anal. Appl. **33** (1999), no. 2, 95–105

[45] M. Grayson, C. Pugh, M. Shub, *Stably ergodic diffeomorphisms*, Ann. Math., **140** (1994), 295–329.

[46] N. Fenichel, *Persistence and smoothness of invariant manifolds of slows*, Indiana Univ. Math. J., **21** (1971–72), 193–226.

[47] M. Field, I. Melbourne and A. Török, *Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets* Ergodic Theory Dynam. Systems, to appear

[48] M. Field and W. Parry, *Stable ergodicity of skew extensions by compact Lie groups*, Topology **38** (1999), no. 1, 167–187.

[49] J. Hadamard, *Sur l'itération et les solutions asymptotiques des équations différentielles*, Bull. Soc. Math. France **29** (1901), 224–228.

[50] P. Halmos, *On automorphisms of compact groups*, Bull. Amer. Math. Soc. **49** (1943), 619–624.

[51] B. Hasselblatt, *Regularity of the Anosov splitting and of Horospheric foliations*, Ergod. Theory and Dyn. Syst., **14** (1994), no. 46, 45–666

[52] Boris Hasselblatt and Amie Wilkinson, *Prevalence of non-Lipschitz Anosov foliations*, Ergodic Theory and Dynamical Systems **19** (1999), no. 3, 643–656

[53] F. Rodriguez Hertz, *Stable ergodicity of certain linear automorphisms of the torus*, preprint, 2002, arXiv.org/abs/math/0212345

[54] M. Hirayama and Ya. Pesin, *Non-absolutely continuous invariant foliations*, preprint PSU 2005

[55] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Bull. Amer. Math. Soc., **76** (1970), 1015–1019.

[56] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lecture Notes in Math., **583**, Springer-Verlag, 1977.

[57] E. Hopf: *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **91** (1939), 261–304.

[58] V. Horita, A. Tahzibi, *Partial hyperbolicity for symplectic diffeomorphisms*, Preprint, 2004

[59] Yu. Ilyashenko and W. Li, *Nonlocal bifurcations*, Amer. Math. Soc., Providence, RI, 1999

[60] M. Irwin, *On the stable manifold theorem*, Bull. London Math. Soc., **2** (1970), 196–198.

[61] M. Jiang, R. de Llave, Y. Pesin, *On the integrability of intermediate distributions for Anosov diffeomorphisms*, Ergod. Theory and Dyn. Syst., **15** (1995), no. 2, 317–331.

[62] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge, 1995.

[63] A. Katok, A. Kononenko, *Cocycle stability for partially hyperbolic systems*, Math. Res. Letters, **3** (1996), 191–210.

[64] Rafael de la Llave, *Smooth conjugacy and S-R-B measures for uniformly and nonuniformly hyperbolic systems*, Communications in Mathematical Physics **150** (1992), 289–320

[65] J. Mather, *Characterization of Anosov diffeomorphisms*, Nederl. Akad. Wetensch. Proc. Ser. A71, Indag. Math., **30** (1968), 479–483.

[66] J. Milnor, *Fubini foiled: Katok's paradoxical example in measure theory*, Math. Intelligencer **19** (1997), 30–32.

[67] V. Niticǎ, A. Török, *An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one*, Topology, **40** (2001), 259–278.

[68] D. Orendovici, Y. Pesin *Chaos in traveling waves of lattice systems of unbounded media*, in *Numerical methods for bifurcation problems and large-scale dynamical systems (Minneapolis, MN, 1997)*, 327–358, Springer, New York.

[69] W. Parry and M. Pollicott, *Stability of mixing for toral extensions of hyperbolic sets*, Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 354–363; translation in Proc. Steklov Inst. Math. **1997**, no. 1 (216), 350–359.

[70] O. Perron, *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systemes endlicher Differenzengleichungen*, J. Reine Angew. Math. **161** (1929), 41–64.

[71] Y. Pesin, *On the existence of invariant fiberings for a diffeomorphism of a smooth manifold*, Matem. Sbornik, **91** (1973), no. 2, 202–210.

[72] Y. Pesin, *Lectures on Partial Hyperbolicity and Stable Ergodicity*, Zürich Lectures in Advanced Mathematics, EMS, 2004

[73] Y. Pesin, Y. Sinai, *Gibbs measures for partially hyperbolic attractors*, Ergod. Theory and Dyn. Syst., **2** (1982), no. 3-4, 417–438.

[74] J. Plante, *Anosov flows*, Amer. J. Math., **94** (1972), 729–754.

[75] C. Pugh, M. Shub, *Ergodicity of Anosov actions*, Invent. Math., **15** (1972), 1–23.

[76] C. Pugh, M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc., **312** (1989), no. 1, 1–54.

[77] C. Pugh, M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. Complexity, **13** (1997), no. 1, 125–179.

[78] C. Pugh, M. Shub, *Stable ergodicity and julienne quasi-conformality*, J. Eur. Math. Soc. (JEMS) **2** (2000), no. 1, 1–52; **6** (2004), no. 1, 149–151.

[79] C. Pugh, M. Shub, *Stable ergodicity*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 1, 1–41

[80] C. Pugh, M. Shub, A. Wilkinson *Hölder foliations*, Duke Math. J. **86** (1997), no. 3 (1989), 517–546; *Correction to: "Hölder foliations"*, Duke Math. J. **105** (2000), 105-106.

[81] D. Ruelle, A. Wilkinson, *Absolutely singular dynamical foliations*, Comm. Math. Phys., **219** (2001), 481–487.
[82] R. Sacker, *A perturbation theorem for invariant Riemannian manifolds*, in *Differential Equations and Dynamical Systems (Proc. Internat. Sympos., Mayaguez, P.R., 1965)*, 43–54, Academic Press, New York.
[83] M. Shub, A. Wilkinson, *A stably Bernoullian diffeomorphism that is not Anosov*, Preprint (1998).
[84] M. Shub, A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math. 139 (2000), no. 3, 495–508.
[85] M. Shub, A. Wilkinson, *Stably ergodic approximation: two examples*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 875–893.
[86] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc., **73** (1967), 747–817.
[87] A. Tahzibi, *Stably ergodic systems which are not partially hyperbolic*, Israel Journal of Mathematics, to appear
[88] A. Talitskaya, *Stable ergodicity of suspension flows*, Preprint, PSU, 2004.
[89] C. Walkden, *Stable ergodic properties of cocycles over hyperbolic attractors*, Comm. Math. Phys. **205** (1999), no. 2, 263–281.
[90] A. Wilkinson, *Stable ergodicity of the time-one map of a geodesic flow*, Ergodic Theory Dynam. Systems **18** (1998), no. 6, 1545–1587.
[91] H. Yamabe, *On an arcwise connected subgroup of a Lie group*, Osaka Math. J. **2** (1950), 13–14.
[92] Jean-Christophe Yoccoz: *Introduction to hyperbolic dynamics*, Real and complex dynamical systems, 265–291. Proceedings of the NATO Advanced Study Institute held in Hillerød, June 20–July 2, 1993. Edited by Bodil Branner and Paul Hjorth. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, **464**. Kluwer Academic Publishers, Dordrecht, 1995

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