ZERO ENTROPY INVARIANT MEASURES FOR SKEW PRODUCT DIFFEOMORPHISMS

PENG SUN

ABSTRACT. In this paper we study some skew product diffeomorphisms with nonuniformly hyperbolic structure along fibers. We show that there is an invariant measure with zero entropy which has atomic conditional measures along fibers. This gives affirmative answer for these diffeomorphisms to the question suggested by Herman whether a smooth diffeomorphism of positive topological entropy fails to be uniquely ergodic. The proof is based on some techniques analogous to those developed by Pesin ([10]) and Katok ([6], [8]) with investigation on some combinatorial properties of the projected return map on the base.

CONTENTS

1. INTRODUCTION

Let f be a $\mathrm{C}^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact s-dimensional smooth manifold M and $\mathrm{d}f : \mathrm{T}M \to \mathrm{T}M$ the derivative of f. f preserves a Borel probability measure μ. For every x in a set Λ of full measure, the Lyapunov exponent

$$\chi(v, f) = \lim_{n\to\infty} \frac{\ln \|\mathrm{d}f^n v\|}{n}$$

exists for every nonzero vector $v \in \mathrm{T}_x M$. This functional takes on at most s values on $\mathrm{T}_x M$ and is independent of $x \in \Lambda$ if μ is ergodic. If all Lyapunov exponents are nonzero, then μ is called a hyperbolic measure. Smooth systems with hyperbolic measures are called nonuniformly hyperbolic. The theory for studying such systems was developed by Pesin and then combined with some powerful techniques by A. Katok to look for invariant orbits and produced a number of profound results. These techniques serve as cornerstones for our discussion. For all necessary definitions, theorems and background facts relevant to this paper, one may see [2] for quick reference or [3] for detailed proofs.

In [6] Katok showed:

Date: March 25, 2009.

Theorem 1.1. *Let f be a $\mathrm{C}^{1+\alpha}(\alpha > 0)$ diffeomorphism of a compact manifold M, and μ a Borel probability f-invariant hyperbolic measure. Then*

$$\overline{Per}(f) \supset \operatorname{supp}(\mu)$$

and

$$\max(0, \limsup_{n\to\infty} \frac{\ln P_n(f)}{n}) \geq h_\mu(f)$$

Where $Per(f)$ is the set of all periodic points of f and $P_n(f)$ the number of periodic points of f with period n. $h_\mu(f)$ is the metric entropy with respect to μ.

In particular, if the manifold M is 2-dimensional, then by Ruelle inequality [11], every ergodic invariant measure μ with positive metric entropy must be hyperbolic. Taking also the variational principle into account, we have

Corollary 1.2. *For any $\mathrm{C}^{1+\alpha}(\alpha > 0)$ diffeomorphism f of a 2-dimensional compact manifold with positive topological entropy,*

$$\limsup_{n\to\infty} \frac{\ln P_n(f)}{n} \geq h(f) \tag{1}$$

Hence f is not minimal or uniquely ergodic.

In general, Equation (1) is not true in high dimensional cases. There can be no periodic orbit for a diffeomorphism with positive topological entropy. Herman [5] constructed a remarkable example as following:

Consider the C^∞ map $A : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ defined by

$$A(\theta) = A_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

where $\lambda > 1$ is a fixed number. Let $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation by $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$.

Theorem 1.3. (Herman, [5]) *There is a dense G_δ subset W of $\mathbb{T}^1$, such that for every $\alpha \in W$, the smooth diffeomorphism $F_\alpha = (R_\alpha, A(\theta))$ on $\mathbb{T}^1 \times \mathrm{SL}(2, \mathbb{R})/\Gamma$, given by $(\theta, y) \mapsto (\theta + \alpha, A(\theta) \cdot y)$, is minimal and has positive topological entropy.*

Herman's example prompted a fruitful research, for example, on generic linear cocycles over compact systems. The phenomenon he discovered turned out to be common for $\mathrm{SL}(2, \mathbb{R})$ extension over rotations [1].

However, the diffeomorphisms in Herman's example fail to be uniquely ergodic. We can find a measurable transformation $S : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ such that for almost every $\theta \in \mathbb{T}^1$, $H_\theta = S_{\theta+\alpha} A_\theta S_\theta^{-1} = \begin{pmatrix} l_\theta & 0 \\ 0 & l_\theta^{-1} \end{pmatrix}$ is diagonal. Then for every measure τ preserved by the geodesic flow which corresponds to the left action by $G_t = \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix}$, $\mu_\tau = \int \tau \circ S_\theta dm$ is F_α-invariant, where m is the Lebesgue measure on $\mathbb{T}^1$. In particular, if τ is supported on a periodic orbit of the geodesic flow, then $h_{\mu_\tau}(F_\alpha) = 0$.

Whether a smooth diffeomorphism of positive topological entropy can be uniquely ergodic is still in question (For homeomorphisms, the answer is yes. See for examples, [4]). We studied some skew product diffeomorphisms and found some invariant measures similar to those in Herman's example.

Let (X, m) be a probability measure space. $g : X \to X$ is an invertible transformation (mod 0) preserving m. M is an l-dimensional compact Riemannian manifold. For every $x \in X$, $h_x : M \to M$ is a $\mathrm{C}^{1+\alpha}$ diffeomorphism. Assume $f = (g, h_x)$ on $X \times M$ preserves a measure $\mu = \int \nu_x dm$. Let $\mathrm{T}_y M = \mathrm{T}_p(\{x\} \times M)$ for $y = (x, p) \in X \times M$. In this paper We prove:

Theorem 1.4. (Main Theorem) *If for almost every $y \in Y$, The Lyapunov exponent $\chi(v, f) \neq 0$ for all $v \in \mathrm{T}_y M \backslash \{0\}$, then f has an invariant measure whose conditional measure on each fiber is atomic.*

From Theorem II in [12], we know if all Lyapunov exponents along fibers have the same sign, then μ itself must be atomic on each fiber. In this paper we mainly discuss the nonuniformly hyperbolic case when the exponents have different signs.

Now suppose that we have a $\mathrm{C}^{1+\alpha}$ diffeomorphism $f = (g, h_x)$ on $X \times M$. Assume that $h(g) = 0$ and M is 2-dimensional. If $h(f) > 0$, we must have $h_\mu(f) > 0$ for some ergodic invariant measure $\mu = \int \nu_x dm$. Then by Ledrappier-Young's formula [9], the Lyapunov exponents along fiber direction must be nonzero almost everywhere. Hence by Theorem 1.4 f has an invariant measure with atomic conditional measures along fibers. The following statement avoids any mention of exponents.

Corollary 1.5. *If f has positive topological entropy, g has zero topological entropy, and M is 2-dimensional, then f has a measure of zero entropy and is not uniquely ergodic.*

2. Shadowing lemma

Now that we have a $\mathrm{C}^{1+\alpha}$ diffeomorphism $f = (g, h_x)$ on $Y = X \times M$ that has nonzero exponents along fibers. We may assume that g is ergodic by considering an ergodic component. Almost all results in [6, 8, 10] can be adapted in this setting with careful modification. By considering the derivative $\mathrm{d}_y h_x = \mathrm{d}_p h_x$ for $y = (x, p)$ as a linear cocycle over f, we have:

Theorem 2.1. *Assume $\dim M = l$. Denote by $B^k(r)$ the standard Euclidean r-ball in $\mathbb{R}^k$ centered at the origin. There exists a set $\Lambda_0 \subset Y$ of full measure such that for every sufficiently small $\epsilon > 0$ and some $\chi > 0$:*

1. *There exists a tempered function $q : \Lambda_0 \to (0, 1]$ and a collection of embeddings $\Psi_y : B^l(q(y)) \to \{x\} \times M$ for each $y = (x, p) \in \Lambda_0$ such that $\Psi_y(0) = y$ and $e^{-\epsilon} < q(y)/q(f(y)) < e^{\epsilon}$.*
2. *There exist a constant $K > 0$ and a measurable function $C : \Lambda_0 \to \mathbb{R}$ such that for $z_1, z_2 \in B^l(q(y))$,*
$$K^{-1} d(\Psi_y(z_1), \Psi_y(z_2)) \leq \|z_1 - z_2\| \leq C(y) d(\Psi(z_1), \Psi(z_2))$$
with $e^{-\epsilon} < C(f(y))/C(y) < e^{\epsilon}$.
3. *The map $f_y := \Psi_{f(y)}^{-1} \circ f \circ \Psi_y : B^s(q(y)) \times B^{l-s}(q(y)) \to \mathbb{R}^l = \mathbb{R}^k \times \mathbb{R}^{l-k}$ has the form*
$$f_y(u, v) = (A_y u + \eta_{2,y}(u, v), B_y v + \eta_{1,y}(u, v))$$
where $\eta_{1,y}(0, 0) = \eta_{2,y}(0, 0) = 0$, $\mathrm{d}\eta_{1,y}(0, 0) = \mathrm{d}\eta_{2,y}(0, 0) = 0$ and
$$\|A_y\| < \exp -(\chi - \epsilon), \|B_y^{-1}\| < \exp -(\chi - \epsilon)$$
For $z = (u, v) \in B^l(q(y))$, $\eta_y(z) = (\eta_{1,y}(z), \eta_{2,y}(z))$:
$$\|\mathrm{d}_z \eta_y\| < \epsilon, \|\eta_y(z)\| < \epsilon$$

Definition 2.2. *The points $y \in \Lambda_0$ are called regular points. For each regular point y, the set $N(y) = \Psi_y(B(q(y)))$ is called a regular neighborhood of y. Let $r(y)$ be the radius of the maximal ball contained in the regular neighbor hood $N(y)$. We say $r(y)$ is the size of $N(y)$.*

Theorem 2.3. *For each $\delta > 0$ and each sufficiently small $\epsilon(\delta) > 0$, there is a set $\Lambda_\delta \subset \Lambda_0$ which has compact intersection $\Lambda_{\delta,x}$ (may be empty) with each fiber $\{x\} \times M$, such that $\mu(\Lambda_\delta) > 1 - \delta$ and the following conditions hold:*

1. *The functions $y \mapsto q(y)$, $y \mapsto C(y)$ and $y \mapsto \Psi_y$ as in Theorem 2.1 for $\epsilon = \epsilon(\delta)$, and $y \mapsto r(y)$ are all continuous on $\Lambda_{\delta,x}$ for each $x \in X$.*
2. *The decomposition $\mathrm{T}_y M = \mathrm{d}_y\Psi_y\mathbb{R}^k \times \mathrm{d}_y\Psi_y\mathbb{R}^{l-k}$ depends continuously on y in $\Lambda_{\delta,x}$.*
3. *On Λ_δ, there are bounds: $q_\delta = \min\{q(y)\}$, $r_\delta = \min\{r(y)\}$, $C_\delta = \max\{C(y)\}$.*

With similar definitions and properties for admissible manifolds, we are able to derive the following version of Shadowing Lemma:

Theorem 2.4. *Given $\delta > 0$, for $\bar{q} < q_\delta$, set $\tilde{\Lambda}_\delta(\bar{q}) = \bigcup_{y\in\Lambda_\delta} \Psi_y(B(0,\bar{q}))$. Given $a \in \mathbb{Z}\bigcup\{-\infty\}$ and $b \in \mathbb{Z}\bigcup\{\infty\}$, a sequence $\{y_n = (x_n, p_n)\}_{a<n<b}$ is called an $(\delta,\bar{q})$-pseudo orbit for $f = (g, h_x)$ if there are $\{z_n \in \Lambda_{\delta,x_n}\}_{a<n<b}$ and $\{k_n\}_{a<n<b}$ such that for every n, $y_n \in \Psi_{z_n}(B(0,\bar{q}))$ and $f^{k_{n+1}-k_n}(y_n) \in \Psi_{z_{n+1}}(B(0,\bar{q}))$. Then there exists $\gamma = \gamma(\delta)$ such that for every (δ,γ)-pseudo orbit, there is a unique point $\tilde{y} \in Y$ such that $f^{k_n}(\tilde{y}) \in \Psi_{z_n}(B(0,q_\delta))$ for all $a < n < b$.*

3. Integrability of return time

Now we would like to take a proper Pesin set on which the shadowing techniques can be carried out.

Definition 3.1. *Let $\pi : Y \to X$ be the projection to the base. A measurable subset $P \subset Y$ is called a "Regular Tube", if for some $\delta > 0$, $\epsilon > 0$, $\nu_0 > 0$ and $\gamma = \gamma(\delta)$ as in Theorem 2.4, there exists for every $x \in B = \pi(P)$, a point $z(x) \in \Lambda_{\delta,x}$ such that $P_x = P\bigcap(\{x\} \times M) \subset \Psi_{z(x)}(B(0,\gamma))$, $\nu_x(P_x) > \nu_0$ and $m(B) > 1 - \epsilon$.*

The existence of such a "Regular Tube" is guaranteed in Section 2. In this "Regular Tube", we can take a measurable section $s : B \to P$, $\pi \circ s = \mathrm{id}_B$. Let $S = s(B)$. We are then going to consider the first return map f_P on P.

Proposition 3.2. *s can be chosen in such a way that the first return time from S to P is integrable with respect to m. In particular, we may assume every point in S returns to P in finite times.*

Proof. For every $y \in P$, denote by $n(y)$ the return time of y. Since μ is f-invariant and $\mu(P) > m(B) \cdot \nu_0 > 0$, we have:

$$0 < \int_P n(y)\mathrm{d}\mu = \mu(\bigcup_{j\geq 0} F^j(P')) \leq 1$$

But

$$\int_P n(y)\mathrm{d}\mu = \sum_{j=0}^{\infty} \mu(P_j)$$

where $P_j = P\backslash(\bigcup_{1\leq k\leq j} F^{-k}(P))$.

We may choose s such that for every $x \in B$, $s(x) \in P_j$ only if $\nu_x(P_x \backslash P_j) = 0$. Let $B_j = \pi_1(S \backslash (\bigcup_{1 \le k \le j} F^{-k}(P)) = \pi_1(S \bigcap P_j)$. By the way s is chosen and the assumption $\nu_x(P_x) > \nu_0$, we have $\mu(P_j) \ge \mu(\pi_1^{-1}(B_j) \bigcap P) > m(B_j) \cdot \nu_0$, hence

$$\int_B n(s(x))\mathrm{d}m = \sum_{j=0}^{\infty} m(B_j) < \sum_{j=0}^{\infty} \frac{1}{\nu_0} \mu(P_j) \le \frac{1}{\nu_0}$$

The return time is integrable. ■

4. Projected return map on the base

Now let us try to find the invariant measure described in Theorem 1.4. We may assume that g has no periodic point, or else the problem can be reduced to the case considered by Katok [6]. Moreover, g is invertible as we assumed earlier.

We are looking for an invariant set I which has finite intersection I_x with almost every fiber $\{x\} \times M$. The measure τ_x supported on I_x is the delta counting measure on $\{x\} \times M$. Then an invariant measure for f can be given by $\int \tau_x dm$.

To find the invariant set I, we start with a fixed "Regular Tube" P and a measurable section s as specified in section 3. Let f_P be the first return map on P and g_B the first return map for g on B. Define $r : B \to B$ by $r(x) = \pi \circ f_P(s(x))$. $r(x)$ is the projection of the return map on the base. g_B is invertible but r may not. For every $x \in B$, let $k(x)$ be such that $f^{k(x)} = f_P(s(x))$.

We can define a partial order on B: $x_1 \prec x_2$ iff there is $n \ge 0$ such that $g_B^n(x_1) = x_2$, i.e. x_2 is an image of x_1 under iterates of g. Since g is invertible and has no periodic point, this partial order is well defined. Moreover, if there is $n \ge 0$ such that $r^n(x_1) = x_2$ then we write $x_1 \prec\prec x_2$, which implies $x_1 \prec x_2$. This is also a partial order.

We can define an equivalence relation on B: $x_1 \backsim x_2$ iff $Q(x_1, x_2) := \{x \in B | x_1 \prec\prec x \text{ and } x_2 \prec\prec x\} \ne \emptyset$, i.e. there are $n_1, n_2 > 0$ such that $r^{n_1}(x_1) = r^{n_2}(x_2)$. If $x \in Q(x_1, x_2)$ then $r^n(x) \in Q(x_1, x_2)$ for all $n > 0$. If $x_1 \backsim x_2$, we must have $x_1 \prec x_2$ or $x_2 \prec x_1$, denoted by $x_1 \precsim x_2$ or $x_2 \precsim x_1$. If $x_1 \precsim x_2$, define $\sigma(x_1, x_2)$ as the minimal (with respect to $\prec$, throughout this paper) element in $Q(x_1, x_2)$. In particular, if $x_1 \prec\prec x_2$ then $x_1 \precsim x_2$ and $\sigma(x_1, x_2) = x_2$.

Remark. The equivalence relation $\sim$ defined here is crucial in this paper. If $x_1 \precsim x_2$, then $s(x_1)$ and $s(x_2)$ return to the same fiber after iteration of f_P. However, $s(x_1)$ does not necessarily return to P_{x_2}, i.e. two points in S may return to P on the same fiber and this may happen all the time. We had trouble dealing with this situation while looking for pseudo orbits. Introduction of this equivalence relation solved this problem. We can then, in each equivalence class, find a unique orbit of r (a sequence of returns, lifted to a pseudo orbit) to construct the invariant set.

Proposition 4.1. *For almost every $x \in B$, $J(x) = \{\sigma(x', x) | x' \precsim x, x' \ne x\}$ is finite. Denote by x^* the maximal element of $J(x)$. Then $x' \prec\prec x^*$ for all $x' \precsim x$. Let $W(x) = \{\bar{x} | x' \prec\prec \bar{x} \text{ for all } x' \precsim x\}$, then $x^* = \min W(x)$. Moreover, if $x_1 \precsim x_2$, then $x_1^* \prec\prec x_2^*$.*

Proof. For every $x \in B$, define the set of "jumps" $J'(x) := \{r(x') | x' \prec x, x' \ne x \text{ and } x \prec r(x')\}$. By integrability of return times (Proposition 3.2), $J'(x)$ must be finite for almost every $x \in B$. To see this, we can consider the set

$$\tilde{S} = \bigcup_{x\in B} \{f(s(x)), f^2(s(x)), \cdots, f^{k(x)}(s(x))\}$$

Let $\tilde{S}_j = \{x \in X||\tilde{S} \bigcap(\{x\} \times M)| = j\}$. We can count the return times and get

$$\sum_{j=0}^{\infty} j \cdot m(\tilde{S}_j) = \int_B k(x)\mathrm{d}m < \infty$$

and $|J'(x)| \le |\tilde{S} \bigcap(\{x\} \times M)| < \infty$ for almost every $x \in B$.

For every $x' \precsim x$ but $x' \ne x$, there must be $\bar{x} \in J'(x)$ such that $x' \prec\prec \bar{x}$ and $\sigma(x', x) = \sigma(x, \bar{x})$. So for different elements $x_1, x_2 \in J(x)$, there must be different elements $\bar{x}_1, \bar{x}_2 \in J'(x)$ such that $\sigma(\bar{x}_i, x) = x_i$, $i = 1, 2$. Hence $|J(x)| \le |J'(x)| < \infty$.

By definition, $x' \prec\prec \sigma(x', x)$ for every $x' \in B$, and $\sigma(x_1, x) \prec\prec \sigma(x_2, x)$ if $\sigma(x_1, x) \prec \sigma(x_2, x)$ since they are both images of x under iteration of r. So for every $x' \precsim x$, we have $x' \prec\prec \sigma(x', x) \prec\prec x^*$.

Since $x^* \in J(x)$, there is some $x' \precsim x$ such that $\sigma(x', x) = x^*$. Then for every $\bar{x} \in W(x)$, $\bar{x} \in Q(x', x)$. But $x^* = \sigma(x', x) = \min Q(x', x) \prec \bar{x}$. $x^* = \min W(x)$.

If $x_1 \precsim x_2$, then $x_1^* = \min\{x|x' \prec\prec x \text{ for all } x' \precsim x_1\} \prec \min\{x|x' \prec\prec x \text{ for all } x' \precsim x_2\} = x_2^*$, because the second set is contained in the first one. But $x_1 \prec\prec x_1^*$ and $x_1 \prec\prec x_2^*$, from previous discussion we must have $x_1^* \prec\prec x_2^*$. ■

Proposition 4.2. *Let $B_0 = \{x \in B|$there is no such $x' \ne x \in B$ that $x' \precsim x\}$. Then $m(B_0) = 0$. Hence by replacing B by $B\backslash(\bigcup_{k\in\mathbb{Z}} g_B^k(B_0))$ and P accordingly, we may assume that for every $x \in B$, there is at least one element $x' \in B$ such that $x' \precsim x$ but $x' \ne x$.*

Proof. If $m(B_0) > 0$, then there must be an element $x_0 \in B_0$ such that $B_0(x_0) = \{g_B^{-n}(x_0), n \in \mathbb{N}\} \bigcap B_0$ has infinitely many elements by Poincaré Recurrence Theorem because g_B is invertible and m-preserving. From the proof of Proposition 4.1, $J'(x_0)$ has finitely many elements. But for every $x \in B_0(x_0)$, there must be $x' \in J'(x_0)$ such that $x \prec\prec x'$. Hence there is an element $\tilde{x} \in J'(x_0)$ such that $\tilde{B}_0(x_0) = \{x \in B_0(x_0)|x \prec\prec \tilde{x}\}$ has infinitely many elements. But $x_1 \sim x_2$ for all $x_1, x_2 \in \tilde{B}_0(x_0) \subset B_0$ because $\bar{x} \in Q(x_1, x_2) \ne \emptyset$, which is a contradiction. ■

For every $x \in B$, let $G(x) := \{x' \in B|x' \sim x\}$ and $G^*(x) := \{(x')^*|x' \in G(x)\}$. If $x_1 \sim x_2$, then we must have $G(x_1) = G(x_2)$ and $G^*(x_1) = G^*(x_2)$.

Pick $H(x)$ in the following way:

1. If $G^*(x)$ is not properly defined: Let $H(x) = \emptyset$. (only for x in a set of measure zero)
2. If $G^*(x)$ has a minimal element $\tilde{x}$: Let $H(x) = \{f^n(s(\tilde{x}))\}_{-\infty<n<\infty}$.
3. If $G^*(x)$ has no minimal element: By Proposition 4.1, $G^*(x)$ can be completed to a full orbit of r, $\bar{G}(x) = \bigcup_{0\le n<\infty} r^n(G^*(x))$. In each equivalence class $G(x)$, $\bar{G}(x)$ is a sequence of returns and is uniquely defined in the sense $\bar{G}(x) = \bigcup_{x_1\sim x} \bigcap_{x_2 \precsim x_1} \{r^n(x_2)\}_{0\le n<\infty}$. $\bar{G}(x)$ ordered by "$\prec\prec$" can be viewed as a sequence $\{\bar{x}_n\}_{-\infty<n<\infty}$ and $r(x_n) = x_{n+1}$ for all n. Then the sequence $\{s(\bar{x}_n)\}_{-\infty<n<\infty}$ is in fact a (δ, γ)-pseudo orbit. Let us call it the pseudo orbit associated to x. Note the pseudo orbits associated to equivalent elements coincide. By the way the "Regular Tube" P was chosen, we can find $\tilde{y} \in Y$ as specified in Theorem 2.4. Let $H(x) = \{f^n(\tilde{y})\}_{-\infty<n<\infty}$.

Let $I = \bigcup_{x \in B} H(x)$. By definition, $H(x)$ is invariant for all $x \in B$. Hence I is f-invariant.

Proposition 4.3. *For almost every $x \in X$, $I_x = I \bigcap (\{x\} \times M)$ is nonempty and contains finitely many elements.*

Proof. For almost every $x \in B$, $I_x \supset (H(x) \bigcap (\{x\} \times M)) \neq \emptyset$ by definition. Note that $H(x_1) = H(x_2)$ if $x_1 \sim x_2$. For different elements $y_1, y_2 \in I_x$, there are $x_1, x_2 \in B$ such that $y_i \in H(x_i)$, $i = 1, 2$. We must have $x_1 \nsim x_2$ and $G(x_1) \bigcap G(x_2) = \emptyset$. But $G(x_i) \bigcap J'(x) \neq \emptyset$, $i = 1, 2$. So we have $|I_x| \leq |J'(x)|$.

Recall that $m(B) > 0$, so by ergodicity, I_x must be nonempty and contain finitely many elements for almost every $x \in X$. ■

Since I is invariant and I_x is finite, $\int \tau_x dm$ is an invariant measure as requested, where τ_x is the delta counting measure on $\{x\} \times M$ supported on I_x, for almost every $x \in X$. The entropy of this measure is the same as the entropy of the transformation g on the base.

5. Measures of intermediate entropies

In [7], Katok showed a stronger result:

Theorem 5.1. *If $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold and μ an ergodic hyperbolic measure for f with $h_\mu(f) > 0$, then for any $\epsilon > 0$ there exists a hyperbolic horseshoe Γ such that $h(f|_\Gamma) > h_\mu(f) - \epsilon$. Hence for any number β between zero and $h_\mu(f)$, there is an ergodic invariant measure μ_β such that $h_{\mu_\beta(f)} = \beta$.*

Detailed proof of this Theorem can be found in [8] and [3]. We are looking for analogous result to the theorem for our skew product diffeomorphisms, where μ is not necessarily a hyperbolic measure but the neutral direction can be factored out such that zero exponents only appear in the base. We would like to show there are ergodic measures with arbitrary intermediate entropy between the entropy on the base and the entropy of the skew product. In this case we should not expect to have any proper closed invariant subset. However, we can find an invariant set Γ that has closed intersection with almost every fiber, on which f acts like a horseshoe map. Moreover, this horseshoe can carry an entropy arbitrarily close to $h_\mu(f)$ in order to produce invariant measures with arbitrary intermediate entropies. This work is in preparation [13].

We may also ask the question if theorem 5.1 holds for any $C^{1+\alpha}$ diffeomorphism without the assumption that μ is hyperbolic. We have not yet found even a zero entropy measure in this general case.

Acknowledgement. The author was introduced to this problem by Anatole Katok and had numerous discussions with him. The author would like to show great appreciation to Katok for his consistent help.

References

[1] A. Avila, R. Krikorian, Reducibility or nonuniform hyperbolicity for quasiperiodic Schrodinger cocycles, *Ann. of Math. (2)*, **164** (2006), 911–940.

[2] L. Barreira and Y. Pesin, Smooth ergodic theory and nonuniformly hyperbolic dynamics, *Handbook of dynamical systems*, Vol. 1B, 57–263, Elsevier B. V., Amsterdam, 2006.

[3] L. Barreira and Y. Pesin, *Nonuniform hyperbolicity: Dynamics of systems with nonzero Lyapunov exponents*, Cambridge University Press, 2007.

[4] F. Béguin, S. Crovisier, F. Le Roux, Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy-Rees technique. *Ann. Sci. École Norm. Sup. (4)* **40** (2007), 251–308.

[5] M. Herman, Construction d'un difféomorphisme minimal d'entropie topologique non nulle, *Ergodic Theory Dyn. Syst.* **1** (1981), 65–76.

[6] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math. I.H.E.S.* **51** (1980), 137–173.

[7] A. Katok, Nonuniform hyperbolicity and structure of smooth dynamical systems, *Proceedings of the International Congress of Mathematicians*, Vol. 1, **2** (Warsaw, 1983), 1245–1253, PWN, Warsaw, 1984.

[8] A. Katok and L. Mendoza, Dynamical systems with nonuniformly hyperbolic behavior, supplement to *Introduction to the Modern Theory of Dynamical Systems*, by A. Katok and B. Hasselblatt, Cambridge University Press, 1995.

[9] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms, *Ann. Math.* (2) **122** (1985) , 509–574.

[10] Y. Pesin, Families of invariant manifolds corresponding to nonzero characteristic exponents, *Math. USSR-Izv*, **40** (1976), 1261–1305.

[11] D. Ruelle, An inequality for the entropy of differentiable map, *Bol. Soc. Brasil. Mat.* **9** (1978), 83–87.

[12] D. Ruelle, A. Wilkinson, Absolutely Singular Dynamical Foliations, *Commun. Math. Phys.* **219** (2001), 481–487.

[13] P. Sun, Measures of intermediate entropies for skew product diffeomorphisms, in preparation.